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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE FIRST INITIAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC EQUATIONS

ПРО АСИМПТОТИЧНУ ПОВЕДІНКУ РОЗВ'ЯЗКІВ ПЕРШИХ ПОЧАТКОВИХ КРАЙОВИХ ЗАДАЧ ДЛЯ ПАРАБОЛІЧНИХ РІВНЯНЬ

We consider the first initial boundary-value problem for a strongly parabolic system on infinite cylinder with non-smooth boundary. We prove some results on the existence, uniqueness and asymptotic behaviour of solutions as $t \to +\infty$.

Розглянуто першу початкову крайову задачу для сильно параболічної системи на нескінченному циліндрі із негладкою межею. Доведено деякі результати про існування, єдиність та асимптотичну поведінку розв'язків при $t \to +\infty$.

1. Introduction. The theory of general boundary-value problems for elliptic systems with smooth and non-smooth boundary is nearly completely considered in [1, 2]. The strongly parabolic systems on a bounded set $\Omega_T = \Omega \times [0,T]$, $\Omega \subset \mathbb{R}^n$, with non-smooth boundary were studied in [3, 4]. However, the boundary-value problem for a parabolic system on cylinder with non-smooth boundary has been rarely considered. O. A. Ladyzhenskaya [5] showed some sufficient conditions for the existence and uniqueness of a generalized solution to the first boundary-value problem for strongly parabolic systems on a finite cylinder Ω_T. The smoothness of generalized solutions for a parabolic equation of order 2 is considered in [6]. The general boundary-value problem on the finite cylinder for a parabolic system in the sense of Petrovski is studied in [7], where the author obtained some results on the uniqueness of a solution in the Sobolev weighted space. In [8], the second boundary-value problem for a parabolic equation of order 2 in infinite cylinder $\Omega_\infty = \Omega \times [-\infty, +\infty)$ is considered. Some results on the asymptotic behaviour of solutions as $t \to +\infty$ was shown in that paper.

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with its boundary $\partial\Omega$. Let us introduce some notation: $\Omega_b = \Omega \times [0,b]$, $\Gamma_b = \partial\Omega \times [0,b]$, $\Omega_\infty = \Omega \times [0,+\infty)$, $\Gamma_\infty = \partial\Omega \times \times[0,\infty)$; $x = (x_1,\ldots,x_n) \in \Omega$; $u(x,t) = (u_1(x,t),\ldots,u_s(x,t))$ is a vector complex function; $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\alpha_i \in \mathbb{N}$, $|\alpha| = \sum_{i=1}^n \alpha_i$; $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$, $|D^\alpha u|^2 = \sum_{i=1}^s |D^\alpha u_i|^2$, $dx = dx_1\ldots dx_n$, $u_{t^j} = (\partial^j u_1/\partial t^j,\ldots,\partial^j u_s/\partial t^j)$; $\overset{\circ}{C}{}^\infty(\Omega)$ is the set of infinitely differentiable functions having their supports compactly embedded in Ω; $H^\ell(\Omega)$ — the space consisting of functions $u(x)$ which have generalized derivatives $D^\alpha u_i$ belonging to $L_2(\Omega)$, $|\alpha| \le \ell$, $1 \le i \le s$ and

$$\|u\|^2_{H^\ell(\Omega)} = \sum_{|\alpha|=0}^{\ell} \int_\Omega \sum_{i=1}^s |D^\alpha u_i|^2 dx < \infty;$$

$\overset{\circ}{H}{}^\ell(\Omega)$ is the completion of $\overset{\circ}{C}{}^\infty(\Omega)$ in the norm of $H^\ell(\Omega)$; $H^{\ell,k}(\Omega_T)$ is the space consisting of functions $u(x,t)$ such that $D^\alpha u_i \in L_2(\Omega_T)$, $\partial^j u_i/\partial t^j \in L_2(\Omega_T)$, $|\alpha| \le \ell$, $1 \le i \le s$, $1 \le j \le k$, with the norm

$$\|u\|^2_{H^{\ell,k}(\Omega_T)} = \sum_{|\alpha|=0}^{\ell} \int_{\Omega_T} |D^\alpha u|^2 dxdt + \sum_{j=1}^{k} \int_{\Omega_T} |u_{t^j}|^2 dxdt;$$

$H^{\ell,0}(\Omega_T)$ is the space consisting of functions $u(x,t)$ with the norm

$$\|u\|^2_{H^{\ell,0}(\Omega_T)} = \sum_{|\alpha|=0}^{\ell} \int_{\Omega_T} |D^\alpha u|^2 dxdt;$$

$\overset{\circ}{H}{}^{\ell,k}(\Omega_T)$ is the completion of $\overset{\circ}{C}{}^{\infty}(\Omega_T)$ in the norm of $H^{\ell,k}(\Omega_T)$; $\hat{\overset{\circ}{H}}{}^{m,0}(\Omega_T) = \left\{\eta(x,t) \in \overset{\circ}{H}{}^{m,0}(\Omega_T) : \eta(x,T) = 0\right\}$.

Consider the differential operator

$$L(x,t,D) = \sum_{|p|,|q|=1}^{m} D^p a_{pq}(x,t) D^q + \sum_{|p|=1}^{m} a_p(x,t) D^p + a(x,t),$$

where a_{pq}, a_p, a are $(s \times s)$-matrices, $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$, and their elements are bounded complex functions on $\bar{\Omega}_\infty$. Moreover, for every non-zero vector $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{C}^s$ we have

$$\sum_{|p|,|q|=m} a_{pq}(x,t) \xi^p \xi^q \eta \bar{\eta} > 0 \quad \forall (x,t) \in \bar{\Omega}_\infty, \tag{1}$$

where $\xi^p = \xi_1^{p_1} \ldots \xi_n^{p_n}$, $\xi^q = \xi_1^{q_1} \ldots \xi_n^{q_n}$.

We set

$$B(u,v) = \sum_{|p|,|q|=1}^{m} (-1)^{|p|} \int_\Omega a_{pq} D^q u \overline{D^p v} dx$$

and

$$B_1(u,u) = B(u,u) + 2\mathrm{Re} \sum_{|p|=1}^{m} \int_\Omega a_p D^p u \bar{u} dx.$$

The following lemma is proved in [9].

Lemma 1. *If a_{pq} satisfy (1) and a_{pq} are continuous on $\bar{\Omega}_\infty$ whenever $|p| = |q| = m$ then there exist positive numbers μ_0, μ_1, λ_0, λ_1 such that*

$$(-1)^m B(u,u) \geq \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L^2(\Omega)}, \tag{2}$$

$$(-1)^m B_1(u,u) \geq \mu_1 \|u\|^2_{H^m(\Omega)} - \lambda_1 \|u\|^2_{L^2(\Omega)} \tag{3}$$

for all $u \in \overset{\circ}{H}{}^m(\Omega)$.

2. The existence of generalized solution. In this section we consider the existence and uniqueness of a solution of the problem

$$(-1)^{m-1} L(x,t,D)u - u_t = f(x,t), \quad (x,t) \in \Omega_\infty, \tag{4}$$

such that

$$u(x,0) = 0 \quad \text{on} \quad \Omega, \tag{5}$$

$$\frac{\partial^j u}{\partial \nu^j}\Big|_{\Gamma_\infty} = 0, \quad j = 0, \ldots, m-1, \tag{6}$$

where ν is the unit outward normal to $\partial\Omega$.

Definition. *A function* $u(x,t)$ *is called a generalized solution of* (4)–(6) *in* $H^{m,1}(\Omega_\infty)$ *if* $u(x,t) \in \overset{\circ}{H}{}^{m,1}(\Omega_\infty)$, $u(x,0) = 0$ *and for all* $T > 0$, $u(x,t)$ *satisfies the following integnal identity on* Ω_T:

$$(-1)^{m-1} \int\limits_{\Omega_T} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u \overline{D^p \eta} + \sum_{|p|=1}^{m} a_p D^p u \overline{\eta} + a u \overline{\eta} \right] dxdt-$$

$$- \int\limits_{\Omega_T} u_t \overline{\eta} dxdt = \int\limits_{\Omega_T} f \overline{\eta} dxdt \quad \forall \eta \in \overset{\circ}{H}{}^{m,0}(\Omega_T), \quad \eta(x,T) = 0. \tag{7}$$

Theorem 1. *If* a_{pq} *are continuous on* $\overline{\Omega}_\infty$ *whenever* $|p| = |q| = m$ *and*

$$\left| \frac{\partial a_{pq}}{\partial t}, \frac{\partial a_p}{\partial t} \right| \leq \mu, \qquad 1 \leq |p|, \quad |q| \leq m, \quad \mu = \text{const},$$

then problem (4)–(6) *has at most one generalized solution in* $H^{m,1}(\Omega_\infty)$.

Proof. Suppose that problem (4)–(6) has two solutions u_1, u_2 and denote $u(x,t) = u_1(x,t) - u_2(x,t)$. For any $T > 0$ and $b \in (0,T)$, we define $\eta(x,t) = \int_b^t u(x,\tau)d\tau$ for $0 \leq t \leq b$ and $\eta(x,t) = 0$ for $b \leq t < T$. One can see that

$$\eta_t(x,t) = u(x,t), \quad 0 \leq t \leq b, \qquad \eta(x,t) \in \overset{\circ}{H}{}^{m,0}(\Omega_T)$$

and by identity (7), we have

$$(-1)^{m-1} \int\limits_{\Omega_b} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q \eta_t \overline{D^p \eta} + \sum_{|p|=1}^{m} a_p D^p \eta_t \overline{\eta} \right] dxdt +$$

$$+ (-1)^{m-1} \int\limits_{\Omega_b} a \eta_t \overline{\eta} dxdt - \int\limits_{\Omega_b} \eta_{tt} \overline{\eta} dxdt = 0. \tag{8}$$

Let $a_1 = a - (-1)^m \lambda_1 I$, where λ_1 is a positive number satisfying (3). Thus,

$$(-1)^{m-1} \int\limits_{\Omega_b} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q \eta_t \overline{D^p \eta} + (-1)^m \lambda_1 \eta_t \overline{\eta} + \right.$$

$$\left. + \sum_{|p|=1}^{m} a_p D^p \eta_t \overline{\eta} + a_1 \eta_t \overline{\eta} \right] dxdt - \int\limits_{\Omega_b} \eta_{tt} \overline{\eta} dxdt = 0. \tag{9}$$

Thus, the integration of the real parts of (9) gives

$$2 \int\limits_{\Omega_b} |\eta_t|^2 dxdt + (-1)^m \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} \int\limits_{\Omega} a_{pq}(x,0) D^q \eta(x,0) \overline{D^p \eta(x,0)} dx + \right.$$

$$+ 2\mathrm{Re}\sum_{|p|=1}^{m}\int_{\Omega} a_p(x,0)D^p\eta(x,0)\overline{\eta(x,0)}dx\Big] + \lambda_1\int_{\Omega}|\eta(x,0)|^2dx =$$

$$= (-1)^{m-1}\int_{\Omega_b}\sum_{|p|,|q|=1}^{m}(-1)^{|p|}\frac{\partial a_{pq}}{\partial t}D^q\eta\overline{D^p\eta}dxdt +$$

$$+(-1)^{m-1}2\mathrm{Re}\int_{\Omega_b}\left\{\sum_{|p|=1}^{m}\left[\frac{\partial a_p}{\partial t}D^p\eta\overline{\eta} + a_pD^p\eta\overline{\eta_t}\right] + a_1\eta_t\overline{\eta}\right\}dxdt.$$

Since $\dfrac{\partial a_{pq}}{\partial t}$, $\dfrac{\partial a_p}{\partial t}$ are bounded, by Cauchy inequality and (3) we obtain

$$\|\eta_t\|^2_{L_2(\Omega_b)} + \|\eta(x,0)\|^2_{H^m(\Omega)} \le C(\varepsilon)\sum_{|p|=0}^{m}\int_{\Omega_b}|D^p\eta|^2dxdt + \varepsilon\|\eta_t\|^2_{L_2(\Omega_b)} \tag{10}$$

and

$$\|\eta(x,0)\|^2_{H^m(\Omega)} \le C\sum_{|p|=0}^{m}\int_{\Omega_b}|D^p\eta|^2dxdt. \tag{11}$$

Setting

$$v_p(x,t) = \int_t^0 D^pu(x,\tau)d\tau, \quad 0<t<b, \qquad J(t) = \sum_{|p|=0}^{m}\int_{\Omega}|v_p(x,t)|^2dx,$$

we have

$$(1-Cb)J(b) \le C\int_0^b J(t)dt, \quad b\in\left[0,\frac{1}{2}C\right]. \tag{12}$$

By Gronwall–Bellman inequality, one has $J(t)\equiv 0$. Thus, (10), (11) lead to $\eta_t\equiv 0$, i.e., $u_1\equiv u_2$ $\forall t\in[0,1/2C]$. Using the same argument as before for functions u_1, u_2 on $[1/2C,T]$, we can show that $u_1\equiv u_2$ $\forall t\in[1/2C,1/C]$. Continuing in this fashion, after finite number of steps, we can prove that $u_1\equiv u_2$ $\forall t\in[0,T]$. Since $T>0$ is arbitrary, $u_1\equiv u_2$ $\forall t\in[0,\infty)$. This completes the proof.

Now we prove the following theorem as a result on existence.

Theorem 2. *Suppose that a_{pq} are continuous on $\overline{\Omega}_\infty$ whenever $|p|=|q|=m$, there exists a $\lambda>0$, a function $B(t)\in L_1(0,+\infty)$ such that $fe^{\lambda t}\in L_2(\Omega_\infty)$ and*

$$\max\left\{\left|\frac{\partial a_{pq}}{\partial t}\right|, |a_p|, |a-(-1)^m(\lambda+\lambda_0)I|\right\} \le B(t) \quad \forall (t,x)\in\Omega_\infty,$$

where λ_0 is determined from (2). Then problem (4)–(6) has a generalized solution $u(x,t)\in H^{m,1}(\Omega_\infty)$ satisfying

$$\|u\|^2_{H^{m,1}(\Omega_\infty)} \le C\|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}.$$

Proof. Suppose that $\{\varphi_k(x)\}_{k=1}^{\infty} \subset \overset{\circ}{H}{}^m(\Omega)$ is orthonormal in $L_2(\Omega)$ and its linear closure is $\overset{\circ}{H}{}^m(\Omega)$. For each natural number N, let us consider the function

$$u^N(x,t) = \sum_{k=1}^{N} C_k^N(t)\varphi_k(x),$$

where $C_k^N(t)$ satisfies $C_k^N(0) = 0$ and

$$\int_\Omega \left[u_t^N \overline{\varphi_\ell} + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p \varphi_\ell} + \lambda_0 u^N \overline{\varphi_\ell} \right] dx +$$

$$+(-1)^m \int_\Omega \left(\sum_{|p|=1}^{m} a_p D^p u^N + a_0 u^N \right) \overline{\varphi_\ell} dx =$$

$$= -\int_\Omega f \overline{\varphi_\ell} dx, \qquad \ell = 1, \ldots, N, \tag{13}$$

with $a_0 = a - (-1)^m \lambda_0 I$. Put $v^N(x,t) = u^N(x,t)e^{\lambda t}$. Multiplying (13) by $e^{\lambda t} \dfrac{d}{dt}\overline{(C_\ell^N(t) e^{\lambda t})}$, then taking the sum in ℓ from 1 to N, one obtains

$$\int_\Omega |v_t^N|^2 dx + \int_\Omega \sum_{|p|,|q|=1}^{m} \left[(-1)^{m+|p|} a_{pq} D^q v^N \overline{D^p v_t^N} + \lambda_0 v^N \overline{v_t^N} \right] dx +$$

$$+(-1)^m \int_\Omega \left[\sum_{|p|=1}^{m} a_p D^p v^N \overline{v_t^N} + (a_0 - (-1)^m \lambda I) v^N \overline{v_t^N} \right] dx = -\int_\Omega f e^{\lambda t} \overline{v_t^N} dx. \tag{14}$$

Since $a_{pq} = (-1)^{|p|+|q|} a_{qp}^*$, we have

$$2\mathrm{Re}\left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q v^N \overline{D^p v_t^N} + (-1)^m \lambda_0 v^N \overline{v_t^N} \right] =$$

$$= \frac{\partial}{\partial t} \left(\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q v^N \overline{D^p v^N} + (-1)^m \lambda_0 |v^N|^2 \right) -$$

$$- \sum_{|p|,|q|=1}^{m} (-1)^{|p|} \frac{\partial a_{pq}}{\partial t} D^q v^N \overline{D^p v^N}. \tag{15}$$

By integrating the real parts of (14) in t from 0 to t and using (15), we get

$$2\int_{\Omega_t} |v_t^N|^2 dx dt +$$

$$+\int\limits_{\Omega}\sum_{|p|,|q|=1}^{m}\left[(-1)^{m+|p|}a_{pq}(x,t)D^{q}v^{N}(x,t)\overline{D^{p}v^{N}(x,t)}+\lambda_0|v^{N}(x,t)|^2\right]dx=$$

$$=\int\limits_{\Omega_t}\sum_{|p|,|q|=1}^{m}(-1)^{m+|p|}\frac{\partial a_{pq}}{\partial t}D^{q}v^{N}\overline{D^{p}v^{N}}dxdt-$$

$$-(-1)^{m}2\mathrm{Re}\int\limits_{\Omega_t}\left[\sum_{|p|=1}^{m}a_pD^pv^N\overline{v_t^N}+\overline{a}v^N\overline{v_t^N}\right]dxdt-2\mathrm{Re}\int_{\Omega_t}fe^{\lambda t}\overline{v_t^N}dxdt,$$

where $\overline{a}=(a_0-(-1)^m\lambda I)$.

By (2) and the Cauchy inequality, we obtain

$$2\int\limits_{\Omega_t}|v_t^N|^2dxdt+\mu_0\|v^N(x,t)\|^2_{H^m(\Omega)}\leq$$

$$\leq C_1(\epsilon)\int\limits_0^t B(t)\|v^N(x,t)\|^2_{H^m(\Omega)}dt+$$

$$+\epsilon\int\limits_{\Omega_t}|v_t^N|^2dxdt+C_2(\epsilon)\int\limits_0^t\|fe^{\lambda t}\|^2_{L_2(\Omega)}dt.$$

Therefore,

$$\int\limits_{\Omega_t}|v_t^N|^2dxdt+\|v^N(x,t)\|^2_{H^m(\Omega)}\leq$$

$$\leq C_3\int\limits_0^t B(t)\|v^N(x,t)\|^2_{H^m(\Omega)}dt+C_4\int\limits_0^\infty\|fe^{\lambda t}\|^2_{L_2(\Omega)}dt. \tag{16}$$

So we get

$$\|v^N(x,t)\|^2_{H^m(\Omega)}\leq C_3\int\limits_0^t B(t)\|v^N(x,t)\|^2_{H^m(\Omega)}dt+C_4\|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}.$$

By Gronwall – Bellman inequality, we have

$$\|v^N(x,t)\|^2_{H^m(\Omega)}\leq C_4\|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}e^{C_3\int_0^t B(t)dt}. \tag{17}$$

Thus,

$$\|v^N(x,t)\|^2_{H^m(\Omega)}\leq C_5\|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)},$$

i.e.,

$$e^{2\lambda t}\|u^N(x,t)\|^2_{H^m(\Omega)}\leq C_5\|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}.$$

Hence,

$$\|u^N(x,t)\|^2_{H^{m,0}(\Omega_\infty)} \le C_6 \|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}. \tag{18}$$

Multiplying (13) by $\dfrac{d(C_\ell^N(t))}{dt}$, taking the sum in ℓ from 1 to N, and integrating in t from 0 to t, we obtain

$$\int\limits_{\Omega_t} \left[u_t^N \overline{u_t^N} + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p u_t^N} + \lambda_0 u^N \overline{u_t^N} \right] dxdt +$$

$$+(-1)^m \int\limits_{\Omega_t} \left(\sum_{|p|=1}^{m} a_p D^p u^N \overline{u_t^N} + a_0 u^N \overline{u_t^N} \right) dxdt = -\int\limits_{\Omega_t} f\overline{u_t^N} dxdt. \tag{19}$$

Hence,

$$2\|u_t^N\|^2_{L_2(\Omega_t)} +$$

$$+\int\limits_{\Omega} \left[\sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq}(x,t) D^q u^N(x,t) \overline{D^p u^N(x,t)} + \lambda_0 \|u^N(x,t)\|^2 \right] dx =$$

$$= \int\limits_{\Omega_t} \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} \frac{\partial a_{pq}}{\partial t} D^q u^N \overline{D^p u^N}\, dx\, dt -$$

$$-(-1)^m 2\mathrm{Re} \int\limits_{\Omega_t} \left[\sum_{|p|=1}^{m} a_p D^p u^N \overline{u_t^N} + a_0 u^N \overline{u_t^N} \right] dxdt - 2\mathrm{Re} \int\limits_{\Omega_t} f\overline{u_t^N} dxdt.$$

By Cauchy inequality and (2) we obtain

$$\|u_t^N\|^2_{L_2(\Omega_t)} + \|u^N(x,t)\|^2_{H^m(\Omega)} \le D_1 \|u^N\|^2_{H^{m,0}(\Omega_t)} + D_2 \|f\|^2_{L_2(\Omega_t)}.$$

Thus,

$$\|u_t^N\|^2_{L_2(\Omega_t)} \le D_1 \|u^N\|^2_{H^{m,0}(\Omega_t)} + D_2 \|f\|^2_{L_2(\Omega_t)} \Rightarrow \|u_t^N\|^2_{L_2(\Omega_\infty)} \le$$

$$\le D_1 \|u^N\|^2_{H^{m,0}(\Omega_\infty)} + D_2 \|f\|^2_{L_2(\Omega_\infty)} \le$$

$$\le D_1 \|u^N\|^2_{H^{m,0}(\Omega_\infty)} + D_2 \|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}. \tag{20}$$

According to (18), we obtain

$$\|u_t^N\|^2_{L_2(\Omega_\infty)} \le (C_6 D_1 + D_2) \|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}. \tag{21}$$

Thus, (18), (21) imply

$$\|u^N\|^2_{H^{m,1}(\Omega_\infty)} \le C \|fe^{\lambda t}\|^2_{L_2(\Omega_\infty)}, \tag{22}$$

where C does not depend on N.

Since the sequence of functions $\{u^N\}$ is uniformly bounded in $H^{m,1}(\Omega_\infty)$, we can take a subsequence which is weakly convergent in $H^{m,1}(\Omega_\infty)$ to some function $u(x,t)$. We will prove that $u(x,t)$ is the solution of (4)–(6). Since $u^N(x,0) = 0$ $\forall x \in \Omega$ and $u^N(x,t) \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$, it follows that $u(x,0) = 0, u(x,t) \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$. For any $T > 0$, multiplying (13) by $d_\ell(t) \in H^1(0,T)$, $d_\ell(T) = 0$, taking the sum in ℓ from 1 to N and integrating in t from 0 to t, we obtain

$$\int\limits_{\Omega_T} u_t^N \bar{\eta} dx dt +$$

$$+(-1)^m \int\limits_{\Omega_T} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u^N \overline{D^p \eta} + (-1)^m u^N \bar{\eta} \right] dx dt +$$

$$+(-1)^m \int\limits_{\Omega_T} \left[\sum_{|p|=1}^{m} a_p D^p u^N \bar{\eta} + a_0 u^N \bar{\eta} \right] dx dt =$$

$$= - \int\limits_{\Omega_T} f \bar{\eta} dx dt. \tag{23}$$

The above equality is true for any function $\eta \in M_N$, where M_N is the set consisting of functions which have the form $\sum_{i=1}^N d_i(t) \varphi_i(x)$, $d_i(t) \in H^1(0,T)$, $d_i(T) = 0$. Since $\{u^N\}$ is weakly convergent, passing to the limit as $N \to \infty$ in equality (23). We get

$$\int\limits_{\Omega_T} u_t \bar{\eta} dx dt + (-1)^m \int\limits_{\Omega_T} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u \overline{D^p \eta} + (-1)^m u \bar{\eta} \right] dx dt +$$

$$+(-1)^m \int\limits_{\Omega_T} \left[\sum_{|p|=1}^{m} a_p D^p u \bar{\eta} + a_0 u \bar{\eta} \right] dx dt = - \int\limits_{\Omega_T} f \bar{\eta} dx dt. \tag{24}$$

Since $M = \bigcup_{N=1}^\infty M_N$ is dense in $\overset{\circ}{H}{}^{m,0}(\Omega_T)$, it follows that (24) holds for any function $\eta \in \overset{\circ}{H}{}^{m,0}(\Omega_T)$, $\eta(x,T) = 0$, i.e., $u(x,t)$ is a generalized solution of (4)–(6). Moreover, the weak convergence of $\{u^N\}$ and (22) imply that

$$\|u(x,t)\|_{H^{m,1}(\Omega_\infty)} = \varliminf_{N \to \infty} \|u^N(x,t)\|_{H^{m,1}(\Omega_\infty)} \le C \|f e^{\lambda t}\|_{L_2(\Omega_\infty)}.$$

The theorem is proved.

3. Asymptotic behaviour of solutions. We now consider the asymptotic behaviour as $t \to \infty$ of generalized solutions of the first boundary-value problem

$$(-1)^{m-1} L(x,t,D) u - u_t = f(x,t), \tag{25}$$

where $(x,t) \in \Omega_\infty$, such that

$$u(x,t)|_{t=0} = \varphi(x) \in \overset{\circ}{H}{}^{2m}(\Omega), \tag{26}$$

$$\left.\frac{\partial^j u}{\partial \nu^j}\right|_{\Gamma_\infty} = 0, \quad j = 0, \ldots, m-1. \tag{27}$$

Theorem 3. *Suppose that*

1) a_{pq} *are continuous on* $\overline{\Omega}_\infty$ *whenever* $|p| = |q| = m$;

2) $\left|\frac{\partial a_{pq}}{\partial t}, \frac{\partial a_p}{\partial t}\right| \leq \mu,\ 1 \leq |p|,\ |q| \leq m,\ \mu = \text{const}$;

3) *there exist* $\lambda_1 > \lambda_2 > 0$ *and a function* $B(t) \in L_1(0, +\infty)$ *such that* $fe^{\lambda_1 t} \in \in L_2(0, +\infty)$, *and*

$$\max\left\{\left|\frac{\partial a_{pq}}{\partial t}\right|, \left|\frac{\partial a_p}{\partial t}\right|, |a - (-1)^m(\lambda_0 + \lambda_2)I|\right\} \leq B(t) \quad \forall (t,x) \in \Omega_\infty.$$

Then problem (25)–(27) *has a generalized solution* $u(x,t)$ *in* $H^{m,1}(\Omega_\infty)$. *Moreover, we have*

$$u(t,x) = \varphi(x)e^{-\lambda_1 t} + o(e^{-\frac{\lambda_2}{2}t}) \quad \textit{as} \quad t \to \infty.$$

Proof. We set $v(x,t) = u(x,t) - \varphi(x)e^{-\lambda_1 t}$. Then problem (25)–(27) can be written as

$$(-1)^{m-1}L(x,t,D)v - v_t = f(x,t) + e^{-\lambda_1 t}[(-1)^m L(x,t,D)\varphi - \lambda_1\varphi(x)], \tag{28}$$

where $(x,t) \in \Omega_\infty$, such that

$$v(x,0) = 0 \quad \text{on} \quad \Omega, \tag{29}$$

$$\left.\frac{\partial^j v}{\partial \nu^j}\right|_{\Gamma_\infty} = 0, \quad j = 0, \ldots, m-1. \tag{30}$$

Consider $g(x,t) = f(x,t) + e^{-\lambda_1 t}[(-1)^m L(x,t,D)\varphi - \lambda_1\varphi(x)]$. It is easy to see that $ge^{\lambda_2 t} \in L_2(\Omega_\infty)$. Thus, by virtue of Theorem 2, problem (28)–(30) has a generalized solution $v(x,t) \in H^{m,1}(\Omega_\infty)$. Furthermore, by the same proof for Theorem 2, as $N \to \infty$, inequality (16) implies that

$$\|e^{\lambda_2 t}v(x,t)\|^2_{H^m(\Omega)} \leq C_1, \tag{31}$$

$$\|(e^{\lambda_2 t}v(x,t))_t\|^2_{L_2(\Omega_\infty)} \leq C_2. \tag{32}$$

It follows from (31), (32) that

$$e^{\lambda_2 t}\left[\|v(x,t)\|^2_{L_2(\Omega)} + \|Dv(x,t)\|^2_{L_2(\Omega)}\right] \leq C_1 e^{-\lambda_2 t},$$

$$\int_0^\infty e^{\lambda_2 t}\|v_t(x,t)\|^2_{L_2(\Omega)}dt \leq$$

$$\leq \int_0^\infty e^{-\lambda_2 t} \|(e^{\lambda_2 t} v(x,t))_t\|^2_{L_2(\Omega)} dt +$$

$$+ \int_0^\infty \lambda_2^2 e^{\lambda_2 t} \|v(x,t)\|^2_{L_2(\Omega)} dt \leq C_3.$$

Thus,

$$\int_0^\infty e^{\lambda_2 t} \Big[\|v_t(x,t)\|^2_{L_2(\Omega)} + \|v(x,t)\|^2_{L_2(\Omega)} + \|Dv(x,t)\|^2_{L_2(\Omega)} \Big] dt < \infty.$$

By virtue of the Nash inequality [10], this gives

$$|v(x,t)| \leq C e^{-\frac{\lambda_2}{2} t} \quad \text{as} \quad t \to \infty.$$

This implies that the generalized solution $u(x,t)$ of problem (25)–(27) satisfies the inequality

$$|u(x,t) - e^{-\lambda_1 t} \varphi(x)| \leq C e^{-\frac{\lambda_2}{2} t}.$$

The proof is completed.

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