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ON NEW PROPERTIES OF GRAPHS WITH MAGIC TYPE LABELING

We have shown the connection between vertex labelings of magic graph and its overgraph. Also, we have introduced binary relation on the set of all D_i-distance magic graphs, where $D_i \subset \{0, 1, ..., d\}$, $i = 1, 2, ...$ and proved, that it is equivalence relation. Therefore, we have explored the properties of the graphs, which are in this relation. Finally, we have offered the algorithm of constructing r-regular handicap graph $G = (V, E)$ of order n, where $n \equiv 0 (mod 8)$, $r \equiv 1,3 (mod 4)$ and $3 \leq r \leq n-5$.

Keywords: *graph, D-distance magic labeling, (a, d)-distance antimagic labeling, handicap labeling, D-distance matrix, equivalence relation, 1-factor.*

Introduction

The search for the new ways of solving certain practical and theoretical problems contributed into the origin of labeling theory. One of those was “minimization of chaos”, which emerged in the middle of the past century from the field of coding for digital computers. The next problem was the development of methods for optimal element distribution of antenna grate in connection the phenomena of interference; this problem became popular at the end of 1960s. The equivalence between half-graceful graph labeling and no-remaider measurement device, called “Golomb’s ruler”, made significant contribution to solving the problem. The problem of designing radio antennas led to mathematical task of building a decomposition of a complete graph into copies, which are isomorphic to the trees. This problem, known as Ringel’s hypothesis, was described for the first time in 1963 by G. Ringel on symposium in Smolenice [1]. He made a hypothesis that for each positive integer q there is a decomposition of complete graph K_{2q+1} into $2q+1$ subgraphs, each of which is isomorphic to the given tree with q edges. To solve this Ringel’s problem, in 1967, A. Rosa introduced α-, β-, σ- and ρ-valuations of graph in his paper “On certain valuations of the vertices of graph” [2]. At present, the term “labeling” is mostly used instead of “valuation”. In 1963, one more type of labeling was proposed by J. Sedljahcek [3] on the same symposium in Smolenice. J. Sedljahcek extended the theoretical-digital notion of magic square to magic graphs; he defined *magic graph* as a finite connected graph G with neither loops nor multiple edges for which there exist real numbers, the edge labels of G, with the following properties: (1) different edges have different labels, and (2) sum of the labels’ values assigned to all edges, which are in incidence to the certain vertex, is the same for all vertices of graph G [3]. B. Stewart imposed constraints on the set of edge labels and proposed to use sequential integers; he also called the edge labeling as supermagic [4]. Currently, there are various types of magic labelings of a graph. Their descriptions and results concerning their existence, construction and enumeration can be found in electronic journal published by J. Gallian [5].

Broad array of applications explains the popularity of this subject. Labeling is used for graph-model analysis to solve problems of automatic programme parallelizing; for studies of extremal properties of graphs, their decomposition and packaging; to solve the problems, which appear while radio frequencies' distribution occurs during the signal communication network setup; to address the certain questions in theory of coding and cryptography, etc. In this study, we describe the vertex labelings of magic type:

1) D-distance magic, which was first introduced by A. O'Neal and P. Slater in 2011 [6];

2) (a, d)-distance antimagic, which was first introduced by S. Amurugam and N. Kamatchi [7] in 2012.

Problem definition

Let's consider the finite undirected graphs with neither loops nor multiple edges. Let $G = (V, E)$ be a graph and $D \subset \{0, 1, 2, \ldots, d\}$ is a set, where d is the diameter of G. *D-neighborhood* of vertex $v \in V(G)$ is denoted $N_D(v)$. $N_D(v)$ is a set of all vertices $u \in V(G)$, which is located on the distance $d(u, v) \in D$ from a vertex v, thus $N_D(v) = \{u \in V(G) : d(u,v) \in D\}$. If $D = \{l\}$, than we obtain (opened) *l-neighborhood* $N_D(v) = N_l(v)$, and if $D = \{0, l\}$ than it's closed *l-neighborhood* of vertex v with denoting $N_D(v) = N_l[v]$. In particular, $N_{\{1\}}(v) = N(v)$ is the (opened) adjacency set of the vertex v and $N_{\{0,1\}}(v) = N[v]$ is the closed adjacency set of the vertex v. Weight $w(v)$ (or $w_f(v)$) of vertex v for labeling f is calculated as a sum of labels *D-neighborhood* of vertex v, thus $w(v) = \sum_{u \in N_D(v)} f(u)$, where $v \in V(G)$.

Definition 1 [6]. A graph G of order n is called *D-vertex magic graph* or *D-distance magic graph*, if there are a bijection $f : V(G) \to \{1, 2, \ldots, n\}$, and such constant k so for each vertex $v \in V(G)$, $w(v) = k$. The integer k is called *D-distance* (or *distance*) *magic* constant of labeling f.

If $D = \{1\}$, then labeling f of graph G is called *distance magic* or *Σ-labeling*, and if $D = \{0, 1\}$ — then f is called *Σ'-labeling*. Also, the term «*l-vertex magic*» labeling is used, if the graph G is *D-distance magic graph* and $D = \{l\}$, where $0 < l < d$.

It is known, if the graph G of order n doesn't contain isolated vertices and is a distance magic graph with labeling f and magic constant k, then complement $\overline{G}$ of the vertices' weight is denoted with formula $\sum_{v \in N(u)} f(v) = n(n+1)/2 - k - f(u)$, and these form arithmetic progression with the first member $a = n(n + 1) / 2 - k - n$ and the difference $d = 1$. These findings prompted *S*. Amurugam and *N*. Kamatchi to introduce new type of distance labeling, described into definition 2.

Definition 2 [7]. (a, d)-*distance antimagic* labeling of a graph G of order n is a bijection f: $V(G) \to \{1, 2, \ldots, n\}$ with a property that set of all vertex weights form arithmetic progression a, $a + d$, $a + 2d$, ..., $a + (n-1)d$ with first member a and difference d, where a, d are fixed nonnegative integers, $a \geq 1$, $d \geq 0$. A graph that admits the labeling is called (a, d)-*distance antimagic graph*.

Definition 3 [8].The (a, d)-distance antimagic labeling f of a graph $G = (V, E)$ of order n is called *balanced distance d-antimagic* (or *d-handicap*) *labeling*, while $f(u_i) = i$ and the sequence of the weights $w_f(u_1), w_f(u_2), \ldots, w_f(u_n)$, of all vertices forms the increasing arithmetic progression with difference d, where $d \geq 0$, $u_1, u_2, \ldots, u_n \in V(G)$ [14]. When $d = 1$, we are taking about handicap distance antimagic labeling or handicap labeling, in short, and the corresponding graph is called *handicap graph*.

The aim of this paper is to study the new graph properties that admit distance magic and antimagic labelings, as well as to design an algorithm of constructing a r-regular handicap graph $G = (V, E)$ of order n, when $n \equiv 0 (\mathrm{mod}\ 8)$, $r \equiv 1,3 (\mathrm{mod}\ 4)$ and $3 \leq r \leq n-5$.

New properties of D-distance magic graphs

It is convenient to deal with matrix representations of labeling graph rather than with graph itself. Labeling matrices are described in references [6, 9–12]. The authors of the study [6] used matrix analysis to solve the problem of existence D-distance magic labeling of (D, r)-regular graph. The results in regards to relation between closed distance magic labeling graph and its spectrum are described in study [10]. In this section, we continue the studies of D-distance magic graphs properties, described in [12].

Square matrix $A_D=(a^D_{ij})$ of order n is *D-distance* (or *distance*) *matrix* of graph G of order n, while $a^D_{ij}=1$, – if and only if – when $d(u_i, u_j)\in D$, where $u_i, u_j\in V(G)$, $1\le i, j\le n$. In other cases, when $d(u_i, u_j)\notin D$, then $a^D_{ij}=0$.

Every matrix A_D, where $0\notin D$, and every matrix $A_D - E$, where $0\in D$, of graph G of order n, are adjacency matrices of the given graph G_D of order n, which is the D-overgraph of graph G. Let us prove that there is the relation between labeling of graph G, which does not have isolated vertices and its D-overgraph.

Theorem 1. If vertex labeling f of graph G is D-distance magic with a magic constant k, then for D-overgraph of G_D, bijection f generates: i) distance magic labeling if $0 \notin D$; ii) $(k - n, 1)$-distance antimagic labeling if $0 \in D$.

Proof. Let us define $\{u_1, u_2, ..., u_n\}$ as a set of vertices of graph G of order n. Suppose bijection f: $V(G)\to\{1, 2, ..., n\}$ is D-distance magic labeling of graph G. There is labeling matrix $X = (f(u_1), f(u_2), ..., f(u_n))^T$, which is permutation of numbers 1, 2,... ..., n, and complies with equation: $A_DX = kI$, where k is a magic constant of labeling f, $I = (1\ 1\ ...\ 1)^T$ is column matrix, every element of which equals 1.

If $0 \notin D$, then D-distance matrix A_D of graph G is a matrix adjacency of D-overgraph of G_D. Thus, bijection f is the distance magic labeling of G_D.

Suppose $0 \in D$. Then $A = A_D - E$ is adjacency matrix of G_D. With $A_DX = k$ get $AX = \begin{pmatrix} k - f(u_1) \\ k - f(u_2) \\ \vdots \\ k - f(u_n) \end{pmatrix}$.

Thus, bijection f is $(k - n, 1)$-distance antimagic labeling of G_D.

The theorem has been proved.

Let us remind definition of (D, r)-regular graph.

Definition 4 [6]. Graph G of order n is called *(D, r)-regular*, if for any vertex $v_i \in V(G)$ and for every $i = 1, 2, ..., n$, equation $\sum_{j=1}^{n} a_{ij}^D = r$ is true, when $0 \notin D$, and equation $\sum_{j=1}^{n} a_{ij}^D = r+1$ is true, when $0 \in D$, where $A_D = (a_{ij}^D)$ is distance matrix of graph G, thus all D-neighborhood of vertices get the same strength.

We use the results of Theorem 1 to find the magic constant k of (D, r)-regular D-distance magic graph. For this we prove the following corollary. Although it is possible to obtain values k di-rectly from the definitions 1 and 4.

Corollary. If G is (D, r)-regular D-distance magic graph of order n with magic constant k, then $k = \frac{r(n+1)}{2}$ when $0\notin D$ and $k = \frac{(r+1)(n+1)}{2}$ when $0\in D$.

Proof. If G is (D, r)-regular D-distance magic graph of order n, then overgraph G_D is r-regular. For r-regular distance magic graph there is $k = \frac{r(n+1)}{2}$ [9]. This fact and theorem 1 proves that the corollary is correct.

Let $0 \in D$ and $\{u_1, u_2, ..., u_n\}$ be the set of vertices of the graph G. According to theorem 1, r-regular overgraph G_D of graph G will be $(k - n, 1)$-distance antimagic. Let us determine the sum of the weights of its vertices: $\sum_{i=1}^{n} w(u_i) = \frac{rn(n+1)}{2}$ or $\sum_{i=1}^{n} w(u_i) = kn - \frac{n(n+1)}{2}$. Thus, $k = \frac{(r+1)(n+1)}{2}$.

The corollary has been proved.

Every labeled graph G corresponds to distance matrix A_D. The opposite statement is false. Two distance matrices A_{D_1} and A_{D_2} of corresponding graphs G and H of order n are similar, if there is the permutation matrix P with the property that $A_{D_1} = P^{-1}A_{D_2}P$. Since the permutation of similarity is an equivalence relation, then the set of all distance matrices is divided into classes of equivalence. In this case, we will say that A_{D_1} and A_{D_2} belong to one class of the similar matrices, so $A_{D_1} \sim A_{D_2}$. Graphs of same order with the similar distance matrices are called *distance similar* [12]. The distance similar graphs G and H that allows D_1-, D_2-distance magic labeling are called *magic distance similar* [12]. Set of all D_i-distance magic graphs is denoted as M, where $D_i\subset \{0, 1, 2, ..., d\}$, $i = 1, 2, ...$ Let binary relation $R\subset M\times M$. The relation of pair of graphs (G, H) to R we denote as $G \underset{R}{\sim} H$.

Lemma 1. Relation R on the set M of all D_i-distance magic graphs, where $D_i \subset \{0, 1, 2, ..., d\}$, $i = 1, 2, ...$, is equivalence relation, which possesses the following properties:

(1) reflexive: $G \underset{R}{\sim} G$;

(2) symmetrical: $G \underset{R}{\sim} H$ leads to $H \underset{R}{\sim} G$;

(3) transitive: $G \underset{R}{\sim} H$ and $H \underset{R}{\sim} Q$ leads to $G \underset{R}{\sim} Q$.

Proof. Execution of the reflexivity property is obvious.

Suppose G and H are distance similar graphs of order n. It is known, graph G is D_1-distance magic, if and only if, when H is D_2-distance magic graph [12]. So, the symmetric property is true.

Suppose, the graphs G, H and Q of same order have corresponding D_1-, D_2-, D_3-distance magic labeling and corresponding distance matrices A_{D_1}, A_{D_1}, A_{D_3}. This leads to the following: if $G \underset{R}{\sim} H$ and $H \underset{R}{\sim} Q$, then $A_{D_1} \sim A_{D_2}$ and $A_{D_2} \sim A_{D_3}$. So, $A_{D_1} \sim A_{D_3}$, thus $G \underset{R}{\sim} Q$. The lemma has been proved.

Let us divide above described set M into subsets M_1, M_2, ..., M_s so $G \in M_j$ and $H \in M_j$ (j = 1, 2, ..., s), if and only if, when G, H are magic distance similar graphs. Obviously, that set M is the disjoining union of sets M_1, M_2, ..., M_s, which do not cross pairwise and generate the set M/R of classes of equivalence relation $R \subset M \times M$. As it is well known, M/R is called *factor set M*. Given the lemma 1 and results, obtained in [9, 11, 12], we can state, that for each graph $G_i \in M_j$ (j = 1, 2, ..., s) magic constant equals the same number.

Lemma 2. Suppose G and H are D_1-, D_2-distance magic graphs, respectively. If $0 \notin D_1$ and $0 \in D_2$, or $0 \in D_1$ and $0 \notin D_2$, then G and H belong to the different equivalence classes of factor set M/R.

Proof. Suppose G and H are D_1-, D_2-distance magic graphs with both $0 \notin D_1$, and $0 \in D_2$. Distance matrix A_{D_1} is matrix of adjacency overgraph G_{D_1} of graph G, and $A_2 = A_{D_2} - E$ is the matric of adjacency overgraph H_{D_2} of graph H. Let us assume that $G \underset{R}{\sim} H$. So, distance matrices A_{D_1} and A_{D_2} are similar, and there is permutation matrix P, for which equation $A_{D_1} = P^{-1}(A_2 + E)P$ or $A_{D_1} - E = P^{-1}A_2P$ is true. This means that every element of the main diagonal of matrix $A_{D_1} - E$ equals -1. Hence, the execution of the last equation is impossible. That allows us to conclude that the assumption is wrong, so G and H belong to the different equivalence classes of factor set M/R.

Same approach can be used in case of $0 \in D_1$ and $0 \notin D_2$. The lemma has been proved.

It is known that graphs appear isomorphic, then and only then, when their adjacency matrices can be obtained one from another by similar permutations of rows and columns, thus adjacency matrices of isomorphic graphs are similar. We can use this fact to prove the validity of theorem 2.

Theorem 2. If $G \underset{R}{\sim} H$, then their respective overgraphs G_{D_1} and H_{D_2} are isomorphic, meaning $G_{D_1} \cong H_{D_2}$.

Proof. Let us assume that $G \underset{R}{\sim} H$. Then graphs G and H possess D_1-, D_2-magic labelings and their matrices A_{D_1}, A_{D_2} are similar; it means that there is such a permutation matrix P, when $A_{D_2} = P^{-1}A_{D_1}P$.

1) Suppose $0 \notin D_1$ and $0 \notin D_2$. Since A_{D_1} and A_{D_2} are adjacency matrices of respective overgraphs G_{D_1} and H_{D_2}, then $G_{D_1} \cong H_{D_2}$.

2) Suppose $0 \in D_1$ and $0 \in D_2$. Matrices $A_1 = A_{D_1} - E$ and $A_2 = A_{D_2} - E$ are adjacency matrices of respective overgraphs G_{D_1} and H_{D_2}. As A_{D_1} and A_{D_2} are similar, then $A_2 + E = P^{-1}(A_1 + E)P \Leftrightarrow A_2 = P^{-1}A_1P$. So, $G_{D_1} \cong H_{D_2}$.

3) If $0 \notin D_1$ and $0 \in D_2$, or $0 \in D_1$ and $0 \notin D_2$, then according to lemma 2, graphs G and H belong to the different equivalence classes of factor set M/R; this means they are not magic distance similar.

The theorem has been proved.

As well known, if graphs G_1 and G_2 are isomorphic and G is subgraph in G_1, then G_2 has subgraph that is isomorphic to G. Given this fact and theorem 2, we will get the following corollary.

Corollary. If G_{D_1} and H_{D_2} are isomorphic overgraphs respective magic distance similar graphs G and H, then there are such graphs $H^* \subseteq G_{D_1}$ and $G^* \subseteq H_{D_2}$, thus $G \cong G^*$ and $H \cong H^*$.

Proof. Let $G_{D_1} = (V_1, E_1)$ and $H_{D_2} = (V_2, E_2)$ are isomorphic overgraphs corresponding magic distance similar graphs $G = (V_1, E_1^*)$ and $H = (V_1, E_2^*)$. Suppose $E_1 = E \cup E^*$, where each edge with E^* is isomorphic image of the respective edge from the set E_2^*. Consider graph $H^* = (V_1, E^*) = G_{D_1} - E$, where $G_{D_1} - E$ is the operation of sequential removal of all edges that belong to set E. Graph H^* is spanning subgraph of the graph G_{D_1}, and $H \cong H^*$. Following this logic, get $G \cong G^*$. The corollary has been proved.

Construction algorithm of the r-regular handicap graph of order n ≡ 0(mod8) with r ≡ 1,3(mod 4)

One of the types of vertex antimagic labelings is the handicap labeling, introduced by T. Kovarova in 2016. The appearance of the handicap graphs was encouraged by planning incomplete round tournament. The review of results obtained from vertex magic labelings, which are used as mathematical models in problems regarding schedules' setup for incomplete tourmanents, is presented in [13].

We remind that 1-*factorization* of graph G is a set Φ of 1-factors of the graph, if each edge of graph G corresponds to one and only one factor from Φ. In turn, 1-factor of graph G is its 1-regular spanning subgraph.

Let us consider the problem of constructing the r-regular handicap graph G of order n, where $n \equiv 0 (\mathrm{mod}\ 8)$, $r \equiv 1,3 (\mathrm{mod}\ 4)$ and $3 \le r \le n-5$. For its solving in [14, 15], constructive 3-step methods were used. The first and second steps are in connection with identifying three 1-factors, which meet the certain requirements, and for execution of the 3d step, the so-called "bubble" structure is introduced. Authors of paper [13] modified this method by eliminating the «bubble-graph» construction in the 3d step. Below, we will show the algorithm of graph G construction based on the method, described in [13]. Constructing G was simplified to the process of finding its 1-factorization. First, we will introduce some denotations.

Let $n = 8k$, $k \in N$. Vertices of graph $G = (V, E)$, we will denote $u_1, u_2, \dots, u_n$, . Since r is less than three and $r \equiv 1,3 (\mathrm{mod}\ 4)$, then let G be such r-regular graph, which has $r = 2s + 3$. With the bijection $f: V(G) \to \{1, 2, \dots, n\}$, with is a vertex labeling f of G we will identify the vertex with its label, then $f(u_i) = f(i)$. We will mean the number $|i-j|$ as the distance between two vertices i and j. In case of adjacency of these vertices, let us consider the number $|i-j|$ to be the length of the edge. Constructing graph G is realized with the help of $2s + 3$ graphs $F_1, F_2, \dots, F_{2s+3}$, each of which is 1-factor of graph G with edges of a given length. Let $t = 4, 6, \dots, 2s + 2$. For constructing 2-factors $F_t \cup F_{t+1} = \bigcup_{i=1}^{2k} C_i$, where every C_i is an isomorphic image of cycle C_4, we will introduce matrix of cycles $C = [C_1, C_2, \dots, C_{2k}]$. For every vertex $i \in V(G)$ we introduce the certain characteristic — weight of vertex $w(i)$ so that G is distance magic graph, when $w(i) = \mathrm{const}$ and G is handicap graph, when $w(i) = \mathrm{const} + i$.

«HANDICAP» *algorithm* (constructing $2s + 3$-regular handicap graph of order $n \equiv 0 (\mathrm{mod}\ 8)$).

The search begins from the empty graph $G = (V, E)$, where $V = \{1, 2, \dots, n\}$, $E = \varnothing$. As input an integer array $(1, 2, \dots, n)$ and matrix $C = [C_1, C_2, \dots, C_{2k}]$ are used, where $C_i = (0, 0, 0, 0)$ for anyone $1 \le i \le 2k$.

1. [Construct 1-factor F_1] For $1 \le i \le 4k$, we calculate $j = 4k + i$. Thus, $(i, j) \in E(F_1)$. In case of constructed 1-factor F_1, go to step 2.

2. [Construct 1-factor F_2] For $t = 1, 2, \dots, 4k$, we calculate $i = 2t - 1$, $j = 2t$. So, $(i, j) \in E(F_2)$. If 1-factor F_2 was constructed, go to step 3.

3. [Construct 1-factor F_3] For $t = 1, 2, \dots, k$, we calculate $i_1 = 2t - 1$, $i_2 = 2t$, $j_1 = 4k - i_1$ and $j_2 = 4k + 2 - i_2$. Thus $(i_1, j_1), (i_2, j_2) \in E(F_3)$.

For $t = 2k + 1, 2k + 2, \dots, 3k$ we calculate $i_1 = 2t - 1$, $i_2 = 2t$, $j_1 = 12k - i_1$, $j_2 = 12k + 2 - i_2$. We have $(i_1, j_1), (i_2, j_2) \in E(F_3)$.

In case of constructed 1-factor of F_3, go to step 4.

4. [Construct 2-factors $F_t \cup F_{t+1}$, where $t = 4, 6, \dots, 2s + 2$].

4.1. [Find dividers of integer $4k$] For $2 \le l < 4k$, if $4k \equiv 0 (\mathrm{mod}\ l)$, then go to step 4.2, otherwise increase the value l into 1.

4.2. [Construct cycles]. For $1 \le i_1 \le 4k$, we calculate $i_2 = i_1 + l$, $j_1 = 8k + 1 - i_1$, $j_2 = 8k + 1 - i_2$. We get $C' = (i_1, i_2, j_1, j_2)$. Go to step 4.3.

4.3. [Construct $F_t \cup F_{t+1}$]. For $1 \le i \le 2k$ and $C_i = (i_1^*, i_2^*, j_1^* j_2^*) \in C$, if $C_i = (0, 0, 0, 0) \in C$, then $C_i = C' \in C$ otherwise we have to compare C_i and C'. If $i_1 \ne i_1^*, i_1 \ne i_2^*, i_1 \ne j_1^*, i_1 \ne j_2^*$, $i_2 \ne i_1^*, i_2 \ne i_2^*, i_2 \ne j_1^*$, $i_2 \ne j_2^*, j_1 \ne i_1^*, j_1 \ne i_2^*$, $j_1 \ne j_1^*, j_1 \ne j_2^*, j_2 \ne i_1^*, j_2 \ne i_2^*$, $j_2 \ne j_1^*, j_2 \ne j_2^*$, then C', C_i are different connection components $F_t \cup F_{t+1}$ i $C_{i+1} = C' \in C$ else $C' \notin C$. Go to step 4.2.

If a matrix $C = [C_1, C_2, \dots, C_{2k}]$ was found, then all variants of 2-factors of type $F_t \cup F_{t+1}$ were taken. Go to step 4.1, in the other case — to step 5.

5. We derive the resulting graph $G = F_1 \cup F_2 \cup \dots \cup F_{2s+3}$.

Let us evaluate the constructing computational cost of $2s + 3$-regular handicap graph of order n. Construction labour output of the first three 1-factors F_1, F_2, F_3 is denoted by value $\Theta(n^2/2)$. Labour output of finding 2-factors $F_t \cup F_{t+1}$, when $t = 4, 6, \dots, 2s + 2$ is limited by value $\Theta(n^2/2)$. So, we got the «HANDICAP» algorithm labour output assesment in the form of $\Theta(n^2/2)$.

For algorithm validity we will prove such theorems.

Theorem 3. The complex 1-factor $F_1, F_2, \dots, F_{2s+3}$ is built by 1-factorization $(2s + 3)$-regular graph $G = (V, E)$.

Proof. We have to prove, that $E_i \cap E_j = \varnothing$ for any pair of 1-factors $F_i = (V, E_i)$, $F_j = (V, E_j)$ of graph $G = F_1 \cup F_2 \cup \ldots \cup F_{2s+3}$, when $i \neq j$ and $1 \leq i, j \leq 2s+3$.

The distance between every pair of adjacent vertices of 1-factor F_1 equals $4k$, F_2 –1, F_3 – 2 or 6, or ..., or $4k$ – 6, or $4k$ – 2, F_5 – 3 and 5, or, 11, or 13, ..., or $8k$ – 13, or $8k$ – 11, or $8k$ – 5, or $8k$ – 3, etc. So, $E_1 \cap E_2 = \varnothing$, $E_1 \cap E_3 = \varnothing$, $E_i \cap E_j = \varnothing$ for any $i \neq j$, where $i = 1, 2, 3, j = 5, 7, \ldots, 2s + 3$. Similarly, $E_1 \cap E_j = \varnothing$, $E_2 \cap E_j = \varnothing$ and $E_i \cap E_j = \varnothing$, for any $i \neq j$ when $4 \leq i, j \leq 2s+3$.

We now only have to determine whether F_3 has common edges with 1-factors F_j for j = 4, 6, 8, ..., $2s$ + 2. In F_3 there are four edges with distance 2: ($2k$ – 1, $2k$ + 1), ($2k$, $2k$ + 2), ($6k$ – 1, $6k$ + 1) , ($6k$, $6k$ + 2). The vertices {1, 2, ..., $2k$} of graph $F_4 \cup F_5$ belong to k of its different components, the vertices {$2k$ + 1, $2k$ + 2, ..., $4k$} of graph $F_4 \cup F_5$ belong to other k of different components, so each edge from ($2k$ – 1, $2k$ + 1), ($2k$, $2k$ + 2) doesn't belong to $F_4 \cup F_5$. The same conclusion we do for edges ($6k$ – 1, $6k$ + +1), ($6k$, $6k$ + 2). Thus, $E_3 \cap E_4 = \varnothing$. Similar logics leads to proving the fact, that $E_3 \cap E_j = \varnothing$ for any j = 6, 8, ..., $2s$ + 2. Theorem has been proved.

Theorem 4. Graph $G = F_1 \cup F_2 \cup \ldots \cup F_{2s+3}$ of order $n = 8k$ is ($2s$ + 3)-regular handicap graph with $w(i) = (8k + 1)(s + 1) + i$, where i = 1, 2, ..., $8k$ and F_1, F_2, ..., F_{2s+3} are 1-factorization of G.

Proof. For 1-factor F_1 every its vertex has weight $w(i) = 4k + i$, when i = 1, 2, ..., $4k$ and $w(i)$ = $= -4k + i$, when $i = 4k+1, 4k+2, \ldots, 8k$. Since $F_2 \cup F_3$ is a disconnected graph each component of which is isomorphic to cycle C_4, then a weight of any its vertex i, where i = 1, 2, ..., $8k$, equals $w(i)$ = = $4k$ + 1. According to construction $F_4 \cup F_5$, the weight of every its vertex i, where i = 1, 2, ..., $8k$, equals $w(i) = 8k + 1$. For all following s – 1 of 2-factors we get $w(i) = 8k + 1$ for any vertex i. Thus, $w(i) = (4k+i) + (4k+1) + s(8k+1) = (8k+1)(s+1) + i$, where i = 1, 2, ..., $8k$ and $G = F_1 \cup F_2 \cup \ldots \cup F_{2s+3}$ is a handicap graph. Theorem has been proved.

As the example, we will check the execution of algorithm using 9-regular graph G of order $n = 8k$, where $k = 4$. According to the algorithm, on the first step, we get 1-factor F_1: (1, 17), (2, 18), (3, 19),... ..., (16,32). On the second step we construct 1- factor F_2: (1, 2), (3, 4), (5, 6), ..., (31,32).

1-factor F_3 consists of edges (1, 15), (2, 16), (3, 13), (4, 14), (5, 11), (6, 12), (7, 9), (8, 10), (17, 31), (18, 32), (19, 29), (20, 30), (21, 27), (22, 28), (23, 25), (24, 26).

The following step is associated with the construction of 2-factors $F_4 \cup F_5$, $F_6 \cup F_7$, $F_8 \cup F_9$ when the weight of each vertex is equal $8k$ + 1 = 33.

2-factor $F_4 \cup F_5$ consists of edges (1, 3, 32, 30), (2, 4, 31, 29), (5, 7, 28, 26), (6, 8, 27, 25), (9, 11, 24, 22), (10, 12, 23, 21), (13, 15, 20, 18), (14, 16, 19, 17).

2-factor $F_6 \cup F_7$ consists of edges (1, 5, 32, 28), (2, 6, 31, 27), (3, 7, 30, 26), (4, 8, 29, 25), (9, 13, 24, 20), (10, 14, 23, 19), (11, 15, 22, 18), (12, 16, 21, 17).

2-factor $F_8 \cup F_9$ consists of edges (1, 9, 32, 24), (2, 10, 31, 23), (3, 11, 30, 22), (4, 12, 29, 21), (5, 13, 28, 20), (6, 14, 27, 19), (7, 15, 26, 18), (8, 16, 25, 17).

As the result, we get 9-regular handicap graph $G = \bigcup_{i=1}^{9} F_i$. The weight of its vertices equals $w(i)$ = =132 + i, where $i \in \{1, 2, \ldots, 8k\}$.

Conclusion

In this research, the connection between vertex labeling of magic graph and its overgraph is identified. Also, the equivalence relation on the set of all D_i-distance magic graphs is described as well as the properties of the graphs are studied, which are in this relation. In addition, the problem of constructing *r-regular* handicap graph of order n is solved, where $n \equiv 0 \pmod 8$, $r \equiv 1,3 \pmod 4$ and $3 \leq r \leq n-5$. Finally, the construction algorithm of such graphs is developed. The research results are useful for the further development of the subject and broaden the usage of graphs with magic type labelings.

REFERENCES

1. Ringel, G., 1963. Problem 25. Theory of Graphs and Its Application. *Proc. Sympos. Smolenice*, 1963, Nakl. CSAV, Praha, 1964, pp. 162.
2. Rosa, A., 1967. On certain valuations of the vertices of a graph. In: Theory of Graphs, *Internat. Symposium*, Rome, July 1966. Gordon and Breach/Dunod, New York/Paris, pp. 349–355.

3. Sedlacek, J., 1963. Problem 27, "Theory of graphs and its applications". Proc. Symposium, Smolenice, Nakl. CSAV, Praha. 1964, pp. 163–164.
4. Stewart, B.M., 1996. "Supermagic complete graphs". Canadian Journal of Mathematics, Vol. 19, pp. 427–438.
5. Gallian, J. A., 2018. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*. DS6:Dec 21. 502 p.
6. O'Neal, A., Slater, P., 2011. "An introduction to distance *D* magic graphs". Journal of the Indonesian Mathematical Society. Special Edition, pp. 89–107.
7. Arumugam, S., Kamatchi, N., 2012. "On (*a*, *d*)-distance antimagic graphs". *Australasian journal of combinatorics*. 2012. Vol. 54, pp. 279–287.
8. Froncek, D., 2013. "Handicap distance antimagic graphs and incomplete tournaments". AKCE International Journal of Graphs and Combinatorics, Vol. 10, N 2, pp. 119–127.
9. Arumugam, S., Froncek, D., Kamatchi, N., 2011. "Distance magic graphs – a survey". Journal of the Indonesian Mathematical Society. Special Edition, pp. 11–26.
10. Anholcer, M., Cichacz, S., Peterin, I., 2016. "Spectra of graphs and closed distance magic labelings". Discrete mathematics, Vol. 339, pp.1915–1923.
11. Arumugam, S., Kamatchi, N., 2014. "On the uniqueness of *D*-vertex magic constant". Discussiones mathematicae graph theory, Vol. 34, pp. 279–286.
12. Semeniuta, M., Shulhin, V., 2019. "Matrices associated with *D*-distance magic graphs and their properties". Cybernetics and Systems Analysis, Vol. 5, Issue 3, pp. 441–448.
13. Semeniuta, M.F., Sherman, Z.A., Dmitriiev, O.N., 2018. "Incomplete tournaments and magic types of labeling". Control Systems and Computers, Vol.5, pp. 13–24. (In Russian)
14. Shepanik, A., 2015. Graph labelings and tournament scheduling. MS Thesis. University of Minnesota Duluth, 55 p.
15. Froncek, D., Shepanik, A., 2018. "Regular handicap graph of order n≡0(mod8)". Electronic Journal of Graph Theory and Applications, Vol. 6, N 2, pp.208–218.

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ПРО НОВІ ВЛАСТИВОСТІ ГРАФІВ З РОЗМІТКАМИ МАГІЧНОГО ТИПУ

Вступ. В даній статті досліджуються властивості графів з вершинними розмітками магічного типу, а саме з *D*-дистанційною магічною, вперше запропонованою А. О'Нілом і П. Слейтером в 2011 р. та з (*a*, *d*)-дистанційною антімагічною, введеною у 2012 р. С. Арумугамом і Н. Камачі. Результати щодо вирішення задач існування, побудови і переліку для різних версій розміток графів можна знайти в електронному журналі *A dynamic survey of graph labeling* під редакцією Д. Галліана.

Мета статті — одержати нові властивості графів, що допускають дистанційні магічні та антимагічні розмітки, а також розробити алгоритм побудови *r*-регулярного гандикап графа $G = (V, E)$ порядку *n*, де $n \equiv 0 (\mathrm{mod}\ 8)$, $r \equiv 1{,}3 (\mathrm{mod}\ 4)$ і $3 \leq r \leq n - 5$.

Методи. Використано методи теорії графів, теорії множин, матричного аналізу при дослідженні властивостей графів, які володіють розмітками магічного типу. При розробці алгоритму побудови регулярного гандикап графа задіяні методи теорії розкладів графів та теорії алгоритмів.

Результати. Проведено дослідження властивостей *D*-дистанційних магічних графів і показано, якщо вершинна розмітка *f* графа *G* є *D*-дистанційною магічною, то для *D*-надграфа G_D біекція *f* породжує дистанційну магічну розмітку при $0 \notin D$ і $(k{-}n, 1)$-дистанційну антимагічну розмітку при $0 \in D$. На множині *M* всіх D_i-дистанційних магічних графів, де i=1, 2, ..., введено бінарне відношення $R \subset M \times M$. З цього приводу одержано ряд результатів:

1) *R* є відношенням еквівалентності на *M*;

2) якщо *G* і *H* є D_1-, D_2-дистанційними магічними графами, відповідно і $0 \notin D_1$, $0 \in D_2$ або $0 \in D_1$, $0 \notin D_2$, то *G* і *H* належать до різних класів еквівалентності фактор множини M/R;

3) якщо $G \underset{R}{\sim} H$, то їх відповідні надграфи G_{D_1} і H_{D_2} ізоморфні;

4) якщо G_{D_1} і H_{D_2} є ізоморфними надграфами відповідних магічних дистанційно-подібних графів G і H, то існують такі графи $H^* \subseteq G_{D_1}$ і, $G^* \subseteq H_{D_2}$, що $G \cong G^*$ і $H \cong H^*$.

Запропоновано алгоритм побудови r-регулярного гандикап графа $G = (V, E)$ порядку n, де $n \equiv 0 (\bmod 8)$, $r \equiv 1,3 (\bmod 4)$ і $3 \le r \le n-5$, наведена оцінка його трудомісткості.

Висновок. Встановлено зв'язок між вершинними розмітками магічного типу графа та його надграфа. Одержано опис відношення еквівалентності на множині всіх D_i-дистанційних магічних графів і досліджено властивості графів, що знаходяться в цьому відношенні. Результати дослідження корисні для подальшого розвитку даної тематики і розширюють коло застосувань графів з розмітками магічного типу.

Ключові слова: *граф, D-дистанційна магічна розмітка, (a, d)-дистанційна антимагічна розмітка, гандикап розмітка, D-дистанційна матриця, відношення еквівалентності, 1-фактор.*

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О НОВЫХ СВОЙСТВАХ ГРАФОВ С РАЗМЕТКАМИ МАГИЧЕСКОГО ТИПА

Введение. В данной статье исследуются свойства графов с вершинными разметками магического типа, а именно с D-дистанционной магической, впервые предложенной А. О'Нилом и П. Слэйтером в 2011 г., и с (a, d)-дистанционной антимагической, введенной в 2012 г. С. Арумугамом и Н. Камачи. Результаты по решению задач существования, построения и перечисления для различных версий разметок графов можно найти в электронном журнале *A dynamic survey of graph labeling* под редакцией Д. Галлиана.

Цель статьи — получить новые свойства графов, допускающих дистанционные магические и антимагические разметки, а также разработать алгоритм построения r-регулярного гандикап графа $G = (V, E)$ порядка n, где $n \equiv 0 (\bmod 8)$, $r \equiv 1,3 (\bmod 4)$ и $3 \le r \le n-5$.

Методы. Использованы методы теории графов, теории множеств, матричного анализа при исследовании свойств графов, обладающих разметками магического типа. При разработке алгоритма построения регулярного гандикап графа задействованы методы теории разложений графов и теории алгоритмов.

Результаты. Проведено исследование свойств D-дистанционных магических графов и показано, если вершинная разметка f графа G является D-дистанционной магической, то для D-надграфа G_D биекция f порождает дистанционную магическую разметку при $0 \notin D$ и $(k-n, 1)$-дистанционную антимагическую разметку при $0 \in D$. На множестве M всех D_i-дистанционных магических графов, где $i = 1, 2, \ldots$, введено бинарное отношение $R \subset M \times M$. Поэтому поводу получен ряд результатов:

1) R является отношением эквивалентности на M;

2) если G и H – D_1-, D_2-дистанционные магические графы, соответственно, и $0 \notin D_1$, $0 \in D_2$ или $0 \in D_1$, $0 \notin D_2$, то G и H принадлежат к разным классам эквивалентности фактор множества M/R;

3) если $G \underset{R}{\sim} H$, то их соответствующие надграфы G_{D_1} и H_{D_2} изоморфны;

4) если G_{D_1} и H_{D_2} являются изоморфными надграфами соответствующих магических дистанционно-подобных графов G и H, то существуют такие графы $H^* \subseteq G_{D_1}$ и $G^* \subseteq H_{D_2}$, что $G \cong G^*$ и $H \cong H^*$.

Предложен алгоритм построения r-регулярного гандикап графа $G = (V, E)$ порядка n, где $n \equiv 0 (\bmod 8)$, $r \equiv 1,3 (\bmod 4)$ и $3 \le r \le n-5$, приведена оценка его трудоемкости.

Выводы. Установлена связь между вершинными разметками магического типа графа и его надграфа. Получено описание отношения эквивалентности на множестве всех D_i-дистанционных магических графов и исследованы свойства графов, находящихся в этом отношении. Результаты исследования полезны для дальнейшего развития данной тематики и расширяют круг применений графов с разметкой магического типа.

Ключевые слова: *граф, D-дистанционная магическая разметка, (a, d)-дистанционная антимагическая разметка, гандикап разметка, D-дистанционная матрица, отношение эквивалентности, 1-фактор.*