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ON THE SPATIAL AND TEMPORAL BEHAVIOUR IN DYNAMICS OF POROUS ELASTIC MIXTURES *

ПРО ПРОСТОРОВУ І ЧАСОВУ ПОВЕДІНКУ В ДИНАМІЦІ ПОРИСТИХ ПРУЖНИХ СУМІШЕЙ

In this paper, we study the spatial and temporal behaviour of dynamic processes in porous elastic mixtures. For the spatial behaviour, we use the time-weighted surface power function method in order to obtain a more precise determination of the domain of influence and establish spatial decay estimates of the Saint–Venant type with respect to time-independent decay rate for the inside of the domain of influence. For the asymptotic temporal behaviour, we use the Cesáro means associated with the kinetic and strain energies and establish the asymptotic equipartition of the total energy. A uniqueness theorem is proved for finite and infinite bodies and we note that it is free of any kind of a priori assumptions on the solutions at infinity.

Вивчається розвиток у часі і просторі динамічних процесів у пористих пружних сумішах. Для аналізу просторової поведінки використано метод поверхневої часово-зрівноваженої енергетичної функції для більш точного визначення області впливу і встановлено оцінку просторового згасання типу Сен-Венана відносно часового згасання в межах області впливу. Для асимптотики часової поведінки використано метод Чезаро, пов'язаний з кінетичною та деформаційною енергіями, та встановлено асимптотичний рівнорозподіл сумарної енергії. Доведено теорему єдиності для скінченних та нескінченних тіл без будь-яких попередніх припущень щодо розв'язків на нескінченності.

1. Introduction. Various theories have been proposed in literature for describing the behaviour of the chemically reacting media (see, for example, Truesdell and Toupin [1], Kelly [2], Eringen and Ingram [3, 4], Green and Naghdi [5, 6], Dunwoody and Müller [8], Bedford and Drumheller [9], etc.)

Recenty Ieşan [10] has developed a theory for binary mixtures of of granular materials in Lagrangian description, in wich the independent constitutive variables are the displacement gradients, displacement fields, volume fractions and volume fraction gradients. The theory takes into account the results established previously by Nunziato and Cowin [11], Goodman and Cowin [12] and Drumheller [13]. The intended applications for such a theory are to granular composites, solid explosives and geological materials.

In [10] a linear theory is also presented and some uniqueness results for bounded bodies are established for the linear dynamic theory with no definiteness assumptions on the elasticities and without any restriction on the initial stresses.

The present paper studies the spatial and temporal behaviour of the solutions to the boundary-initial value problems in the linear dynamic theory of porous elastic mixtures as developed in [10].

For the spatial behaviour of the dynamic processes in porous elastic mixtures we use the time-weighted surface power method developed in [14]. Thus, we introduce a time-weighted surface measure associated with the dynamic process in question and then we establish a first-order partial differential inequality, whose integration gives a good information upon the spatial behavior. Then we obtain a more precisely version of the domain of influence in the sense that for each fixed $t \in [0,T]$ the whole activity is vanishing at distances to the support of the given data on $[0,T]$ greater than ct, where c is a constant characteristic to the elastic mixture. A spatial decay estimate of the Saint-Venant's type is established for describing the spatial behaviour of the dynamic process inside the domain of influence.

As regards the temporal behaviour of the dynamic processes in porous elastic mix-

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tures, we introduce the Cesáro means of various energies and then establish the relations describing the asymptotic equipartition of energy. To this end we use some Lagrange identities and the method developed by Day [15] and Levine [16].

The plan of our paper is the folowing one. In the Section 2 we present the basic equations of the linear dynamic theory of porous mixtures developed in [10]. Some constitutive assumptions and other useful results are also presented. The auxiliary identities are established in the Section 3, while in the Section 4 a time-weighted surface measure is defined and its properties are studied. Moreover, a first-order partial differential inequality is established for this measure. The main result concerning the spatial behaviour is presented in the Section 5 and some uniqueness results are obtained as a direct consequence. In the Section 6 we introduce the Cesáro means of various energies and establish the asymptotic equipartition of the total energy.

2. Basic equations. Throughout this article, the motions of continuum are studied with respect to a fixed orthonormal frame in $\mathbb{R}^3$. Then, we deal with functions of position and time. Moreover, it is useful to stress that in the following text the tensor components of order $p \geq 1$ will appear with Latin subscripts, ranging over the integers $\{1, 2, 3\}$, and summation over repeated subscripts will be implied. Greek indices are understood to range over $\{1, \ldots, 9\}$ if they are lower case letters, or over $\{1, 2\}$ if they are upper case letters; the summation convention is not used for these indices. Occasionally, we shall use bold-face character and typical notations for vectors and operations upon them. Superposed dots or subscripts preceded by a comma will mean partial derivative with respect to time or corresponding coordinates.

Let B be a bounded or unbounded regular region in the physical 3-dimensional space, whose boundary ∂B is a piecewise smooth surface. A chemically inert binary mixture of two interacting porous elastic solids, c_1 and c_2, in a given reference configuration, is into B.

The positions of particles of c_1 and c_2 at time t are $\mathbf{x}$ and $\mathbf{y}$ respectively, i.e.,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad \mathbf{y} = \mathbf{y}(\mathbf{Y}, t) \qquad \mathbf{X}, \mathbf{Y} \in B, \quad t \in I,$$

in wich $\mathbf{X}$ and $\mathbf{Y}$ are reference positions of these particles, $I = [0, \infty)$. By following Bedford and Stern [17], we assume that $\mathbf{X} = \mathbf{Y}$.

Let the top label α refers the various fields to the constituent c_α. Taking into account the linear theory, the behaviour of a binary mixture of elastic solids is governed by the local balance equations (see Ieşan [10])

$$\begin{gathered} S_{ji,j}^{(\alpha)} + (-1)^\alpha p_i + \rho^{(\alpha)} f_i^{(\alpha)} = \rho^{(\alpha)} \ddot{u}_i^{(\alpha)}, \\ h_{i,i}^{(\alpha)} + g^{(\alpha)} + \rho^{(\alpha)} \ell^{(\alpha)} = \rho^{(\alpha)} \chi^{(\alpha)} \ddot{\varphi}^{(\alpha)}, \qquad \text{on } B \times (0, \infty). \end{gathered} \tag{1}$$

In these equations, $\mathbf{S}^{(\alpha)}$, $\mathbf{f}^{(\alpha)}$ are the stress tensor and the body force associated with c_α; $\mathbf{p}$ is the vector field for characterizing the mechanical interaction between the constituents c_1 and c_2; $\mathbf{h}^{(\alpha)}$, $g^{(\alpha)}$, $\ell^{(\alpha)}$ are the equilibrated stress vector, intrinsic and extrinsic equilibrated body force associated to c_α, respectively.

Moreover, $\mathbf{u}^{(\alpha)}$ is the displacement vector fields associated with c_α; $\varphi^{(\alpha)}$ is the change in volume fraction for the constituent c_α starting from the reference configuration.

Finally, $\rho^{(\alpha)}$, $\chi^{(\alpha)}$ are the bulk mass density and the equilibrated inertia of the material c_α in the reference state.

According to the classical interpretation of system (1), we assume that

i) $u_i^{(\alpha)}$, $\varphi^{(\alpha)} \in C^{2,2}(\overline{B} \times I)$;

ii) $S_{ji}^{(\alpha)}$, $h_i^{(\alpha)} \in C^{1,0}(\overline{B} \times I)$, $\quad p_i \in C^{0,0}(\overline{B} \times I)$;

iii) $f_i^{(\alpha)},\ g^{(\alpha)},\ \ell^{(\alpha)} \in C^{0,0}(\bar{B}\times I),\quad \rho^{(\alpha)},\ \chi^{(\alpha)} \in C^0(\bar{B})$,
where, $\bar{B}$ is the closure of B.

Then, we introduce the 29-dimensional vector field

$$\mathbf{E}(\mathbf{U}) \equiv \left\{e_{ij}(\mathbf{U}), g_{ij}(\mathbf{U}), \varphi^{(1)}, \varphi^{(2)}, d_i(\mathbf{U}), \varphi_{,i}^{(1)}, \varphi_{,i}^{(2)}\right\},$$

with

$$\mathbf{U} \equiv \left\{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \varphi^{(1)}, \varphi^{(2)}\right\}$$

and

$$e_{ij}(\mathbf{U}) = \frac{1}{2}\left(u_{i,j}^{(1)} + u_{j,i}^{(1)}\right), \qquad g_{ij}(\mathbf{U}) = u_{j,i}^{(1)} + u_{i,j}^{(2)},$$
$$d_i(\mathbf{U}) = u_i^{(1)} - u_i^{(2)}, \quad \text{on } \bar{B}\times I. \tag{2}$$

Now we define the magnitude of $\mathbf{E}(\mathbf{U})$ by

$$|\mathbf{E}(\mathbf{U})| \equiv \left\{e_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + g_{ij}(\mathbf{U})g_{ij}(\mathbf{U}) + d_i^2(\mathbf{U}) + \right.$$
$$\left. + \sum_{\alpha=1}^{2}\left[\varphi^{(\alpha)}(\mathbf{U})\varphi^{(\alpha)}(\mathbf{U}) + \varphi_{,i}^{(\alpha)}(\mathbf{U})\varphi_{,i}^{(\alpha)}(\mathbf{U})\right]\right\}^{1/2}.$$

Our attention is focused on a homogeneous, centrosymmetric mixture, by supposing that the initial continuum is free from stresses. Thus, in the context of our theory, the internal energy density associated with $\mathbf{U}$ is given by [10]

$$W(\mathbf{U}) = \frac{1}{2}\left[A_{ijrs}e_{ij}(\mathbf{U})e_{rs}(\mathbf{U}) + C_{ijrs}g_{ij}(\mathbf{U})g_{rs}(\mathbf{U}) + \zeta\varphi^{(1)}\varphi^{(1)} + \right.$$
$$\left. + \mu\varphi^{(2)}\varphi^{(2)} + \alpha_{ij}\varphi_{,i}^{(1)}\varphi_{,j}^{(1)} + \gamma_{ij}\varphi_{,i}^{(2)}\varphi_{,j}^{(2)} + a_{ij}d_i(\mathbf{U})d_j(\mathbf{U})\right] +$$
$$+ B_{ijrs}e_{ij}(\mathbf{U})g_{rs}(\mathbf{U}) + D_{ij}e_{ij}(\mathbf{U})\varphi^{(1)} + E_{ij}e_{ij}(\mathbf{U})\varphi^{(2)} +$$
$$+ M_{ij}g_{ij}(\mathbf{U})\varphi^{(1)} + N_{ij}g_{ij}(\mathbf{U})\varphi^{(2)} + \beta_{ij}\varphi_{,i}^{(1)}\varphi_{,j}^{(2)} +$$
$$+ b_{ij}d_i(\mathbf{U})\varphi_{,j}^{(1)} + c_{ij}d_i(\mathbf{U})\varphi_{,j}^{(2)} + \tau\varphi^{(1)}\varphi^{(2)}. \tag{3}$$

The material coefficients, appearing in the previous equations (3), are constants and they obey the following symmetry relations:

$$A_{ijrs} = A_{jirs} = A_{rsij}, \quad B_{ijrs} = B_{jirs}, \quad C_{ijrs} = C_{rsij}, \quad a_{ij} = a_{ji},$$
$$\alpha_{ij} = \alpha_{ji}, \quad \gamma_{ij} = \gamma_{ji}, \quad D_{ij} = D_{ji}, \quad E_{ij} = E_{ji}. \tag{4}$$

The constitutive equations are

$$S_{ji}^{(1)}(\mathbf{U}) = (A_{ijrs} + B_{rsji})e_{rs}(\mathbf{U}) + (B_{ijrs} + C_{jirs})g_{rs}(\mathbf{U}) +$$
$$+ (D_{ij} + M_{ij})\varphi^{(1)} + (E_{ij} + N_{ij})\varphi^{(2)},$$
$$S_{ji}^{(2)}(\mathbf{U}) = B_{rsij}e_{rs}(\mathbf{U}) + C_{jirs}g_{rs}(\mathbf{U}) + M_{ij}\varphi^{(1)} + N_{ij}\varphi^{(2)},$$
$$g^{(1)}(\mathbf{U}) = -D_{rs}e_{rs}(\mathbf{U}) - M_{rs}g_{rs}(\mathbf{U}) - \zeta\varphi^{(1)} - \tau\varphi^{(2)},$$
$$g^{(2)}(\mathbf{U}) = -E_{rs}e_{rs}(\mathbf{U}) - N_{rs}g_{rs}(\mathbf{U}) - \tau\varphi^{(1)} - \mu\varphi^{(2)}, \tag{5}$$

$$p_i(\mathbf{U}) = a_{ij}\, d_j(\mathbf{U}) + b_{ij}\, \varphi_j^{(1)} + c_{ij}\, \varphi_j^{(2)},$$

$$h_i^{(1)}(\mathbf{U}) = \alpha_{ij}\, \varphi_j^{(1)} + \beta_{ij}\, \varphi_j^{(2)} + b_{ji}\, d_j(\mathbf{U}),$$

$$h_i^{(2)}(\mathbf{U}) = \beta_{ji}\, \varphi_j^{(1)} + \gamma_{ij}\, \varphi_j^{(2)} + c_{ji}\, d_j(\mathbf{U}).$$

Let $\mathcal{A}_1 = \|\tilde{a}_{KL}\|$, $K, L = 1, \ldots, 20$,

$$\begin{gathered}\tilde{a}_{\Gamma\Delta} = A_{\Gamma\Delta}, \quad \tilde{a}_{\Gamma(9+\Delta)} = B_{\Gamma\Delta}, \quad \tilde{a}_{\Gamma 19} = D_\Gamma, \quad \tilde{a}_{\Gamma 20} = E_\Gamma, \\ \tilde{a}_{(9+\Gamma)\Delta} = B_{\Delta\Gamma}, \quad \tilde{a}_{(9+\Gamma)(9+\Delta)} = C_{\Gamma\Delta}, \quad \tilde{a}_{(9+\Gamma)19} = M_\Gamma, \quad \tilde{a}_{(9+\Gamma)20} = N_\Gamma, \\ \tilde{a}_{19\Delta} = D_\Delta, \quad \tilde{a}_{19(9+\Delta)} = M_\Delta, \quad \tilde{a}_{1919} = \zeta, \quad \tilde{a}_{1920} = \tau, \\ \tilde{a}_{20\Delta} = E_\Delta, \quad \tilde{a}_{20(9+\Delta)} = N_\Delta, \quad \tilde{a}_{2019} = \tau, \quad \tilde{a}_{2020} = \mu,\end{gathered} \tag{6}$$

where we have called the nine index combinations $(i\,j)$ or $(r\,s)$ by capital Greek letters (i.e. Γ, Δ, and so on). Now, let O be the null matrix 20×9 and $\mathcal{A}_2 = \|\tilde{b}_{KL}\|$, $K, L = 1, \ldots, 9$, be

$$\begin{gathered}\tilde{b}_{ij} = a_{ij}, \quad \tilde{b}_{i(3+j)} = b_{ij}, \quad \tilde{b}_{i(6+j)} = c_{ij}, \\ \tilde{b}_{(3+i)j} = b_{ji}, \quad \tilde{b}_{(3+i)(3+j)} = \alpha_{ij}, \quad \tilde{b}_{(3+i)(6+j)} = \beta_{ij}, \\ \tilde{b}_{(6+i)j} = c_{ji}, \quad \tilde{b}_{(6+i)(3+j)} = \beta_{ji}, \quad \tilde{b}_{(6+i)(6+j)} = \gamma_{ij}.\end{gathered} \tag{7}$$

Then, the energy density (3) assumes the form

$$W(\mathbf{U}) = \frac{1}{2}\sum_{K,L=1}^{29} \tilde{A}_{KL} E_K(\mathbf{U}) E_L(\mathbf{U}) = \frac{1}{2}\mathbf{E}(\mathbf{U}) \cdot \mathcal{A}\mathbf{E}(\mathbf{U}), \tag{8}$$

where the matrix $\mathcal{A} = \|\tilde{A}_{KL}\|$, $K, L = 1, \ldots, 29$, is defined by

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & O \\ O^T & \mathcal{A}_2 \end{bmatrix}, \tag{9}$$

and O^T is the transposed matrix of O.

In that follows we assume that $\rho^{(\alpha)}$, $\chi^{(\alpha)}$ are strictly positive and $W(\mathbf{U})$ is a positive definite quadratic form; thus, there exist the positive constants ξ_m and ξ_M so that

$$\xi_m |\mathbf{E}(\mathbf{U})|^2 \le 2W(\mathbf{U}) \le \xi_M |\mathbf{E}(\mathbf{U})|^2, \tag{10}$$

where ξ_m is the minimum elastic moduli and ξ_M is the maximum elastic moduli.

Let

$$\mathbf{S}(\mathbf{U}) \equiv \left\{S_{ji}^{(1)}(\mathbf{U}), S_{ji}^{(2)}(\mathbf{U}), g^{(1)}(\mathbf{U}), g^{(2)}(\mathbf{U}), p_i(\mathbf{U}), h_i^{(1)}(\mathbf{U}), h_i^{(2)}(\mathbf{U})\right\},$$

then the magnitude of $S(\mathbf{U})$ is defined by

$$|\mathbf{S}(\mathbf{U})| \equiv \left\{\sum_{\alpha=1}^{2}\left[S_{ji}^{(\alpha)}(\mathbf{U})S_{ji}^{(\alpha)}(\mathbf{U}) + h_i^{(\alpha)}(\mathbf{U})h_i^{(\alpha)}(\mathbf{U}) + g^{(\alpha)}(\mathbf{U})g^{(\alpha)}(\mathbf{U})\right] + p_i(\mathbf{U})p_i(\mathbf{U})\right\}^{1/2}.$$

Taking into account the equations (5) – (9), as in [18] it follows that

$$|\mathbf{S}(\mathbf{U})|^2 = \mathcal{A}\mathbf{E} \cdot \mathcal{A}\mathbf{E} = \mathbf{E} \cdot \mathcal{A}^2\mathbf{E} \le \xi_M \mathbf{E} \cdot \mathcal{A}\mathbf{E} = 2\xi_M W(\mathbf{U}). \tag{11}$$

The surface tractions $\mathbf{s}^{(\alpha)}(\mathbf{U})$ and $h^{(\alpha)}(\mathbf{U})$ are defined by

$$s_i^{(\alpha)}(\mathbf{U}) = S_{ji}^{(\alpha)}(\mathbf{U})n_j, \quad h^{(\alpha)}(\mathbf{U}) = h_j^{(\alpha)}(\mathbf{U})n_j, \tag{12}$$

where $\mathbf{n}$ is the outward unit normal vector to boundary surface. The relations (11), (12) imply that

$$\sum_{\alpha=1}^{2}\left[s_i^{(\alpha)}(\mathbf{U})s_i^{(\alpha)}(\mathbf{U}) + h^{(\alpha)}(\mathbf{U})h^{(\alpha)}(\mathbf{U})\right] \le |\mathbf{S}(\mathbf{U})|^2 \le 2\xi_M W(\mathbf{U}). \tag{13}$$

If we introduce the notations

$$\begin{gathered} a_{ijrs} = A_{jirs} + B_{rsji} + B_{jisr} + C_{jisr}, \\ b_{ijrs} = B_{jirs} + C_{jirs}, \quad d_{ijrs} = C_{ijrs}, \\ \tau_{ij} = D_{ij} + M_{ij}, \quad \sigma_{ij} = E_{ij} + N_{ij}, \end{gathered} \tag{14}$$

the equation (3) becomes

$$\begin{gathered} W(\mathbf{U}) = \frac{1}{2}\Big[a_{ijrs}u_{i,j}^{(1)}u_{r,s}^{(1)} + d_{ijrs}u_{i,j}^{(2)}u_{r,s}^{(2)} + \zeta\varphi^{(1)}\varphi^{(1)} + \mu\varphi^{(2)}\varphi^{(2)} + \\ + \alpha_{ij}\varphi_{,i}^{(1)}\varphi_{,j}^{(1)} + \gamma_{ij}\varphi_{,i}^{(2)}\varphi_{,j}^{(2)} + a_{ij}d_i(\mathbf{U})d_j(\mathbf{U})\Big] + b_{ijrs}u_{i,j}^{(1)}u_{r,s}^{(2)} + \\ + \tau_{ij}u_{i,j}^{(1)}\varphi^{(1)} + \sigma_{ij}u_{i,j}^{(1)}\varphi^{(2)} + M_{ij}u_{i,j}^{(2)}\varphi^{(1)} + N_{ij}u_{i,j}^{(2)}\varphi^{(2)} + \\ + \beta_{ij}\varphi_{,i}^{(1)}\varphi_{,j}^{(2)} + b_{ij}d_i(\mathbf{U})\varphi_{,j}^{(1)} + c_{ij}d_i(\mathbf{U})\varphi_{,j}^{(2)} + \tau\varphi^{(1)}\varphi^{(2)}. \end{gathered} \tag{15}$$

Using the symmetry relation (4), we get

$$\begin{gathered} a_{ijrs} = a_{rsij}, \quad d_{ijrs} = d_{rsij}, \quad a_{ij} = a_{ji}, \\ \alpha_{ij} = \alpha_{ji}, \quad \gamma_{ij} = \gamma_{ji}. \end{gathered} \tag{16}$$

The constitutive equations (5) become

$$\begin{gathered} S_{ji}^{(1)}(\mathbf{U}) = a_{ijrs}u_{r,s}^{(1)} + b_{ijrs}u_{r,s}^{(2)} + \tau_{ij}\varphi^{(1)} + \sigma_{ij}\varphi^{(2)}, \\ S_{ji}^{(2)}(\mathbf{U}) = b_{rsij}u_{r,s}^{(1)} + d_{ijrs}u_{r,s}^{(2)} + M_{ij}\varphi^{(1)} + N_{ij}\varphi^{(2)}, \\ g^{(1)}(\mathbf{U}) = -\tau_{rs}u_{r,s}^{(1)} - M_{rs}u_{r,s}^{(2)} - \zeta\varphi^{(1)} - \tau\varphi^{(2)}, \\ g^{(2)}(\mathbf{U}) = -\sigma_{rs}u_{r,s}^{(1)} - N_{rs}u_{r,s}^{(2)} - \tau\varphi^{(1)} - \mu\varphi^{(2)}, \\ p_i(\mathbf{U}) = a_{ij}d_j(\mathbf{U}) + b_{ij}\varphi_{,j}^{(1)} + c_{ij}\varphi_{,j}^{(2)}, \\ h_i^{(1)}(\mathbf{U}) = \alpha_{ij}\varphi_{,j}^{(1)} + \beta_{ij}\varphi_{,j}^{(2)} + b_{ji}d_j(\mathbf{U}), \\ h_i^{(2)}(\mathbf{U}) = \beta_{ji}\varphi_{,j}^{(1)} + \gamma_{ij}\varphi_{,j}^{(2)} + c_{ji}d_j(\mathbf{U}). \end{gathered} \tag{17}$$

It follows from the equations (15), (16), (17) that

$$2W(\mathbf{U}) = \sum_{\alpha=1}^{2}\left[S_{ji}^{(\alpha)}(\mathbf{U})u_{i,j}^{(\alpha)} + h_i^{(\alpha)}(\mathbf{U})\varphi_{,i}^{(\alpha)} - g^{(\alpha)}(\mathbf{U})\varphi^{(\alpha)}\right] + p_i(\mathbf{U})d_i(\mathbf{U}) \tag{18}$$

and

$$\dot{W}(\mathbf{U}) = \sum_{\alpha=1}^{2}\left[S_{ji}^{(\alpha)}(\mathbf{U})\dot{u}_{i,j}^{(\alpha)} + h_i^{(\alpha)}(\mathbf{U})\dot{\varphi}_{,i}^{(\alpha)} - g^{(\alpha)}(\mathbf{U})\dot{\varphi}^{(\alpha)}\right] + p_i(\mathbf{U})\dot{d}_i(\mathbf{U}). \tag{19}$$

We consider the initial-boundary-value problem $\mathcal{P}$ defined by the equations of motion (1), the geometrical equations (2) and the constitutive equations (5) and the following initial-boundary conditions

$$u_i^{(\alpha)} = a_i^{(\alpha)}, \qquad \dot{u}_i^{(\alpha)} = \dot{a}_i^{(\alpha)},$$
$$\varphi^{(\alpha)} = \varphi_0^{(\alpha)}, \qquad \dot{\varphi}^{(\alpha)} = \dot{\varphi}_0^{(\alpha)} \qquad \text{on } \overline{B} \times \{0\} \tag{20}$$

and

$$u_i^{(\alpha)} = \tilde{u}_i^{(\alpha)} \quad \text{on } \overline{\Sigma}_1 \times I, \qquad s_i^{(\alpha)} = \tilde{s}_i^{(\alpha)} \quad \text{on } \Sigma_2 \times I,$$
$$\varphi^{(\alpha)} = \tilde{\varphi}^{(\alpha)} \quad \text{on } \overline{\Sigma}_3 \times I, \qquad h^{(\alpha)} = \tilde{h}^{(\alpha)} \quad \text{on } \Sigma_4 \times I, \tag{21}$$

where Σ_i, $i = 1, \ldots, 4$, are the subsets of ∂B such that

$$\overline{\Sigma}_1 \cup \Sigma_2 = \overline{\Sigma}_3 \cup \Sigma_4 = \partial B, \qquad \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \varnothing.$$

The items on right-hand in the equations (20) and (21) are prescribed continuous functions; along with $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\ell^{(1)}$, $\ell^{(2)}$ these constitute the external data of the problem $\mathcal{P}$.

An array field $\mathbf{U} = \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \varphi^{(1)}, \varphi^{(2)}\}$, meeting all equation (1), (2), (5), (20) and (21), will be referred to a (regular) solution of the problem $\mathcal{P}$.

3. Auxiliary identities. In this section we establish some integral identities that we will use in next sections.

Lemma 1. *Let* $\mathbf{U}$ *be a solution of the initial-boundary-value problem* $\mathcal{P}$. *Then, for every regular region* $P \subset B$ *with a regular boundary* ∂P *it follows that*

$$\frac{1}{2}\int_P e^{-\lambda t}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t) + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right] + 2W(\mathbf{U}(t))\right\}dv +$$
$$+ \frac{\lambda}{2}\int_0^t\int_P e^{-\lambda s}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right] + 2W(\mathbf{U}(s))\right\}dv\,ds =$$
$$= \int_0^t\int_P e^{-\lambda s}\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)} f_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)}\ell^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right]dv\,ds +$$
$$+ \frac{1}{2}\int_P\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(0)\dot{u}_i^{(\alpha)}(0) + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(0)\dot{\varphi}^{(\alpha)}(0)\right] + 2W(\mathbf{U}(0))\right\}dv +$$
$$+ \int_0^t\int_{\partial P} e^{-\lambda s}\sum_{\alpha=1}^{2}\left[s_i^{(\alpha)}(\mathbf{U}(s))\dot{u}_i^{(\alpha)}(s) + h^{(\alpha)}(\mathbf{U}(s))\dot{\varphi}^{(\alpha)}(s)\right]da\,ds, \tag{22}$$

where λ *is a positive parameter and* $t \in I$.

Proof. The equations (1) and (19) lead to

$$e^{-\lambda s}\frac{\partial}{\partial s}\left\{\frac{1}{2}\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right] + 2W(\mathbf{U}(s))\right\} =$$
$$= e^{-\lambda s}\sum_{\alpha=1}^{2}\left\{\rho^{(\alpha)} f_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)}\ell^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s) +\right.$$
$$\left. + \left[S_{ji}^{(\alpha)}(\mathbf{U}(s))\dot{u}_i^{(\alpha)}(s) + h_j^{(\alpha)}(\mathbf{U}(s))\dot{\varphi}^{(\alpha)}(s)\right]_{,j}\right\}. \tag{23}$$

By integration of the equation (23) over $P\times[0,t]$ and by using the divergence theorem, we obtain the equation (22).

If we introduce

$$\mathcal{E}(t) = \int_B \frac{1}{2}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t)+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right]+2W(\mathbf{U}(t))\right\}dv, \quad (24)$$

then, for $\lambda=0$ and $P=B$ the equation (22) reduces to

$$\mathcal{E}(t) = \mathcal{E}(0)+\int_0^t\int_B\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}f_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s)+\rho^{(\alpha)}\ell^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right]dv\,ds+$$

$$+\int_0^t\int_{\partial B}\sum_{\alpha=1}^{2}\left[s_i^{(\alpha)}(\mathbf{U}(s))\dot{u}_i^{(\alpha)}(s)+h^{(\alpha)}(\mathbf{U}(s))\dot{\varphi}^{(\alpha)}(s)\right]da\,ds. \quad (25)$$

We note that $\mathcal{E}(t)$ is a measure of the energy stored in B at time t.

Lemma 2. *Let* $\mathbf{U}$ *be a solution of the initial-boundary-value problem* $\mathcal{P}$. *Then, for every regular region* $P\subset B$ *with a regular boundary* ∂P *it follows that*

$$\int_P\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}u_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t)+\rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right]dv =$$

$$=\int_0^t\int_P\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s)+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right]-2W(\mathbf{U}(s))\right\}dv\,ds+$$

$$+\int_P\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}u_i^{(\alpha)}(0)\dot{u}_i^{(\alpha)}(0)+\rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(0)\dot{\varphi}^{(\alpha)}(0)\right]dv+$$

$$+\int_0^t\int_{\partial P}\sum_{\alpha=1}^{2}\left[s_i^{(\alpha)}(\mathbf{U}(s))u_i^{(\alpha)}(s)+h^{(\alpha)}(\mathbf{U}(s))\varphi^{(\alpha)}(s)\right]da\,ds+$$

$$+\int_0^t\int_P\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}f_i^{(\alpha)}(s)u_i^{(\alpha)}(s)+\rho^{(\alpha)}\ell^{(\alpha)}(s)\varphi^{(\alpha)}(s)\right]dv\,ds, \qquad t\in I. \quad (26)$$

Proof. The relations (1) and (18) imply that

$$\frac{\partial}{\partial s}\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}u_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s)+\rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right] =$$

$$=\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s)+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right]+$$

$$+\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}f_i^{(\alpha)}(s)u_i^{(\alpha)}(s)+\rho^{(\alpha)}\ell^{(\alpha)}(s)\varphi^{(\alpha)}(s)\right]-2W(\mathbf{U}(s))+$$

$$+\sum_{\alpha=1}^{2}\left[S_{ji}^{(\alpha)}(\mathbf{U}(s))u_i^{(\alpha)}(s)+h_j^{(\alpha)}(\mathbf{U}(s))\varphi^{(\alpha)}(s)\right]_{,j}. \quad (27)$$

The relation (26) follows from (27) by integration over $P\times[0,t]$ and by using the divergence theorem.

Lemma 3. *Let* $\mathbf{U}$ *be a solution of the initial-boundary-value problem. Then, for every regular region* $P\subset B$ *with a regular boundary* ∂P *it follows that*

$$2\int_P \sum_{\alpha=1}^{2} \left[\rho^{(\alpha)} u_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t) + \rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right] dv =$$

$$= \int_P \sum_{\alpha=1}^{2} \left[\rho^{(\alpha)} u_i^{(\alpha)}(0)\dot{u}_i^{(\alpha)}(2t) + \rho^{(\alpha)}\dot{u}_i^{(\alpha)}(0)\dot{u}_i^{(\alpha)}(2t) + \right.$$

$$\left. + \rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(0)\dot{\varphi}^{(\alpha)}(2t) + \rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(2t)\dot{\varphi}^{(\alpha)}(0)\right] dv +$$

$$+ \int_0^t \int_P \sum_{\alpha=1}^{2} \left[\rho^{(\alpha)} f_i^{(\alpha)}(t-s) u_i^{(\alpha)}(t+s) - \rho^{(\alpha)} f_i^{(\alpha)}(t+s) u_i^{(\alpha)}(t-s) + \right.$$

$$\left. + \rho^{(\alpha)}\ell^{(\alpha)}(t-s)\varphi^{(\alpha)}(t+s) - \rho^{(\alpha)}\ell^{(\alpha)}(t+s)\varphi^{(\alpha)}(t-s)\right] dv\, ds +$$

$$+ \int_0^t \int_{\partial P} \sum_{\alpha=1}^{2} \left[s_i^{(\alpha)}(\mathbf{U}(t-s)) u_i^{(\alpha)}(t+s) - s_i^{(\alpha)}(\mathbf{U}(t+s)) u_i^{(\alpha)}(t-s) + \right.$$

$$\left. + h^{(\alpha)}(\mathbf{U}(t-s))\varphi^{(\alpha)}(t+s) - h^{(\alpha)}(\mathbf{U}(t+s))\varphi^{(\alpha)}(t-s)\right] da\, ds, \qquad t \in I. \tag{28}$$

Proof. For every function $\phi \in C^2(I)$, the following identity holds

$$\ddot{\phi}(t-s)\phi(t+s) - \ddot{\phi}(t+s)\phi(t-s) =$$

$$= -\frac{\partial}{\partial s}\left\{\dot{\phi}(t-s)\phi(t+s) + \phi(t-s)\dot{\phi}(t+s)\right\}, \qquad s \in [0,t], \qquad t \in I. \tag{29}$$

On the other hand, in view of the relation (1), we have

$$\sum_{\alpha=1}^{2} \left[\rho^{(\alpha)}\ddot{u}_i^{(\alpha)}(t-s) u_i^{(\alpha)}(t+s) - \rho^{(\alpha)}\ddot{u}_i^{(\alpha)}(t+s) u_i^{(\alpha)}(t-s)\right] =$$

$$= \sum_{\alpha=1}^{2} \left[\rho^{(\alpha)} f_i^{(\alpha)}(t-s) u_i^{(\alpha)}(t+s) - \rho^{(\alpha)} f_i^{(\alpha)}(t+s) u_i^{(\alpha)}(t-s) + \right.$$

$$+ \left[S_{ji}^{(\alpha)}(\mathbf{U}(t-s)) u_i^{(\alpha)}(t+s) - S_{ji}^{(\alpha)}(\mathbf{U}(t+s)) u_i^{(\alpha)}(t-s)\right]_{,j} +$$

$$\left. + \left[S_{ji}^{(\alpha)}(\mathbf{U}(t+s)) u_{i,j}^{(\alpha)}(t-s) - S_{ji}^{(\alpha)}(\mathbf{U}(t-s)) u_{i,j}^{(\alpha)}(t+s)\right]\right\} -$$

$$- p_i(\mathbf{U}(t-s)) d_i(\mathbf{U}(t+s)) + p_i(\mathbf{U}(t+s)) d_i(\mathbf{U}(t-s)), \tag{30}$$

and

$$\sum_{\alpha=1}^{2} \left[\rho^{(\alpha)}\chi^{(\alpha)}\ddot{\varphi}^{(\alpha)}(t-s)\varphi^{(\alpha)}(t+s) - \rho^{(\alpha)}\chi^{(\alpha)}\ddot{\varphi}^{(\alpha)}(t+s)\varphi^{(\alpha)}(t-s)\right] =$$

$$= \sum_{\alpha=1}^{2} \left\{\left[\rho^{(\alpha)}\ell^{(\alpha)}(t-s)\varphi^{(\alpha)}(t+s) - \rho^{(\alpha)}\ell^{(\alpha)}(t+s)\varphi^{(\alpha)}(t-s)\right] + \right.$$

$$+ \left[h_j^{(\alpha)}(\mathbf{U}(t-s))\varphi^{(\alpha)}(t+s) - h_j^{(\alpha)}(\mathbf{U}(t+s))\varphi^{(\alpha)}(t-s)\right]_{,j} +$$

$$+ \left[h_j^{(\alpha)}(\mathbf{U}(t+s))\varphi_{,j}^{(\alpha)}(t-s) - h_j^{(\alpha)}(\mathbf{U}(t-s))\varphi_{,j}^{(\alpha)}(t+s)\right] +$$

$$\left. + \left[g^{(\alpha)}(\mathbf{U}(t-s))\varphi^{(\alpha)}(t+s) - g^{(\alpha)}(\mathbf{U}(t+s))\varphi^{(\alpha)}(t-s)\right]\right\}. \tag{31}$$

Further, with the help of (15) – (17) we prove that

$$\sum_{\alpha=1}^{2}\Big\{\Big[S_{ji}^{(\alpha)}(\mathbf{U}(t+s))u_{i,j}^{(\alpha)}(t-s)-S_{ji}^{(\alpha)}(\mathbf{U}(t-s))u_{i,j}^{(\alpha)}(t+s)\Big]+$$
$$+\Big[h_j^{(\alpha)}(\mathbf{U}(t+s))\varphi_{,j}^{(\alpha)}(t-s)-h_j^{(\alpha)}(\mathbf{U}(t-s))\varphi_{,j}^{(\alpha)}(t+s)\Big]+$$
$$+\Big[g^{(\alpha)}(\mathbf{U}(t-s))\varphi^{(\alpha)}(t+s)-g^{(\alpha)}(\mathbf{U}(t+s))\varphi^{(\alpha)}(t-s)\Big]\Big\}+$$
$$+\big[p_i(\mathbf{U}(t+s))d_i(\mathbf{U}(t-s))-p_i(\mathbf{U}(t-s))d_i(\mathbf{U}(t+s))\big]=0. \qquad (32)$$

Then, the equations (29) – (32) imply that

$$\sum_{\alpha=1}^{2}\Big\{\rho^{(\alpha)}f_i^{(\alpha)}(t-s)u_i^{(\alpha)}(t+s)-\rho^{(\alpha)}f_i^{(\alpha)}(t+s)u_i^{(\alpha)}(t-s)+$$
$$+\Big[S_{ji}^{(\alpha)}(\mathbf{U}(t-s))u_i^{(\alpha)}(t+s)-S_{ji}^{(\alpha)}(\mathbf{U}(t+s))u_i^{(\alpha)}(t-s)\Big]_{,j}+$$
$$+\rho^{(\alpha)}\ell^{(\alpha)}(t-s)\varphi^{(\alpha)}(t+s)-\rho^{(\alpha)}\ell^{(\alpha)}(t+s)\varphi^{(\alpha)}(t-s)+$$
$$+\Big[h_j^{(\alpha)}(\mathbf{U}(t-s))\varphi^{(\alpha)}(t+s)-h_j^{(\alpha)}(\mathbf{U}(t+s))\varphi^{(\alpha)}(t-s)\Big]_{,j}\Big\}=$$
$$=-\frac{\partial}{\partial s}\sum_{\alpha=1}^{2}\Big\{\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t-s)u_i^{(\alpha)}(t+s)+\rho^{(\alpha)}u_i^{(\alpha)}(t-s)\dot{u}_i^{(\alpha)}(t+s)+$$
$$+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t-s)\varphi^{(\alpha)}(t+s)+\rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(t-s)\dot{\varphi}^{(\alpha)}(t+s)\Big\}. \qquad (33)$$

The equation (28) is reached by performing integration of the equation (33) over $P\times[0,t]$ and then by using the divergence theorem.

4. A time weighted surface measure. Having fixed time $T\in I$, for the external given data of the problem $\mathcal{P}$ we define the set $\hat{D}_T$ by:

i) if $\mathbf{x}\in B$, then

$$a_i^{(1)}(\mathbf{x})\neq 0 \quad \text{or} \quad \dot{a}_i^{(1)}(\mathbf{x})\neq 0 \quad \text{or} \quad a_i^{(2)}(\mathbf{x})\neq 0 \quad \text{or} \quad \dot{a}_i^{(2)}(\mathbf{x})\neq 0$$

or

$$\varphi_0^{(1)}(\mathbf{x})\neq 0 \quad \text{or} \quad \dot{\varphi}_0^{(1)}(\mathbf{x})\neq 0 \quad \text{or} \quad \varphi_0^{(2)}(\mathbf{x})\neq 0 \quad \text{or} \quad \dot{\varphi}_0^{(2)}(\mathbf{x})\neq 0$$

or there exists such $\tau\in[0,T]$ that

$$f_i^{(1)}(\mathbf{x},\tau)\neq 0 \quad \text{or} \quad f_i^{(2)}(\mathbf{x},\tau)\neq 0 \quad \text{or} \quad \ell^{(1)}(\mathbf{x},\tau)\neq 0 \quad \text{or} \quad \ell^{(2)}(\mathbf{x},\tau)\neq 0;$$

ii) if $\mathbf{x}\in\partial B$, then there exists such $\tau\in[0,T]$ that

$$s_i^{(1)}(\mathbf{x},\tau)\dot{u}_i^{(1)}(\mathbf{x},\tau)\neq 0 \quad \text{or} \quad s_i^{(2)}(\mathbf{x},\tau)\dot{u}_i^{(2)}(\mathbf{x},\tau)\neq 0$$

or

$$h^{(1)}(\mathbf{x},\tau)\dot{\varphi}^{(1)}(\mathbf{x},\tau)\neq 0 \quad \text{or} \quad h^{(2)}(\mathbf{x},\tau)\dot{\varphi}^{(2)}(\mathbf{x},\tau)\neq 0.$$

The set $\hat{D}_T$ represents the support of the initial and boundary data and the body force on the time interval $[0,T]$. In what follows, we assume that $\hat{D}_T$ is a bounded set.

We consider a nonempty set $\hat{D}_T^*$ which is such that $\hat{D}_T\subset\hat{D}_T^*\subset\overline{B}$ and

i) if $\hat{D}_T\cap B\neq\varnothing$, then we choose $\hat{D}_T^*$ is the smallest bounded regular region in $\overline{B}$ that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^*=\hat{D}_T$ if $\hat{D}_T$ it also happens to be a regular region;

ii) if $\varnothing\neq\hat{D}_T\subset\partial B$, then we choose $\hat{D}_T^*$ as the smallest regular subsurface of

∂B that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if $\hat{D}_T$ is a regular subsurface of ∂B;

iii) if $\hat{D}_T = \varnothing$, then we choose $\hat{D}_T^*$ as arbitrary nonempty regular subsurface of ∂B.

Now, we mean the set D_r, by

$$D_r = \left\{ \mathbf{x} \in \bar{B} : \hat{D}_T^* \cap \overline{\Sigma(\mathbf{x}, r)} \neq \varnothing \right\}, \quad r \geq 0, \tag{34}$$

where $\overline{\Sigma(\mathbf{x}, r)}$ is the closed ball with the radius r and the center at $\mathbf{x}$. Further, we use the notation B_r for the part of B contained in $\bar{B} \setminus D_r$ and $B(r_1, r_2) = B_{r_2} \setminus B_{r_1}$, $r_1 > r_2$; S_r denotes the subsurface of ∂B_r contained into inside of B and whose outward unit normal vector $\mathbf{n}$ is directed to the exterior of D_r. Surely, taking into account that for each $r > 0$, $\hat{D}_T \subset D_r$ and $\hat{D}_T \cap B_r = \varnothing$, we get

$$\begin{gathered} a_i^{(\alpha)} = 0, \quad \dot{a}_i^{(\alpha)} = 0, \quad \varphi_0^{(\alpha)} = 0, \quad \dot{\varphi}_0^{(\alpha)} = 0 \quad \text{on } B_r, \\ f_i^{(\alpha)} = 0, \quad \ell^{(\alpha)} = 0 \quad \text{on } B_r \times [0, T], \\ s_i^{(\alpha)} \dot{u}_i^{(\alpha)} = 0, \quad h^{(\alpha)} \dot{\varphi}^{(\alpha)} = 0 \quad \text{on } (B_r \cap \partial B) \times [0, T]. \end{gathered} \tag{35}$$

For a fixed positive parameter λ and for any $r \geq 0$, $t \in [0, T]$, we define the time-weighted surface power function $P(r, t)$ by

$$P(r, t) = -\int_0^t \int_{S_r} e^{-\lambda s} \sum_{\alpha=1}^{2} \left[s_i^{(\alpha)}(\mathbf{U}(s)) \dot{u}_i^{(\alpha)}(s) + h^{(\alpha)}(\mathbf{U}(s)) \dot{\varphi}^{(\alpha)}(s) \right] da\, ds. \tag{36}$$

In the following lemmas, we show some relevant properties of the function $P(r, t)$.

Lemma 4. *Let* $\mathbf{U}$ *be a solution of the initial-boundary-value problem* $\mathcal{P}$ *and* $\hat{D}_T$ *is the bounded support of the external data on the time interval* $[0, T]$. *Then, the corresponding time-weighted surface power function* $P(r, t)$ *is a continuous differentiable function on* $r \geq 0$, $t \in [0, T]$ *and*

$$\frac{\partial}{\partial t} P(r, t) = -\int_{S_r} e^{-\lambda t} \sum_{\alpha=1}^{2} \left[s_i^{(\alpha)}(\mathbf{U}(t)) \dot{u}_i^{(\alpha)}(t) + h^{(\alpha)}(\mathbf{U}(t)) \dot{\varphi}^{(\alpha)}(t) \right] da, \tag{37}$$

$$\begin{aligned} \frac{\partial}{\partial r} P(r, t) = & -\frac{1}{2} \int_{S_r} e^{-\lambda t} \left\{ \sum_{\alpha=1}^{2} \left[\rho^{(\alpha)} \dot{u}_i^{(\alpha)}(t) \dot{u}_i^{(\alpha)}(t) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(t) \dot{\varphi}^{(\alpha)}(t) \right] + \right. \\ & \left. + 2W(\mathbf{U}(t)) \right\} da - \frac{\lambda}{2} \int_0^t \int_{S_r} e^{-\lambda s} \left\{ \sum_{\alpha=1}^{2} \left[\rho^{(\alpha)} \dot{u}_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s) + \right. \right. \\ & \left. \left. + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s) \right] + 2W(\mathbf{U}(s)) \right\} da\, ds. \end{aligned} \tag{38}$$

Moreover, for fixed $t \in [0, T]$, $P(r, t)$ *is a nonincreasing function with respect to* r.

Proof. The equation (37) is an immediate consequence of the definition of $P(r, t)$. On the other hand, on the basis of the relations (35) and (36), we obtain

$$\begin{gathered} P(r_1, t) - P(r_2, t) = \\ = -\int_0^t \int_{\partial B(r_1, r_2)} e^{-\lambda s} \sum_{\alpha=1}^{2} \left[s_i^{(\alpha)}(\mathbf{U}(s)) \dot{u}_i^{(\alpha)}(s) + h^{(\alpha)}(\mathbf{U}(s)) \dot{\varphi}^{(\alpha)}(s) \right] dv\, ds. \end{gathered} \tag{39}$$

Then, by taking $P \equiv B(r_1, r_2)$ into Lemma 1 and by using the relation (35), we have

$$P(r_1, t) - P(r_2, t) =$$

$$= -\frac{1}{2}\int_{B(r_1,r_2)} e^{-\lambda t}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t) + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right] + \right.$$

$$\left. + 2W(\mathbf{U}(t))\right\}dv - \frac{\lambda}{2}\int_0^t\int_{B(r_1,r_2)} e^{-\lambda s}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + \right.\right.$$

$$\left.\left. + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right] + 2W(\mathbf{U}(s))\right\}dv\,ds. \tag{40}$$

This relation straightly leads to (38).

Since we have assumed that $\rho^{(\alpha)}$, $\chi^{(\alpha)}$ are strictly positive, λ is positive and $W(\mathbf{U})$ is a positive definite quadratic form, the equation (40) gives

$$P(r_1, t) \le P(r_2, t) \qquad \text{with} \quad r_1 \ge r_2, \quad t \in [0, T]. \tag{41}$$

Lemma 5. *Let* $\mathbf{U}$ *is a solution of initial-boundary-value problem* $\mathcal{P}$ *and* $\hat{D}_T$ *is the bounded support of the external data on the time interval* $[0, T]$. *Then for any* $r \ge 0$, $t \in [0, T]$, *the function* $P(r, t)$ *satisfies the following first-order differential inequalities*

$$\frac{\lambda}{c}|P(r,t)| + \frac{\partial}{\partial r}P(r,t) \le 0, \tag{42}$$

$$\frac{1}{c}\left|\frac{\partial}{\partial t}P(r,t)\right| + \frac{\partial}{\partial r}P(r,t) \le 0, \tag{43}$$

where

$$c = \sqrt{\frac{\xi_M}{m}} \quad \textit{with} \quad m = \min\left\{\rho^{(1)}, \rho^{(2)}, \rho^{(1)}\chi^{(1)}, \rho^{(2)}\chi^{(2)}\right\}, \tag{44}$$

and ξ_M *is the maximum elastic moduli.*

Proof. It follows from the Schwarz's inequality and the arithmetic-geometric mean inequality that

$$\left|\sum_{\alpha=1}^{2}\left[s_i^{(\alpha)}(\mathbf{U}(t))\dot{u}_i^{(\alpha)}(t) + h^{(\alpha)}(\mathbf{U}(t))\dot{\varphi}^{(\alpha)}(t)\right]\right| \le$$

$$\le \frac{1}{2}\sum_{\alpha=1}^{2}\left[\varepsilon\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t) + \frac{1}{\varepsilon\rho^{(\alpha)}}s_i^{(\alpha)}(\mathbf{U}(t))s_i^{(\alpha)}(\mathbf{U}(t)) + \right.$$

$$\left. + \varepsilon\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t) + \frac{1}{\varepsilon\rho^{(\alpha)}\chi^{(\alpha)}}h^{(\alpha)}(\mathbf{U}(t))h^{(\alpha)}(\mathbf{U}(t))\right], \tag{45}$$

where ε is an arbitrary positive constant.

Using the relations (35), (45), (44), (13) and taking $\varepsilon = c$, we deduce

$$|P(r,t)| \le \frac{c}{2}\int_0^t\int_{S_r} e^{-\lambda s}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s)+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right]+ + 2W(\mathbf{U}(s))\right\}da\,ds \tag{46}$$

for $r \ge 0$ and $0 \le t \le T$.

Similarly, from (37) we obtain

$$\left|\frac{\partial}{\partial t}P(r,t)\right| \le \frac{c}{2}\int_{S_r} e^{-\lambda t}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t)+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right]+ + 2W(\mathbf{U}(t))\right\}da \tag{47}$$

for $r \ge 0$ and $t \in [0,T]$. By the equation (38) and the relations (46) and (47) we obtain the result.

Lemma 6. *Let* $\mathbf{U}$ *is a solution of the initial-boundary-value problem* $\mathcal{P}$ *and* $\hat{D}_T$ *is the bounded support of the external data on the time interval* $[0,T]$. *Then, it follows that*

$$P(r,t) \ge 0 \quad \textit{for} \quad r \ge 0, \quad 0 \le t \le T; \tag{48}$$

moreover,

$$P(r,t) = E(r,t), \tag{49}$$

where

$$E(r,t) = \frac{1}{2}\int_{B_r} e^{-\lambda t}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t)+\rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right]+ + 2W(\mathbf{U}(t))\right\}dv + \frac{\lambda}{2}\int_0^t\int_{B_r} e^{-\lambda s}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right]+2W(\mathbf{U}(s))\right\}dv\,ds. \tag{50}$$

Proof. If B is a bounded body, then the variable r ranges on $[0,L]$, where

$$L = \max\left\{\min\left\{[(x_i-y_i)(x_i-y_i)]^{1/2} : \mathbf{y}\in\hat{D}_T^*\right\} : \mathbf{x}\in\overline{B}\right\} < \infty. \tag{51}$$

Starting from the definition of $\hat{D}_T$ and by using the relation (36), we obtain

$$P(L,t) = 0, \quad 0 \le t \le T; \tag{52}$$

thus, the equation (41) implies the relation (48).

If B is an unbounded body, then the variable r ranges on $[0,\infty)$. The inequality (43) is equivalent to

$$\frac{1}{c}\frac{\partial}{\partial t}P(r,t) + \frac{\partial}{\partial r}P(r,t) \le 0, \tag{53}$$

and

$$-\frac{1}{c}\frac{\partial}{\partial t}P(r,t) + \frac{\partial}{\partial r}P(r,t) \le 0. \tag{54}$$

If we choose the pair (r_0, t_0) such that $t_0 \in [0, T]$ and $r_0 \geq ct_0$ and we put $t = t_0 + \frac{r - r_0}{c}$ in the inequality (53), then

$$\frac{d}{dr}\left[P\left(r,\, t_0 + \frac{r - r_0}{c}\right)\right] \leq 0, \tag{55}$$

and thus,

$$P\left(r,\, t_0 + \frac{r - r_0}{c}\right) \leq P\left(r_1,\, t_0 + \frac{r_1 - r_0}{c}\right) \quad \text{with} \quad r \geq r_1. \tag{56}$$

For $r = r_0$ and $r_1 = r_0 - ct_0$, we get

$$P(r_0, t_0) \leq P(r_0 - ct_0, 0). \tag{57}$$

Similarly, by setting $t = t_0 - \frac{r - r_0}{c}$ in (54), it follows

$$\frac{d}{dr}\left[P\left(r,\, t_0 - \frac{r - r_0}{c}\right)\right] \leq 0, \tag{58}$$

so that

$$P(r_0 + ct_0, 0) \leq P(r_0, t_0). \tag{59}$$

Taking into account $P(r_0 - ct_0, 0) = 0$ and $P(r_0 + ct_0, 0) = 0$, the relations (57), (59) imply that

$$P(r_0, t_0) = 0.$$

Surely, for $r_0 \to \infty$ in the above relation, it follows

$$P(\infty, t_0) = \lim_{r_0 \to \infty} P(r_0, t_0) = 0, \tag{60}$$

and, by (41), we conclude that the relation (48) is true.

The equation (49) follows from the relation (40) by means of the use of the relations (52) and (60).

5. Spatial behaviour. By the properties of the time-weighted surface power function P, we establish the theorem that gives a complete description of the spatial behaviour of the elastic process in question outside of the support of the external data.

Theorem 1. *Let* $\mathbf{U}$ *is a solution of the initial-boundary-value problem,* $\hat{D}_T$ *is the bounded support of the external data on the time interval* $[0, T]$, *and let* $P(r, t)$ *is the time-weighted surface power measure associated with* $\mathbf{U}$.

i. *Spatial behaviour: For each fixed* $t \in [0, T]$ *and* $0 \leq r \leq ct$, *we have*

$$P(r, t) \leq P(0, t)\exp\left(-\frac{\lambda}{c}r\right); \tag{61}$$

ii. *Domain of influence results: For each fixed* $t \in [0, T]$ *and* $r \geq ct$, *we have*

$$\begin{gathered} u_i^{(1)} = 0, \qquad u_i^{(2)} = 0, \\ \varphi^{(1)} = 0, \qquad \varphi^{(2)} = 0 \qquad \text{on } B_r. \end{gathered} \tag{62}$$

Proof. The equations (42), (48) give

$$\frac{\partial}{\partial r}\left[\exp\left(\frac{\lambda}{c}r\right)P(r, t)\right] \leq 0, \quad 0 \leq r \leq ct, \quad 0 \leq t \leq T, \tag{63}$$

so that we obtain the equation (61).

If we choose $t \in [0,T]$ and we set $r = ct$ in (53), then

$$\frac{d}{dt}[P(ct,t)] \le 0, \tag{64}$$

and therefore, by means of the relations (41) and (64), it results in that for all $r' \ge ct$ we have

$$P(r',t) \le P(ct,t) \le P(0,0) = 0. \tag{65}$$

Thus, we have for all $r \ge ct$

$$P(r,t) \le 0, \tag{66}$$

and, taking into account (48), we obtain

$$P(r,t) = 0. \tag{67}$$

Now, the equations (67), (49), (50) imply

$$E(r,t) = \frac{1}{2}\int_{B_r} e^{-\lambda t}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(t)\dot{u}_i^{(\alpha)}(t) + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(t)\dot{\varphi}^{(\alpha)}(t)\right] + \right.$$

$$\left. + 2W(\mathbf{U}(t))\right\}dv + \frac{\lambda}{2}\int_0^t\int_{B_r} e^{-\lambda s}\left\{\sum_{\alpha=1}^{2}\left[\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(s)\dot{u}_i^{(\alpha)}(s) + \right.\right.$$

$$\left.\left. + \rho^{(\alpha)}\chi^{(\alpha)}\dot{\varphi}^{(\alpha)}(s)\dot{\varphi}^{(\alpha)}(s)\right] + 2W(\mathbf{U}(s))\right\}dv\,ds = 0. \tag{68}$$

Since $\rho^{(\alpha)}$ and $\chi^{(\alpha)}$ are strictly positive, λ is positive and $W(\mathbf{U})$ is a positive definite quadratic form, we have

$$\dot{u}_i^{(1)} = 0, \quad \dot{u}_i^{(2)} = 0, \quad \dot{\varphi}^{(1)} = 0, \quad \dot{\varphi}^{(2)} = 0 \quad \text{on} \quad B_r, \quad r \ge ct, \quad t \in [0,T]; \tag{69}$$

so, by (35), we obtain the equations (62).

The equations (62) imply that the set D_{cT} covers a domain of elastic disturbances produced by the data at time T, i. e.

$$u_i^{(1)} = 0, \quad u_i^{(2)} = 0, \quad \varphi^{(1)} = 0, \quad \varphi^{(2)} = 0 \quad \text{on} \quad B_{cT} \times [0,T]; \tag{70}$$

this result is known as a so-called domain of influence theorem (see Gurtin [18]).

As an immediate consequence of the equations (62), we establish the following uniqueness result valid for a bounded or unbounded body.

Theorem 2 (*Uniqueness*). *It exists at most one (regular) solution for the initial-boundary-value problem.*

Proof. Thanks to the linearity of the problem, we have only to show that the null data imply null solution. Let $\tilde{\mathbf{U}} = \{\tilde{\mathbf{u}}^{(1)}, \tilde{\mathbf{u}}^{(2)}, \tilde{\varphi}^{(1)}, \tilde{\varphi}^{(2)}\}$ a solution corresponding to null data. Since the set $\hat{D}_T = \varnothing$ for each $T \in (0, +\infty)$ and the function $P(r,t) = 0$, we can conclude that

$$\tilde{u}_i^{(1)} = 0, \quad \tilde{u}_i^{(2)} = 0, \quad \tilde{\varphi}^{(1)} = 0, \quad \tilde{\varphi}^{(2)} = 0 \quad \text{on} \quad B \times I.$$

We note that if B is a bounded regular region, for suffficiently great values of T, then such a value of $t \in [0,T]$ having the property that $D_{ct} \supseteq B$ exists, the relation (62) becomes superfluous and behaviour of solutions is completely described by the relation (61). On the other hand, if the values of T are sufficiently small, the behaviour of solutions is described by the relation (62) almost as in B. Similar arguments are valid for an unbounded regular region.

6. Asymptotic equipartition of energy. Throughout this section, we study time asymptotic behaviour of the problem $\mathcal{P}_0$ for the bounded regular region B. The problem $\mathcal{P}_0$ is defined by the following equations of motion

$$S_{ji,j}^{(\alpha)} + (-1)^{\alpha} p_i = \rho^{(\alpha)} \ddot{u}_i^{(\alpha)},$$
$$h_{i,i}^{(\alpha)} + g^{(\alpha)} = \rho^{(\alpha)} \chi^{(\alpha)} \ddot{\varphi}^{(\alpha)}, \quad \text{on } B \times (0, \infty), \tag{71}$$

the geometrical equations (2) and the constitutive equations (5), the initial conditions (20) and the boundary conditions

$$u_i^{(\alpha)} = 0 \quad \text{on } \overline{\Sigma}_1 \times I, \qquad s_i^{(\alpha)} = 0 \quad \text{on } \Sigma_2 \times I,$$
$$\varphi^{(\alpha)} = 0 \quad \text{on } \overline{\Sigma}_3 \times I, \qquad h^{(\alpha)} = 0 \quad \text{on } \Sigma_4 \times I. \tag{72}$$

Now, we introduce the Cesáro means of various energies associated with the solution $\mathbf{U}$ of the problem $\mathcal{P}_0$:

$$\mathcal{K}_C^u(t) = \sum_{\alpha=1}^{2} \frac{1}{2t} \int_0^t \int_B \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s)\, dv\, ds,$$
$$\mathcal{K}_C^{\varphi}(t) = \sum_{\alpha=1}^{2} \frac{1}{2t} \int_0^t \int_B \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s)\, dv\, ds, \tag{73}$$
$$S_C(t) = \frac{1}{t} \int_0^t \int_B W(\mathbf{U}(s))\, dv\, ds,$$

and

$$\mathcal{K}_C(t) = \mathcal{K}_C^u(t) + \mathcal{K}_C^{\varphi}(t). \tag{74}$$

If $\operatorname{meas} \Sigma_1 = 0$, then there exists a family of rigid motions and null change in volume fraction, which satisfy the equations (71), (2), (5), (20) and (72). We decompose the initial data $a_i^{(\alpha)}$ and $\dot{a}_i^{(\alpha)}$ as

$$a_i^{(\alpha)} = \overline{a}_i^{(\alpha)} + A_i^{(\alpha)}, \qquad \dot{a}_i^{(\alpha)} = \dot{\overline{a}}_i^{(\alpha)} + \dot{A}_i^{(\alpha)}, \tag{75}$$

where $\overline{a}_i^{(\alpha)}$ and $\dot{\overline{a}}_i^{(\alpha)}$ are the rigid displacements determined so that $A_i^{(\alpha)}$ and $\dot{A}_i^{(\alpha)}$ satisfy the normalization restrictions

$$\int_B \rho^{(\alpha)} A_i^{(\alpha)}\, dv = 0, \qquad \int_B \rho^{(\alpha)} \varepsilon_{ijk} x_j A_k^{(\alpha)}\, dv = 0,$$
$$\int_B \rho^{(\alpha)} \dot{A}_i^{(\alpha)}\, dv = 0, \qquad \int_B \rho^{(\alpha)} \varepsilon_{ijk} x_j \dot{A}_k^{(\alpha)}\, dv = 0, \tag{76}$$

and ε_{ijk} is the alternating symbol.

We put

$$\hat{\mathbf{C}}^1(B) \equiv \Big\{ \mathbf{v} \text{ with } v_i \in C^1(\overline{B}) \colon v_i = 0 \text{ on } \Sigma_1 \text{ if } \operatorname{meas} \Sigma_1 \neq 0,$$
$$\text{or } \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} v_i\, dv = 0, \quad \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \varepsilon_{ijk} x_j v_k\, dv = 0 \quad \text{if } \operatorname{meas} \Sigma_1 = 0 \Big\},$$

and

$$\hat{C}^1(B) \equiv \big\{ \zeta \in C^1(B) \colon \zeta = 0 \quad \text{on} \quad \Sigma_3 \big\},$$

and

$$\hat{\mathbf{W}}_1(B) \equiv \text{the completion of } \hat{\mathbf{C}}^1(B) \text{ by means of } \|\cdot\|_{\mathbf{W}_1(B)},$$

and

$$\hat{W}_1(B) \equiv \text{the completion of } \hat{C}^1(B) \text{ by means of } \|\cdot\|_{W_1(B)}.$$

The space $W_m(B)$ represents the familiar Sobolev space and $\mathbf{W}_m(B) \equiv [W_m(B)]^3$.

The equation (10) assures that the following Korn's inequality [19] holds

$$\int_B 2W(\mathbf{V})dv \ \geq \ m_1 \int_B \sum_{\alpha=1}^{2} \left(v_i^{(\alpha)} v_i^{(\alpha)} + \chi^{(\alpha)} \phi^{(\alpha)} \phi^{(\alpha)} \right) dv, \qquad m_1 = \text{const} > 0, \tag{77}$$

for every $\mathbf{V} = \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \phi^{(1)}, \phi^{(2)}\}$: $\mathbf{v}^{(\alpha)} \in \hat{\mathbf{W}}_1(B)$, $\phi^{(\alpha)} \in \hat{W}_1(B)$.

If meas $\Sigma_1 = 0$, then we find a convenient practice to decompose the solution $\mathbf{U} = \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \varphi^{(1)}, \varphi^{(2)}\}$ of the problem $\mathcal{P}_0$ in the form

$$u_i^{(\alpha)} = \bar{a}_i^{(\alpha)} + t\dot{\bar{a}}_i^{(\alpha)} + v_i^{(\alpha)}, \qquad \varphi^{(\alpha)} = \phi^{(\alpha)}, \tag{78}$$

where $\mathbf{V} = \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \phi^{(1)}, \phi^{(2)}\} \in \hat{\mathbf{W}}_1(B) \times \hat{\mathbf{W}}_1(B) \times \hat{W}_1(B) \times \hat{W}_1(B)$ represents the solution of the problem $\mathcal{P}_0$ with the initial data $\{\bar{a}_i^{(\alpha)}, \varphi_0^{(\alpha)}\}$ and $\{\dot{\bar{a}}_i^{(\alpha)}, \dot{\varphi}_0^{(\alpha)}\}$.

Theorem 3. *Let* $\mathbf{U}$ *is a solution of the problem* $\mathcal{P}_0$. *Then, for all choices of initial data with* $\mathbf{a}^{(\alpha)} \in \mathbf{W}_1(B)$, $\dot{\mathbf{a}}^{(\alpha)} \in \mathbf{W}_0(B)$ $\varphi_0^{(\alpha)} \in W_1(B)$, $\dot{\varphi}_0^{(\alpha)} \in W_0(B)$. *Then, the following asymptotic behaviour of the solution* $\mathbf{U}$ *holds:*

i) *if* meas $\Sigma_1 \neq 0$, *we have*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} S_C(t) = \frac{1}{2}\mathcal{E}(0); \tag{79}$$

ii) *if* meas $\Sigma_1 = 0$, *we have*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} S_C(t) + \frac{1}{2}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} dv =$$

$$= \frac{1}{2}\mathcal{E}(0) + \frac{1}{2}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} dv, \tag{80}$$

where $\mathcal{E}(t)$ *is defined by* (24).

Proof. Relative to the problem $\mathcal{P}_0$ it follows that $\mathbf{f}^{(1)} = \mathbf{0}$, $\mathbf{f}^{(2)} = \mathbf{0}$, $\ell^{(1)} = 0$, $\ell^{(2)} = 0$ and the boundary conditions (72) are verified. Thus, the equation (25) becomes

$$\mathcal{E}(t) = \mathcal{E}(0), \qquad t \geq 0, \tag{81}$$

so that

$$\mathcal{K}_C(t) + S_C(t) = \mathcal{E}(0), \qquad \text{for all} \quad t \geq 0. \tag{82}$$

On the other hand, the relations (26) and (28) imply

$$\mathcal{K}_C(t) - S_C(t) = \frac{1}{4t}\int_B \sum_{\alpha=1}^{2} \Big\{ -\left[2\rho^{(\alpha)} u_i^{(\alpha)}(0)\dot{u}_i^{(\alpha)}(0) + 2\rho^{(\alpha)}\chi^{(\alpha)}\varphi^{(\alpha)}(0)\dot{\varphi}^{(\alpha)}(0)\right] +$$

$$+ \rho^{(\alpha)}\left[u_i^{(\alpha)}(0)\dot{u}_i^{(\alpha)}(2t) + \dot{u}_i^{(\alpha)}(0)u_i^{(\alpha)}(2t)\right] +$$

$$+ \rho^{(\alpha)}\chi^{(\alpha)}\left[\varphi^{(\alpha)}(0)\dot{\varphi}^{(\alpha)}(2t) + \varphi^{(\alpha)}(2t)\dot{\varphi}^{(\alpha)}(0)\right]\Big\} dv, \qquad t > 0. \tag{83}$$

The relations (10), (24) and (81) imply

$$\int_B \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s) dv \le 2\mathscr{E}(0),$$

$$\int_B \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \varphi^{(\alpha)}(s) dv \le 2\mathscr{E}(0), \tag{84}$$

$$\int_B \varphi^{(\alpha)}(s)\varphi^{(\alpha)}(s) dv \le \frac{2}{\xi_m}\int_B \sum_{\alpha=1}^{2} W(\mathbf{U}(s)) dv \le \frac{2}{\xi_m}\mathscr{E}(0),$$

thus

$$\lim_{s\to\infty} \frac{1}{s}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s) dv = 0,$$

$$\lim_{s\to\infty} \frac{1}{s}\int_B \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \varphi^{(\alpha)}(s) dv = 0, \tag{85}$$

$$\lim_{s\to\infty} \frac{1}{s}\int_B \varphi^{(\alpha)}(s) \varphi^{(\alpha)}(s) dv = 0.$$

By using the Schwarz's inequality and the relation (85) in (83), we obtain

$$\lim_{t\to\infty} \mathcal{K}_C(t) - \lim_{t\to\infty} S_C(t) = \lim_{t\to\infty} \frac{1}{4t}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(0) u_i^{(\alpha)}(2t) dv. \tag{86}$$

When meas $\Sigma_1 \ne 0$, then for $\mathbf{u} \in \hat{\mathbf{W}}_1(B)$, $\varphi^{(\alpha)} \in \hat{W}_1(B)$, the relations (24), (77) and (81) imply

$$\int_B \sum_{\alpha=1}^{2} u_i^{(\alpha)}(s) u_i^{(\alpha)}(s) dv \le \frac{1}{m_1}\int_B \sum_{\alpha=1}^{2} 2W(\mathbf{U}(s)) dv \le \frac{2}{m_1}\mathscr{E}(0), \tag{87}$$

and, by means of the Schwarz's inequality, we obtain

$$\lim_{t\to\infty} \left\{ \frac{1}{4t}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(0) u_i^{(\alpha)}(2t) dv \right\} = 0. \tag{88}$$

Then, by using the equations (86) and (88) we have

$$\lim_{t\to\infty} \mathcal{K}_C(t) - \lim_{t\to\infty} S_C(t) = 0. \tag{89}$$

The relations (82) and (89) imply (79).

When meas $\Sigma_1 = 0$, then, the equations (75), (76) and (78) lead to

$$\frac{1}{4t}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(0) u_i^{(\alpha)}(2t) dv = \frac{1}{4t}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} dv +$$

$$+ \frac{1}{4t}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} (\dot{\bar{a}}_i^{(\alpha)} + \dot{A}_i^{(\alpha)}) v_i(2t) dv + \frac{1}{2}\int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} dv. \tag{90}$$

Since $\mathbf{V} = \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \phi^{(1)}, \phi^{(2)}\} \in \hat{\mathbf{W}}_1(B) \times \hat{\mathbf{W}}_1(B) \times \hat{W}_1(B) \times \hat{W}_1(B)$, from (24), (77) and (81), we deduce that

$$\int_B \sum_{\alpha=1}^{2} v_i^{(\alpha)}(s) v_i^{(\alpha)}(s) dv \le \frac{2}{m_1}\int_B \sum_{\alpha=1}^{2} W(\mathbf{V}(s)) dv \le \frac{2}{m_1}\mathscr{E}(0). \tag{91}$$

Taking into account the equations (90) and (91), we have

$$\lim_{t\to\infty}\frac{1}{4t}\int_B\sum_{\alpha=1}^{2}\rho^{(\alpha)}\dot{u}_i^{(\alpha)}(0)u_i^{(\alpha)}(2t)dv = \frac{1}{2}\int_B\sum_{\alpha=1}^{2}\rho^{(\alpha)}\dot{\bar{a}}_i^{(\alpha)}\dot{\bar{a}}_i^{(\alpha)}dv. \tag{92}$$

Thanks to the equations (86) and (92), we get

$$\lim_{t\to\infty}\mathcal{K}_C(t) = \lim_{t\to\infty}S_C(t)+\frac{1}{2}\int_B\sum_{\alpha=1}^{2}\rho^{(\alpha)}\dot{\bar{a}}_i^{(\alpha)}\dot{\bar{a}}_i^{(\alpha)}dv. \tag{93}$$

In addition, the equations (82) and (93) lead to (80).

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