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THE LYAPUNOV – SCHMIDT APPROACH TO STUDYING HOMOCLINICS SPLITTING IN WEAKLY PERTURBED LAGRANGIAN AND HAMILTONIAN SYSTEMS

ПРО ЗАСТОСУВАННЯ МЕТОДУ ЛЯПУНОВА – ШМІДТА ДО ДОСЛІДЖЕННЯ ГОМОКЛІНІЧНИХ РОЗЩЕПЛЕНЬ СЛАБКОЗБУРЕНИХ ЛАГРАНЖЕВИХ І ГАМІЛЬТОНОВИХ СИСТЕМ

We analyze the geometric structure of the Lyapunov – Schmidt approach to studying critical manifolds of weakly perturbed Lagrangian and Hamiltonian systems.

Наведено аналіз геометричної структури методу Ляпунова – Шмідта для вивчення критичних многовидів слабкозбурених лагранжевих і гамільтонових систем.

1. Setting. Consider a real Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$, norm $\|\cdot\|$ and a family of Gateau-differentiable functionals $\mathcal{L}_\varepsilon \in C^2(\mathcal{H};\mathbf{R})$ smooth with respect to a real parameter $\varepsilon \in \mathbf{R}$. In further we will use tho following definitions.

Definition 1. *The set* $\operatorname{crit}(\mathcal{L}_\varepsilon) = \{u \in \mathcal{H}\colon \mathcal{L}'_\varepsilon(u) = 0\}$ *is called critical.*

Definition 2. *One says the C^1-functional $\mathcal{L}_\varepsilon\colon \mathcal{H} \to \mathbf{R}^1$ satisfies the Palais – Smale condition* [1] *if any sequence $\{u_n \in \mathcal{H}\colon n \in \mathbf{Z}_+\}$ such that*

$$\sup_{n\in\mathbf{Z}_+} |\mathcal{L}_\varepsilon(u_n)| < +\infty$$

and $\lim\limits_{n\to+\infty} \mathcal{L}'_\varepsilon(u_n) = 0$ *contains a convergent subsequence.*

Definition 3. *A set $Z^d_\varepsilon \subset \operatorname{crit}(\mathcal{L}_\varepsilon)$ is called regular if: a) $Z^d_\varepsilon \subset \mathcal{L}_\varepsilon^{-1}(c)$ for some $c \in \mathbf{R}$ and b) Z^d_ε is isolated, that is there exists a neighborhood $U(Z^d_\varepsilon)$ of the set Z^d_ε* such that $U(Z^d_\varepsilon) \cap (\operatorname{crit}(\mathcal{L}_\varepsilon) \setminus Z^d_\varepsilon) = \varnothing$.

Let's assume further that:

i) the set Z^d of critical points of the functional $\mathcal{L}_0 \subset C^2(\mathcal{H};\mathbf{R})$ is a d-dimensional C^2-manifold;

ii) for all $z \in \mathbf{Z}^d$ the linear operator $\mathcal{L}''_0(z)$ is Fredholmian;

iii) for all $z \in \mathbf{Z}^d$ one has $T_z(Z^d) = \operatorname{Ker} \mathcal{L}''_0(z)$.

Remark 1. In general it is evident, that $T_z(Z^d) \subset \operatorname{Ker} \mathcal{L}''_0(z)$ for all $z \in \mathbf{Z}^d$, that is conditions iii) reflects the nondegeneracy of the mapping $\mathcal{L}_0 \in C^2(\mathcal{H};\mathbf{R})$ amounting to the following: if any $\alpha \in \mathcal{H}$ solves the equation $\mathcal{L}''_0(z)\alpha = 0$, then $\alpha \in T_z(Z^d)$ for any $z \in Z^d$.

The first property being of interest for us is the existence near any $z \in Z^d$ of a manifold Z^d_ε diffeomorphic to Z^d, and such that for any $u \in Z^d_\varepsilon$ the condition $\mathcal{L}'_\varepsilon\big|_{Z^d_\varepsilon}(u) = 0$ implies $\mathcal{L}'_\varepsilon(u) = 0$. In this way the search of regular critical points of

$\mathcal{L}_\varepsilon$ on $\mathcal{H}$ (near Z^d) is reduced to search of regular critical points of the mapping $\mathcal{L}'_\varepsilon\big|_{Z^d_\varepsilon}$. This procedure is carried out in the lemma below via the well know implicit function theorem construction [2].

Lemma 1. *Assume for convenience that* $Z^d = \xi(\mathbf{R}^d)$ *with* $\xi \in C^2(\mathbf{R}^d; \mathcal{H})$, $B_r = \{\theta \in \mathbf{R}^d : \|\theta\| < r\}$ *and* $Z^d_{(r)} := \xi(B_r)$. *Then for given* $r > 0$ *there exists* $\varepsilon_0 > 0$ *and a smooth function* $w\colon M_r \to \mathcal{H}$, *where* $M_r = Z^d_{(r)} \times (-\varepsilon_0, \varepsilon_0)$, *such that*

1^0) $w(z, 0) = 0 \quad \forall\, z \in Z^d_{(r)}$;

2^0) $\mathcal{L}'_\varepsilon(z + w(z;\varepsilon)) \in T_z(Z^d) \quad \forall\, (z;\varepsilon) \in M_r := Z^d_{(r)} \times (-\varepsilon_0, \varepsilon_0)$;

3^0) $w(z;\varepsilon)$ *is orthogonal to* $T_z\big(Z^d_{(r)}\big) \quad \forall\, (z;\varepsilon) \in M_r$.

Proof. Let $q_i = q_i(z)$, $i = \overline{1, d}$, $z \in Z^d_{(r)}$, denote an orthogonal basis for $T_z\big(Z^d_{(r)}\big)$. We will find the mapping $w\colon M_r \to \mathcal{H}$ by means of the local inversion theorem applied to the map $F\colon M_r \times \big(\mathcal{H} \times \mathbf{R}^d\big) \to \mathcal{H} \times \mathbf{R}^d$ defined as follows:

$$F(z;\varepsilon \,|\, w, c) \;:=\; \left(\mathcal{L}'(z+w) - \sum_{i=1}^{d} c_i q_i, (w, q_1), (w, q_2), \ldots, (w, q_d)\right) \;=\; (F_1, F_2). \quad (1)$$

Let us notice here there $F_1 = 0$ means that $\mathcal{L}'_\varepsilon(z+w) \in T_z\big(Z^d_{(r)}\big)$, namely that condition 2^0 holds, while $F_2 = 0$ means that $\big(w, T_z\big(Z^d_{(r)}\big)\big) = 0$, namely that condition 3^0 holds too.

It is easy to see also that $F_1(z; 0|0, 0) = 0$ and $F_2(z; 0|0, 0) = 0$ for all $z \in Z^d_{(r)}$. Fix $z^* \in Z^d_{(r)}$ and consider the Frechet derivative

$$F'\big(z^*; 0|0, 0\big) \;:=\; \left(\frac{\partial F_1}{\partial(w, c)}, \frac{\partial F_2}{\partial(w, c)}\right)$$

of map $F\colon M_r \times \big(\mathcal{H} \times \mathbf{R}^d\big) \to \mathcal{H} \times \mathbf{R}^d$ at point $\big(z^*; 0|0, 0\big)$ with respect to variable $(w, c) \in \mathcal{H} \times \mathbf{R}^d$. One easily finds that for any $(\upsilon, s) \in \mathcal{H} \times \mathbf{R}^d$

$$\left\langle \frac{\partial F_1}{\partial(w, c)}, (\upsilon, s) \right\rangle \;=\; \mathcal{L}''_0(z^*) \cdot \upsilon - \sum_{i=1}^{d} s_i q_i,$$

$$\left\langle \frac{\partial F_2}{\partial(w, c)}, (\upsilon, s) \right\rangle \;=\; ((\upsilon, q_1), (\upsilon, q_2), \ldots, (\upsilon, q_d)),$$

where $\langle \cdot, \cdot \rangle$ is the usual Euclidean scalar product.

In order to prove that $F'\big(z^*; 0|0, 0\big)$ is invertible, we notice that condition ii) implies that the operator $\mathcal{L}''_0(z^*)\colon \mathcal{H} \to \mathcal{H}$ is Fredholmian, so it is enough to prove that its kernel is trivial. Then assume that $F'\big(z^*; 0|0, 0\big)(\upsilon, s) = 0$, i.e.

$$\mathcal{L}''_0(z^*)\,\upsilon \;=\; \sum_{i=1}^{d} s_i q_i(z^*). \quad (2)$$

Taking the inner product of (2) with $q_i(z^*) \in T_z\big(Z^d_{(r)}\big)$, we infer that

$$\big(\mathcal{L}''_0(z^*)\,\upsilon, q_j\big) \;\equiv\; \big(\upsilon, \mathcal{L}''_0(z^*) q_j\big) \;=\; s_j \|q_j\|^2 \;=\; s_j \quad (3)$$

for all $j = \overline{1, d}$ since $\mathcal{L}''_0(z^*) = \mathcal{L}''^{*}_0(z^*)$.

Making now use of iii), that is the condition $q_j \in \operatorname{Ker} \mathcal{L}_0''(z^*)$, or $\mathcal{L}_0''(z^*) q_j = 0$ for all $j = \overline{1, d}$, one gets due to (3) the condition $s_j = 0$ for all $j = \overline{1, d}$. Thereby, the equation (2) is reduced to $\mathcal{L}_0''(z^*)\upsilon = 0$. Making use once more of condition iii) one gets that $\upsilon \in T_{z^*}\left(Z_{(r)}^d\right)$.

On the other side, the condition $\dfrac{\partial F_2}{\partial (w, c)}(\upsilon, \beta) = 0$ implies that $\left(\upsilon, T_{z^*}\left(Z_{(r)}^d\right)\right) = 0$, and thus $\upsilon = 0$. This shows that $F'(z; 0|0, 0)$ is really invertible. So, one can apply the implicit function theorem [2] giving rise to the existence of smooth, unique functions $(w, c)\colon M_r \to \mathcal{H} \times \mathbf{R}^d$, defined in a neighborhood $U_\delta(z^*)$ (relative to $Z_{(r)}^d$) for $\varepsilon \in \mathbf{R}$ small enough, satisfying there the condition

$$F(z; \varepsilon | w(z; \varepsilon), c(z, \varepsilon)) = 0 \tag{4}$$

for all $z \in U_\delta(z^*)$.

Since $Z_{(r)}^d$ is a finite-dimensional compact manifold, one can extend by compactness the function $w\colon U_\delta(z^*) \to \mathcal{H}$ on the whole set M_r that completes the proof.

***Remark* 2.** The found function $w\colon M_r \to \mathcal{H}$ is smooth and $w(z; 0) = 0$ for all $z \in Z_{(r)}^d$. In particular, it follows that $w(z, \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$, uniformly in $z \in Z_{(r)}^d$.

Define the following set

$$Z_\varepsilon^d = \{\pi_\varepsilon(z) := z + w(z; \varepsilon)\colon (z, \varepsilon) \in M_r\}$$

for all small enough $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Lemma 2. *The set Z_ε^d is an n-dimensional manifold diffeomorphic to Z^d and enjoys the natural constraint for $\mathcal{L}_\varepsilon'(z_\varepsilon)$, namely, if $z_\varepsilon \in Z_\varepsilon^d$ and $\mathcal{L}_\varepsilon'\big|_{Z_\varepsilon^d}(z_\varepsilon) = 0$, then $\mathcal{L}_\varepsilon'(z_\varepsilon) = 0$ too.*

Proof. Suppose that $\mathcal{L}_\varepsilon'\big|_{Z_\varepsilon^d}(z_\varepsilon) = 0$ for some $z_\varepsilon = \pi_\varepsilon(z) \in Z_\varepsilon^d$. Then $\mathcal{L}_\varepsilon'(z_\varepsilon)$ is obviously orthogonal to $T_{z_\varepsilon}\left(Z_\varepsilon^d\right)$ since the following commutative diagram

$$\begin{array}{ccc} T\left(Z_{(r)}^d\right) & \xrightarrow{\pi_{\varepsilon,*}} & T\left(Z_\varepsilon^d\right) \\ \downarrow & & \downarrow \\ Z_{(r)}^d & \xrightarrow{\pi_\varepsilon} & Z_\varepsilon^d \end{array}$$

implies that the mapping $\pi_{\varepsilon,*}\colon T\left(Z_{(r)}^d\right) \to T\left(Z_\varepsilon^d\right)$ is a local diffeomorphism and for any $\alpha_\varepsilon = \pi_{\varepsilon,*}\alpha \in T\left(Z_\varepsilon^d\right)$ with $\alpha \in T\left(Z_{(r)}^d\right)$ the following expression

$$\left(\mathcal{L}_\varepsilon'(\pi_\varepsilon(z)), \alpha_\varepsilon\right) = \left(\mathcal{L}_\varepsilon'(\pi_\varepsilon(z)), \pi_{\varepsilon,*}\alpha\right) =$$

$$= \left(\mathcal{L}_\varepsilon'\big|_{Z_r^d}(\pi_\varepsilon(z))\Big|_{\mathcal{L}_0'(z)=0}, \alpha\right) := \left(\mathcal{L}_\varepsilon'\big|_{Z_\varepsilon^d}(z_\varepsilon), \alpha\right) = 0 \tag{5}$$

holds.

On the other hand, from (1) one has that for all $\pi_\varepsilon(z) \in Z_\varepsilon^d$

$$\mathcal{L}_\varepsilon'(\pi_\varepsilon(z)) = \sum_{i=1}^d c_i(z) q_i(z) \tag{6}$$

for all $z \in Z_{(r)}^d$.

Having substituted the expression (6) into (5) one gets for any $\alpha_\varepsilon \in T\left(Z_\varepsilon^d\right)$ and $j = \overline{1, d}$ that

$$\sum_{i=1}^{d} c_i(z)(q_i, \alpha_\varepsilon) = 0 \Leftrightarrow \sum_{i=1}^{d} c_i(z)\left(q_i, \pi_{\varepsilon,*} q_j\right) = \sum_{i=1}^{d} c_i(z)\left(q_i, w' \cdot q_j\right) + c_j(z) \quad (7)$$

since by definition $\operatorname{span}_{\mathbf{R}}\left\{q_j(z): j = \overline{1, d}\right\} = T_z\left(Z_{(r)}^d\right)$, $z \in Z_{(r)}^d$.

As a result of (7) we get the linear vector equation $(1 + Q)c = 0$, where a vector $c := (c_1(z), c_2(z), \ldots, c_d(d))^{\mathrm{T}}$ and the matrix $Q = \{ Q_{ij} := \left(q_i, w'(z, \varepsilon) q_j(z)\right): i, j = \overline{1, d} \}$. Since the condition $(q_i, w) = 0$ holds for all $i = \overline{1, d}$, one finds easily that the matrix $Q: \mathbf{R}^d \to \mathbf{R}^d$ is symmetric with norm $\| Q \| < 1$. The latter follows from the fact that the mapping $\pi_\varepsilon: Z_{(r)}^d \to Z_\varepsilon^d$ is a diffeomorphism, since then the matrix mapping $1 + w' = \pi_{\varepsilon,*}$ is invertible implying the norm $\left\| w' \right\| < 1$.

On the other hand, the matrix norm $\| Q \| = \left\| w' \right\| < 1$, since the matrix Q: $\mathbf{R}^d \to \mathbf{R}^d$ is symmetric and the vector $(q_1, q_2, \ldots, q_d)^{\mathrm{T}} \in T\left(Z_{(r)}^d\right)$ is orthonormal. Thereby, the equation $(1 + Q)c = 0$ can be solved as $c = (1 + Q)^{-1} \cdot 0 = 0$ since the matrix $(1 + Q)$ is in virtue of the condition $\| Q \| < 1$ invertible too.

Summarizing the results stated above, one gets easily from (6) that

$$\mathcal{L}_\varepsilon'(\pi_\varepsilon(z)) = \mathcal{L}_\varepsilon'(z_\varepsilon) = 0$$

for all $z_\varepsilon \in Z_\varepsilon^d$, solving the equation $\mathcal{L}_\varepsilon' \big|_{Z_\varepsilon^d}(z_\varepsilon) = 0$.

Remark **3.** The mapping $\pi_\varepsilon: Z_{(r)}^d \to Z_\varepsilon^d$, where $\pi_\varepsilon(z) = z + w(z; \varepsilon)$ for all $z \in Z_{(r)}^d$, is smooth and enjoys the condition $w(z; \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly in $z \in Z_{(r)}^d$. So, all of solutions to the equation $\mathcal{L}_\varepsilon' \big|_{Z_\varepsilon^d}(z_\varepsilon) = 0$ due to the Palais – Smale condition must enjoy the above condition $z_\varepsilon \to z^*$ as $\varepsilon \to 0$, where $z^* \in Z_{(r)}^d$ solves the equation $\mathcal{L}_0'(z^*) = 0$ that can serve as a tool for localization of the proper critical points under search.

Remark **4.** The finite-dimensional manifold Z_ε^d constructed above by means of the local diffeomorphism $\pi_\varepsilon: Z_{(r)}^d \to Z_\varepsilon^d$ enjoying the equation (4) was specified by its natural extension on the whole compact set $Z_{(r)}^d$ which obviously is not unique. Otherwise, given a local diffeomorphism $\pi_\varepsilon: Z_{(r)}^d \to Z_\varepsilon^d$ where the set $Z_\varepsilon^d := \operatorname{Im} \pi_\varepsilon$, what conditions have to be put on the sets $Z_{(r)}^d$ and Z_ε^d as metric spaces that this local diffeomorphism be a global diffeomorphism of sets $Z_{(r)}^d$ and Z_ε^d? As a part of answer on this question one can claim that some nontrivial topological constraints on the local diffeomorphism $\pi_\varepsilon: Z_{(r)}^d \to Z_\varepsilon^d$ should be involved on what we shall not dwell here in more details, only pointing out this important problem.

2. Time-dependent weakly perturbed systems: separatrix splitting criterion. Denote now by $\mathcal{H}$ the Sobolev space $H_0^1\left(\mathbf{R}, \mathbf{R}^n\right)$ with the usual scalar product

$$(\alpha, \beta)_{1,2} = \int_{\mathbf{R}} \left(\langle \alpha, \beta \rangle + \left\langle \dot{\alpha}, \dot{\beta} \right\rangle\right) dt$$

for any $(\alpha, \beta) \in H_0^1\left(\mathbf{R}, \mathbf{R}^n\right)$.

Assume that the set $Z^d \subset \mathcal{H}$ of critical points of the nondegenerate functional $\mathcal{L}_0 \in C^2(\mathcal{H}; \mathbf{R})$ enjoys conditions i) and ii) allows the representation

$$Z^d = \mathbf{R}_t \times \bar{Z}^{d-1},$$

where $\bar{Z}^{d-1} \subset \mathbf{R}^n$ is a $(d-1)$-dimensional compact submanifold.

The closed subset $\mathbf{R}_t \times \bar{Z}_0^{k-1}$ will be called *homoclinic* if its α- and β-limiting points [3] subject to evolution system $\mathcal{L}_0'(u)=0$, $u \in \mathcal{H}$, are hyperbolic, coincide and $\dim \operatorname{Ker} \mathcal{L}_0''(u) = k$ for some $k \le d$.

In the case when α- and β-limiting points don't coincide the corresponding subset $\mathbf{R}_t \times \bar{Z}_0^{k-1}$ is called *heteroclinic*. In general, these subsets are manifolds and called *separatrices*.

Proceed now to studying the behavior of these separatrix manifolds in the case when a functional $\mathcal{L}_\varepsilon : \mathcal{H} \to \mathbf{R}$ is a weak periodic perturbation of the functional $\mathcal{L}_0 : \mathcal{H} \to \mathbf{R}$ described above, that is

$$\mathcal{L}_\varepsilon := \mathcal{L}_0 + \varepsilon f(t; u|\upsilon), \qquad f(t+2\pi; u|\upsilon) = f(t; u|\upsilon), \tag{8}$$

where $f: \mathbf{R}/(2\pi\mathbf{Z}) \times \mathcal{H} \times \mathbf{R}^k \to \mathbf{R}$ is smooth for any $t \in \mathbf{R}$, $u \in H_0^1(\mathbf{R}, \mathbf{R}^n)$ with $\varepsilon \in \mathbf{R}$ being a small enough parameter.

For the further convenience let us assume that functional

$$\mathcal{L}_0 = \int_{\mathbf{R}} (\langle \varphi(u), u_t \rangle - H(u))\, dt,$$

where $H \in C^2(\mathbf{R}^n, \mathbf{R})$, a mapping $\varphi: \mathbf{R}^n \to \mathbf{R}$, is such that for any $u \in \mathbf{R}^n$ condition $\varphi' - \varphi^{*\prime} = \Omega^{-1}$ with the invertible symplectic matrix $\Omega: \mathbf{R}^n \to \operatorname{Sp}(\mathbf{R}^n)$ holds, that is the differential 2-form $\langle du, \wedge \Omega(u)\, du \rangle$ is closed.

This amounts to the following equation $u_t = -\Omega^{-1}(u) H'$ being equivalent [2] to the equation $\mathcal{L}_0'(u) = 0$. Thus, the corresponding equation equivalent to $\mathcal{L}_\varepsilon'(u) = 0$ brings about the following Hamiltonian system:

$$u_t = -\Omega^{-1}(u) H'(u) + \varepsilon \Omega^{-1}(u) f'(t; u|\upsilon), \tag{9}$$

where $u \in \mathbf{R}_t \times Z_\varepsilon^{d-1}$ due to conditions i) and ii) implied in the setting chapter. Moreover, since $\Omega^* = -\Omega$ and $\det \Omega \ne 0$, one gets easily that $\dim \mathbf{R}^n = n = 2m$, that is even.

Making use of the approach described in the setting, we can investigate the corresponding critical set Z_ε^d and its homoclinic subsets of the functional (8) by means of Lemma 2.

Denote first the stable and unstable manifolds of $\mathbf{R}_t$-invariant critical hyperbolic points $\bar{u}_0 \in \bar{Z}^{k-1}$ of the functional $\mathcal{L}_0 : \mathcal{H} \to \mathbf{R}$ as $W^{(s)}(\bar{u}_0)$ and $W^{(u)}(\bar{u}_0)$ respectively. Define now the following projectors: for a given $s \in \mathbf{R}_t$ and a homoclinic $\gamma: \mathbf{R} \to \bar{Z}^{k-1}$

$$P(s): T_{\gamma(s)}(\mathbf{R}^n) \to T_{\gamma(s)}(W^{(s)}(\bar{u}_0)),$$

$$Q(s): T_{\gamma(s)}(\mathbf{R}^n) \to T_{\gamma(s)}(W^{(u)}(\bar{u}_0)),$$

satisfying the properties

$$\begin{aligned} \big(1 - P^*(s)\big) T^*_{\gamma(s)}\big(\mathbf{R}^n\big) &= T^{\perp}_{\gamma(s)}\big(W^{(s)}(\overline{u}_0)\big), \\ \big(1 - Q^*(s)\big) T^*_{\gamma(s)}\big(\mathbf{R}^n\big) &= T^{\perp}_{\gamma(s)}\big(W^{(u)}(\overline{u}_0)\big), \end{aligned} \tag{10}$$

where the conjugation "$*$" is taken with respect to the standard scalar product in $\mathbf{R}^n$.

Conditions (10) are equivalent evidently to these: $\mathcal{L}_0'(\gamma) = 0$, $\lim_{t\to\pm\infty} \gamma(t) = \overline{u}_0$ and

$$(1 - P(s))\, T_{\gamma(s)}\big(W^{(s)}(\overline{u}_0)\big) = 0, \qquad (1 - Q(s))\, T_{\gamma(s)}\big(W^{(u)}(\overline{u}_0)\big) = 0,$$

for all $s \in \mathbf{R}$.

If now $\alpha_0 \in T_{\gamma(s)}\big(W_0^{(s)}\big)$ at $s \in \mathbf{R}$, then due to existing the transition matrix $G(t, s)\colon T_{\gamma(s)}\big(W^{(s)}(\overline{u}_0)\big) \to T_{\gamma(t)}\big(W_0^{(s)}(\overline{u}_0)\big)$ for $t \in [s, \infty)$ of the tangent evolution system $\mathcal{L}_0''(\gamma)\alpha = 0 \Leftrightarrow u_t = -\Omega^{-1}(u) H'$, the vector $\alpha = G(t, s)\alpha_0 \to 0$ exponentially as $t \to \infty$.

In the case of the weakly perturbed functional (8) as is well known, the hyperbolic $\mathbf{R}_t$-invariant points of the critical set $\overline{Z}^{k-1}$ transform into hyperbolic periodic orbits with corresponding time-dependent stable and unstable manifolds [1, 3, 4].

Let $\pi_\varepsilon(s)\colon \mathbf{R}^n \to \mathbf{R}^n$, $s \in \mathbf{R}_t$, be a Poincare mapping associated with the evolution system $\mathcal{L}_\varepsilon'(u) = 0$, $u \in H^1\big(\mathbf{R}, \mathbf{R}^n\big)$, equivalent to (9). Then it can be characterized as follows.

Proposition 1. *In a vicinity of a homoclinic orbit* $\gamma \in \mathbf{R}_t \times \overline{Z}^{k-1}$ *the stable* $W^{(s)}_{\text{loc}}(\overline{u}_0)$ *and unstable* $W^{(u)}_{\text{loc}}(\overline{u}_0)$ *manifolds of the deformed orbit* $u_\varepsilon \in \mathbf{R}_t \times \overline{Z}_\varepsilon^{k-1}$ *subject to the Poincare mapping* $\pi_\varepsilon(s)$, $s \in \mathbf{R}^n$, *have the following local expression*

$$W^{(s)}_{\text{loc}}(\overline{u}_0) = \bigcup_{s \in R} \big\{\gamma(s) + \varepsilon\, g^{(s)}\big(s; \eta^{(s)}; \varepsilon\big)\big\},$$

$$W^{(u)}_{\text{loc}}(\overline{u}_0) = \bigcup_{s \in \mathbf{R}} \big\{\gamma(s) + \varepsilon\, g^{(u)}\big(s; \eta^{(u)}; \varepsilon\big)\big\},$$

where

$$\begin{aligned} g^{(s)}\big(s; \eta^{(s)}; \varepsilon\big) &:= \eta^{(s)} + (1 - P(s)) \int_{-\infty}^{s} G(s, \tau)\, \Omega^{-1}(\gamma(\tau))\, f'(\tau - s; \gamma(\tau))\, d\tau, \\ g^{(u)}\big(s; \eta^{(u)}; \varepsilon\big) &:= \eta^{(u)} + (1 - Q(s)) \int_{-\infty}^{s} G(s, \tau)\, \Omega^{-1}(\gamma(\tau))\, f'(\tau - s; \gamma(\tau))\, d\tau, \end{aligned} \tag{11}$$

with

$$\eta^{(s)} \in T_{\gamma(s)}\big(W^{(s)}(\overline{u}_0)\big) / T_{\gamma(s)}(\gamma), \quad \big\|\eta^{(s)}\big\| << 1,$$

and

$$\eta^{(u)} \in T_{\gamma(s)}\big(W^{(u)}(\overline{u}_0)\big) / T_{\gamma(s)}(\gamma), \quad \big\|\eta^{(u)}\big\| << 1,$$

for any $t \in \mathbf{R}$.

Due to the hyperbolicity of the perturbed periodic orbit $u_\varepsilon \in \mathbf{R}_t \times \overline{Z}_\varepsilon^{k-1}$ its stable manifolds $W^{(s)}_{\text{loc}}(\overline{u}_\varepsilon)$ with respect to the Poincare mapping $\pi_\varepsilon(s)\colon \mathbf{R}^n \to \mathbf{R}^n$, $s \in \mathbf{R}$, is generated by initial values of the corresponding bounded solutions for $t \in [s, \infty)$ of the tangent evolution system $\mathcal{L}_\varepsilon''(\gamma)\big|_{t=t-s} \cdot \alpha = 0$, $\alpha\big|_{t=s} = \alpha_0 \in T\big(W^{(s)}_{\text{loc}}(\overline{u}_\varepsilon)\big)$. Thus, one can write down that for all $t \in [s, \infty)$

$$\alpha(t, s; \alpha_0) = G(t, s) P(s)\alpha_0 + G(t, s) P(s) \int_s^{\infty} G(s, \tau)\Omega^{-1}(\gamma(\tau)) f'(\tau - s; \gamma(\tau))\, d\tau +$$

$$+ G(t, s)[1 - P(s)] \int_{-\infty}^{t} G(s, \tau)\Omega^{-1}(\gamma(\tau)) f'(\tau - s; \gamma(\tau))\, d\tau. \tag{12}$$

Put now

$$\eta^{(s)} := P(s)(\alpha_0) \in T_{\gamma(s)}\left(W^{(s)}(\bar{u}_0)\right) / T_{\gamma(s)}(\gamma).$$

Then making use of the contraction mapping principle one gets that integral equation (12) has a unique bounded solution $\alpha(t, s; \alpha_0) \in T\left(W_{\text{loc}}^{(s)}(\bar{u}_\varepsilon)\right)$ for $\left\|\eta^{(s)}\right\| << 1$ and all $t \in [s, \infty)$. Thereby, putting $t = s$ we obtain that

$$g^{(s)}\left(s; \eta^{(s)}; \varepsilon\right) = \eta^{(s)} + (1 - P(s)) \int_{-\infty}^{s} G(s, \tau)\Omega^{-1}(\gamma(\tau)) f'(\tau - s; \gamma(\tau))\, d\tau$$

coinciding with the first expression of (11). On the other hand, for any $u_\varepsilon \in T\left(W_{\text{loc}}^{(s)}(\bar{u}_\varepsilon)\right)$ the representation

$$u_\varepsilon(t) = \gamma(t + s) + \varepsilon\alpha(t, s; \alpha_0) + O(\varepsilon^2)$$

holds in a vicinity of the homoclinic orbit $\gamma \in \mathbf{R}_t \times \bar{Z}^{k-1}$ for all $t \in [s, \infty)$.

By the same way one can find the second expression in (11) for unstable manifold $W_{\text{loc}}^{(u)}(\bar{u}_\varepsilon)$.

Proceed now to studying the separation of $W_{\text{loc}}^{(s)}(\bar{u}_\varepsilon)$ and $W_{\text{loc}}^{(u)}(\bar{u}_\varepsilon)$ making use of the Lyapunov – Schmidt procedure [1, 2].

Lemma 3. *The following direct sum decomposition of the tangent vector bundle* $T_{\gamma(s)}\left(\mathbf{R}^n\right)$ *of the following form*

$$T_{\gamma(s)}\left(\mathbf{R}^n\right) = (\text{Range}\, P(s) \cap \text{Range}\, Q(s)) \oplus (\text{Range}\, P(s) \cap \text{Range}\,(1 - Q(s))) \oplus$$

$$\oplus (\text{Range}\,(1 - P(s)) \cap \text{Range}\, Q(s)) \oplus (\text{Range}\,(1 - P(s)) \cap \text{Range}\,(1 - Q(s))) \tag{13}$$

holds for any $s \in \mathbf{R}$.

The proof is based on facts about projectors that

$$(1 - P(s)) P(s) = 0 = Q(s)(1 - Q(s))$$

amounting to the properties:

$$\text{Range}\, P(s) \oplus \text{Range}\,(1 - P(s)) = T_{\gamma(s)}\left(\mathbf{R}^n\right),$$

$$\text{Range}\, Q(s) \oplus \text{Range}\,(1 - Q(s)) = T_{\gamma(s)}\left(\mathbf{R}^n\right),$$

and $T_{\gamma(s)}\left(\mathbf{R}^n\right) \cap T_{\gamma(s)}\left(\mathbf{R}^n\right) = T_{\gamma(s)}\left(\mathbf{R}^n\right)$ for any $s \in \mathbf{R}$.

Subject to the decomposition (13) points $g^{(s)} = g^{(s)}\left(s; \eta^{(s)}; \varepsilon\right)$ and $g^{(u)} = g^{(u)}\left(s; \eta^{(u)}; \varepsilon\right)$ of the corresponding stable $W_{\text{loc}}^{(s)}(\bar{u}_\varepsilon)$ and unstable $W_{\text{loc}}^{(u)}(\bar{u}_\varepsilon)$ manifolds are decomposed as follows:

$$g^{(s)}\left(s; \eta^{(s)}; \varepsilon\right) = \left(s, \sigma; \eta_1^{(s)}; m_1^{(s)}\left(s, \sigma; \eta_1^{(s)}\right); m_2^{(s)}\left(s, \sigma; \eta_1^{(s)}\right)\right),$$

$$g^{(u)}\left(s; \eta^{(u)}; \varepsilon\right) = \left(s, \sigma; \eta_1^{(u)}; m_1^{(u)}\left(s, \sigma; \eta_1^{(u)}\right); m_2^{(u)}\left(s, \sigma; \eta_1^{(u)}\right)\right),$$

where $(s, \sigma) \in \mathrm{Range}\, P(s) \bigcap \mathrm{Range}\, Q(s)$, $\eta^{(s)} := \left(\sigma, \eta_1^{(s)}\right)$, $\eta_1^{(s)} \in \mathrm{Range}(1 - Q(s)) \bigcap \bigcap \mathrm{Range}\, P(s)$, $\eta^{(u)} := \left(\sigma, \eta_1^{(u)}\right)$ and $\eta_1^{(u)} \in \mathrm{Range}(1 - P(s)) \bigcap \mathrm{Range}\, Q(s)$ for any $s \in \mathbf{R}$.

Notice now that the projections of $g^{(s)}$ and $g^{(u)}$ into the subspace $\mathrm{Range}\, P(s) \bigcap \bigcap \mathrm{Range}(1 - Q(s)) \bigcap \mathrm{Range}(1 - P(s)) \bigcap \mathrm{Range}\, Q(s)$ intersect transversally. This means that equations

$$m_1^{(s)}\left(s, \sigma; \eta_1^{(s)} \middle| \upsilon\right) = \eta_1^{(u)}, \qquad m_1^{(u)}\left(s, \sigma; \eta_1^{(u)} \middle| \upsilon\right) = \eta_1^{(s)}$$

can be solved as $\eta_1^{(s)} = \eta_1^{(s)}(s, \sigma | \upsilon)$ and $\eta_1^{(u)} = \eta_1^{(u)}(s, \sigma | \upsilon)$ for any $(s, \sigma) \in \in \mathrm{Range}\, P(s) \bigcap \mathrm{Range}\, Q(s)$, $s \in \mathbf{R}$.

Therefore, to measure the separation of manifolds $W_{\mathrm{loc}}^{(s)}(\overline{u}_0)$ and $W_{\mathrm{loc}}^{(u)}(\overline{u}_0)$, it is enough to measure the separation in the subspace $\mathrm{Range}(1 - P(s)) \bigcap \mathrm{Range}(1 - Q(s))$, $s \in \mathbf{R}$, that is just a geometrical interpretation of the Lyapunov – Schmidt reduction [2, 4] procedure.

Denote by $\delta(s, \sigma | \upsilon) \in T_{\gamma(s)}\left(R^n\right)$, $s \in \mathbf{R}$, the separation of $W_{\mathrm{loc}}^{(s)}(\overline{u}_0)$ and $W_{\mathrm{loc}}^{(u)}(\overline{u}_0)$, that is the vector

$$\delta(s, \sigma | \upsilon) := m_2^{(u)}\left(s, \sigma; \eta_1^{(u)}(s, \sigma | \upsilon)\right) - m_2^{(s)}\left(s, \sigma; \eta_1^{(s)}(s, \sigma | \upsilon)\right). \tag{14}$$

Since, evidently, the vector $\delta(s, \sigma | \upsilon) \in \mathrm{Range}(1 - P(s)) \bigcap \mathrm{Range}(1 - Q(s))$, $s \in \mathbf{R}$, we can coordinate it by means of elements of the linear space $\Phi(s)$ of bounded solutions to the equation

$$\frac{d\varphi}{dt} + K'^{*}\varphi = 0, \qquad \sup_{t \in \mathbf{R}} \|\varphi\| < \infty, \tag{15}$$

adjoint to that $u_t = K(u) := -\Omega^{-1}(u) H'$, $u \in H^1\left(\mathbf{R}; \mathbf{R}^n\right)$. Really, the space

$$\Phi(s) = \mathrm{Range}\left(1 - P^*(s)\right) \bigcap \mathrm{Range}\left(1 - Q^*(s)\right),$$

is that of initial values of bounded on $s \in \mathbf{R}$ solutions to (15). Since,

$$\dim\left(\mathrm{Range}\left(1 - P^*(s)\right) \bigcap \mathrm{Range}\left(1 - Q^*(s)\right)\right) =$$

$$= \dim\left(\mathrm{Range}(1 - P(s)) \bigcap \mathrm{Range}(1 - Q(s))\right) = q,$$

one gets easily that $\dim \Phi(s) = q$, $s \in \mathbf{R}$.

Let $\{\varphi_1, \varphi_2, \ldots, \varphi_q\} \in T_\gamma^*\left(\mathbf{R}^n\right)$ be a basis of the space $\Phi(s)$, $s \in \mathbf{R}$. Then one can determine [1, 4, 5] the coordinates of the separation vector (14) with respect to the basis fixed above as follows:

$$\mu_j(s, \sigma | \upsilon) = \left\langle \varphi_j(s, \upsilon), \delta(s, \sigma | \upsilon) \right\rangle = \int_{\mathbf{R}} \left\langle \varphi_j(t, \sigma), \Omega^{-1}(\gamma(t)) f'(t - s; \gamma(t) | \upsilon) \right\rangle dt,$$

where $j = \overline{1, q}$, $\upsilon \in \mathbf{R}^k$ and $s \in \mathbf{R}$.

The vector $\mu(s, \sigma | \upsilon) := \left(\mu_1(s, \sigma | \upsilon),\ \mu_2(s, \sigma | \upsilon), \ldots,\ \mu_q(s, \sigma | \upsilon)\right)$, $s \in \mathbf{R}$, is usually called *a Mel'nikov vector* being of fundamental importance when studying chaotic behavior [1] of trajectories in a vicinity of the separatrix to a hyperbolic critical point.

The numbers q and $d \in \mathbf{Z}_+$ introduced above can be estimated as follows:

$$q + d = \dim\left\{T_\gamma(\mathbf{R}^n) \big/ \mathrm{Range}(P(s) + Q(s))\right\} = \dim\left\{T_\gamma\left(W^{(s)}(\overline{u}_0)\right) + T_\gamma\left(W^{(u)}(\overline{u}_0)\right)^{\perp}\right\} \le$$

$$\le n = \dim \mathbf{R}^n.$$

Since $n = 2m$, one gets finally that $q + d \le 2m$.

The following theorem as like as in [4] holds.

Theorem. *Let a point* $(s_0, \sigma_0 | \upsilon_0) \in \mathbf{R}_t \times Z^d \times \mathbf{R}^k$ *be such that the Mel'nikov vector* $\mu(s_0, \sigma_0 | \upsilon_0) = 0$. *If:*

i) vectors $\dfrac{\partial \mu}{\partial \sigma_{0,j}}$, $j = \overline{1, d}$, *are nonvanishing;*

ii) $\operatorname{rank} \left\| \dfrac{\partial \mu}{\partial \sigma} \right\| (s_0, \sigma_0 | \upsilon_0) = d \le k$,

then for small enough values of ε $(|\varepsilon| << 1)$ *the local stable* $W_{\mathrm{loc}}^{(s)}(\overline{u}_0)$ *and unstable* $W_{\mathrm{loc}}^{(u)}(\overline{u}_0)$ *manifolds intersect transversally at some point* $p \in W_{\mathrm{loc}}^{(s)}(\overline{u}_0) \cap \cap W_{\mathrm{loc}}^{(u)}(\overline{u}_0)$.

Thus, given a nonperturbed homoclinic manifold $\mathbf{R} \times Z^d$ of the critical points of a nondegenerate smooth functional $\mathcal{L}_0^t : \mathcal{H} \to \mathcal{H}$, then stable and unstable manifolds of its nonautonomous perturbation $\mathcal{L}_\varepsilon : \mathcal{H} \to \mathbf{R}$, where

$$\mathcal{L}_\varepsilon = \int_{\mathbf{R}} (\langle \varphi(u), u_t \rangle - H(u) + \varepsilon f(u))\, dt,$$

intersect transversally if the conditions i) and ii) are enjoyed at some point $(s_0, \sigma_0 | \upsilon_0) \in \mathbf{R}_t \times Z^d \times \mathbf{R}^k$, at which the Mel'nikov vector $\mu(s_0, \sigma_0 | \upsilon_0) = 0$.

The statement above can be effectively used in many important for application studies of nonregular behavior of trajectories [4, 6, 7] in vicinity of homoclinic hyperbolic stable points manifolds.

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