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EQUILIBRIUM IN QUANTUM SYSTEMS OF PARTICLES WITH MAGNETIC INTERACTION. FERMI AND BOSE STATISTICS*

РІВНОВАГА В КВАНТОВИХ СИСТЕМАХ ЧАСТИНОК З МАГНІТНОЮ ВЗАЄМОДІЄЮ. СТАТИСТИКА ФЕРМІ, БОЗЕ

Quantum systems of particles interacting via an effective electromagnetic potential with zero electrostatic component are considered (magnetic interaction). It is assumed that the j-th component of the effective potential for n particles equals the partial derivative with respect to the coordinate of the j-th particle of "magnetic potential energy" of n particles almost everywhere. The reduced density matrices for small values of the activity are computed in the thermodynamic limit for d-dimensional systems with short-range pair magnetic potentials and for one-dimensional systems with the long-range pair magnetic interaction, which is an analog of the interaction of three-dimensional Chern–Simons electrodynamics ("magnetic potential energy" coincides with the one-dimensional Coulomb (electrostatic) potential energy).

Розглядаються квантові системи частинок, що взаємодіють за допомогою ефективного електромагнітного потенціалу з нульовою електростатичною компонентою (магнітна взаємодія). Припускається, що j-та компонента ефективного потенціалу n частинок збігається з частинною похідною за координатою j-ї частинки „магнітної потенціальної енергії" n частинок майже скрізь. Обчислено редуковані матриці густини в термодинамічній границі при малих значеннях активності частинок для d-вимірних систем з короткодіючим парним потенціалом взаємодії та одновимірних систем з далекосяжною магнітною взаємодією, яка є аналогом взаємодії у 3-вимірній електродинаміці Черна–Саймонса („магнітна потенціальна енергія" збігається з одновимірною кулонівською (електростатичною) потенціальною енергією).

1. Introduction. The quantum system of n d-dimensional particles with magnetic interaction can be defined by the Hamiltonian of a system of n charged particles moving in the electromagnetic collective (effective) field characterized by the vector potential $a_j(X_n)$ depending on the position vectors $X_n = (x_1, \ldots, x_n)$, $x_j = (x_j^1, \ldots, x_j^d)$, from the dn-dimensional space $\mathbb{R}^{nd}$. We assume that a singularity of the function appears only if the Euclidean distance between particles vanishes. Then the initial Hamiltonian $\dot{H}_n$ as a symmetric operator defined on $C_0(\mathbb{R}^{dn}_{[0]})$,

$$\mathbb{R}^{dn}_{[0]} = \bigcup_{\nu=1}^{d} \mathbb{R}^{nd}_{\nu}, \qquad \mathbb{R}^{nd}_{\nu} = \mathbb{R}^{dn} \setminus \bigcup_{k<j} \quad (x_k^\nu = x_k^\nu),$$

is given by

$$\dot{H}_n = \frac{1}{2}\sum_{j=1}^{n} (p_j - a_j(X_n))^2, \tag{1}$$

$$p_j = i^{-1}\partial_j, \quad (p_j - a_j)^2 = \sum_{\nu=1}^{d} (p_j^\nu - a_j^\nu)^2,$$

and ∂_j is the partial derivative in x_j.

The motivation to study such systems appeared recently when it was realized that such systems can be derived in 3d topological or Chern – Simons (CS) nonrelativistic electrodynamics. Then vector potentials a_j are given by a skew partial derivative with respect to x_j of the Coulomb potential energy of a system of n charged particles [1].

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It is believed that the system with Fermi statistic can describe the phenomena of high-temperature superconductivity [2, 3] and there are phase transitions [4, 5]. There is a nontrivial problem of description of equilibrium states for such systems in the thermodynamic limit for the case

$$a_j(X_n) = \sum_{k\in(1,\ldots,j-1,j+1,\ldots,n)} a(x_j - x_k). \tag{2}$$

For such classical systems, the grand partition function coincides with the grand partition function of the free particle system, but the Gibbs (grand canonical) correlation functions are easily computed only in the case of short-range interactions ($a(x)$ is an integrable function). The existence of functions for long-range magnetic interactions ($a(x)$ is not an integrable functions) is an open problem (C-S interaction is long-range). We showed [6] that, for the classical C-S system in the mean-field type limit (the thermodynamic limit transition is performed simultaneously), the correlation functions converge to functions that depend only on the momenta of particles and do no factorize into a product of one-particle correlation functions. In the quantum case, the situation is more complicated. Up to now, there are no results concerning the existence of reduced density matrices (RDMs) even for the short-range magnetic interaction with general pair vector magnetic potential $a(x)$. Substantial simplification is achieved if

$$a_j(X_n) = \partial_j U(X_n), \quad X_n \in \mathbb{R}^{dn}_{[0]}, \tag{3}$$
$$U(X_n) = \sum_{1\le k<j\le n} \varphi(x_j - x_k),$$

where φ and U can be multi-valued functions. It is remarkable that the C-S interaction allows such the representation for its vector potential with U_{CS} and

$$\varphi_{CS}(x) = \operatorname{arc\,tan}\frac{x^2}{x^1}, \quad x = (x^1, x^2).$$

For such systems (quasiintegrable), the following equality is true:

$$\dot{H}_n = \exp\{i\hat{U}\}\,\dot{H}_n^0 \exp\{-i\hat{U}\}, \tag{4}$$

where $\hat{U}$ is the operator of multiplication by $U(X_n)$, $\dot{H}_n^0 = -\dot{\Delta}_n/2$, $\dot{\Delta}_n$ is the dn-dimensional Laplacian restricted to $C_0^\infty(\mathbb{R}^{dn}_{[0]})$. It is obvious that there exists the following operator H_n with the domain $D(H_n) = \exp\{i\hat{U}\}\,D(\Delta_n)$, which is the self-adjoint extension of $\dot{H}_n$:

$$H_n = \exp\{i\hat{U}\}\,H_n^0 \exp\{-i\hat{U}\}, \quad H_n^0 = -\frac{1}{2}\Delta_n. \tag{5}$$

In the case of this extension, the grand partition function for the Maxwell–Boltzmann (M-B) statistics coincides with the grand partition function of the free particle system. Moreover, if the magnetic potential is short-range, the RDMs $\rho(X_m|Y_m)$ for the Dirichlet boundary condition are easily computed in the thermodynamic limit [7]. For long-range magnetic interactions, it is difficult to prove the existence of RDMs in the thermodynamic limit. But in the one-dimensional case, there is an exceptional system for which RDMs $\rho(X_m|Y_m)$ can be found in the limit for M-B statistics. It is defined by U expressed as the Coulomb potential energy of a system of n charged particles. We established that RDMs are nonzero in the thermodynamic limit if $x_j - y_j$ sit on a lattice [7]. In this paper, we confirm this result for RDMs $\rho_{+(-)}(X_m|Y_m)$ for the one-dimensional system with Bose (Fermi) statistics and the above selfadjoint extension with the Dirichlet boundary condition. We also find the expression for the RDMs for d-dimensional system in the thermodynamic limit for short-range (integrable) magnetic potential. We utilize Ginibre's loop ensemble in deriving our results [8]. Our paper consists of two sections. In the first section, we formulate the results and, in the second one, we give the proofs.

2. Main results. Let us consider the system with magnetic interaction that satisfies conditions (2) and (3) in the compact domain Λ and the Dirichlet boundary condition. An analog of Hamiltonian (4) is the Hamiltonian $H_{n,\Lambda}$

$$H_{n,\Lambda} = \exp\{i\hat{U}\} H^0_{n,\Lambda} \exp\{-i\hat{U}\}, \tag{6}$$

where

$$H^0_{n,\Lambda} = -\frac{1}{2}\Delta_{n,\Lambda},$$

$\Delta_{n,\Lambda}$ is the dn-dimensional Laplacian with the Dirichlet boundary condition on the boundary of the domain Λ^n, which is the n-fold Cartesian product of the domain Λ. The RDMs of the systems with the Fermi $(\backslash\varepsilon = -, \varepsilon = -1)$ and Bose $(\backslash\varepsilon = +, \varepsilon = 1)$ statistics are given by

$$\rho^{\Lambda}_{\backslash\varepsilon}(X_m | Y_m) =$$

$$= \Xi^{-1}_{\varepsilon,\Lambda} \sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda^n} dX'_n \sum_{\pi\in S_{n+m}} e^{|\pi|}(\exp\{-\beta H_{n,\Lambda}\})(X_m, X'_n; \pi(Y_m, X'_n)), \qquad m > 0, \tag{7}$$

where S_n is the group of permutations of n elements, $|\pi|$ is the signature of the permutation π, $(\exp\{-\beta H_{n,\Lambda}\})(X_n; Y_n)$ is the kernel of the strongly continuous contraction semigroup generated by the Hamiltonian, $\Xi_{\backslash\varepsilon,\Lambda}$ is given by the numerator on the right-hand side of the above equality for $m = 0$.

Lemma. *RDMs* $\rho^{\Lambda}_{\backslash\varepsilon}(X_m|Y_m)$ *in the compact domain with Hamiltonian* (6) *are given by*

$$\rho^{\Lambda}_{\backslash\varepsilon}(X_m|Y_m) = \exp\{i[U(X_m) - U(Y_m)]\}\, \rho^{0(\Lambda)}_{\backslash\varepsilon}(X_m, Y_m) \exp\{G^{\Lambda}_{\backslash\varepsilon}(X_m|Y_m)\}, \tag{8}$$

where

$$G^{\Lambda}_{\backslash\varepsilon}(X_m|Y_m) = \sum_{j>0} \frac{e^{j-1}z^j}{j} \int_{\Lambda} dx \int P^{j\beta}_{x,x}(d\omega)\,\chi_\Lambda(\omega) \times$$

$$\times \left[\exp\left\{i \sum_{k=1}^{m}\sum_{s=1}^{j} (\varphi(x_s - \omega(k\beta)) - \varphi(y_s - \omega(k\beta)))\right\} - 1\right], \tag{9}$$

$P^{j\beta}_{x,y}(d\omega)$ *is the Wiener measure concentrated on paths starting at the initial moment from* x *and arriving at* y *at time* $j\beta$, $\chi_\Lambda(\omega)$ *is the characteristic function of paths that do not intersect the complement of* Λ.

$$\rho^{0(\Lambda)}_{\backslash\varepsilon}(X_m|Y_m) = \sum_{\pi\in S_m} \prod_{j=1}^{m} \rho^{0(\Lambda)}_{\backslash\varepsilon}(x_j|y_j), \tag{10}$$

$$\rho^{0(\Lambda)}_{\backslash\varepsilon}(x|y) = \sum_{j>0} e^{j-1}z^j \int P^{j\beta}_{x,y}(d\omega)\,\chi_\Lambda(\omega).$$

Theorem 1. *Let*

$$|\varphi|_{L^1} = \int |\varphi(x)|\,dx < \infty.$$

Then for $|z| < 1$, *there exist limits of* $\rho^{0(\Lambda)}_{\setminus\varepsilon}$ *and* $G^{\Lambda}_{\setminus\varepsilon}$ *as* $\Lambda \to \mathbb{R}^d$, *denoted by* $\rho^0_{\setminus\varepsilon}$ *and* $G_{\setminus\varepsilon}$, *respectively. The latter are found from equations* (9), (10), *by substituting* $\mathbb{R}^d$ *for* Λ *in them. RDMs in the limit are expressed as follows:*

$$\rho_{\setminus\varepsilon}(X_m|Y_m) = \exp\{i[U(X_m) - U(Y_m)]\}\,\rho^0_{\setminus\varepsilon}(X_m|Y_m)\exp\{G_{\setminus\varepsilon}(X_m|Y_m)\}. \quad (11)$$

Theorem 2. *Let* $d = 1$ *and*

$$U(x_m) = \alpha \sum_{1 \le k < j \le n} |x_j - x_k|.$$

If $|x_j - y_j| \in 2\pi\alpha^{-1}\mathbb{Z}$, *then RDMs in the thermodynamic limit are given by*

$$\rho_{\setminus\varepsilon}(X_m|Y_m) = \exp\{i[U(X_m) - U(Y_m)]\}\,\rho^0_{\setminus\varepsilon}(X_m|Y_m)\exp\{G_{\setminus\varepsilon}(X_m|Y_m)\},$$

where

$$G_{\setminus\varepsilon}(X_m|Y_m) = \sum_{j>0} \frac{e^{j-1}z^j}{j} \int dx \int P^{j\beta}_{x,x}(d\omega) \times$$

$$\times \sum_{k=1}^{j} \sum_{(l_1,\dots,l_k)\in(1,\dots,j)} \prod_{q\in(l_1,\dots,l_k)} \chi_{[a^-,a^+]}(\omega(q\beta)) \times$$

$$\times \left[\exp\left\{ik_0 \sum_{k=1}^{m}\sum_{s=1}^{j} \left(|x_j - \omega(s\beta)| - |y_j - \omega(s\beta)|\right)\right\} - 1\right], \quad (12)$$

$$a^- = \min(X_m, Y_m), \qquad a^+ = \max(X_m, Y_m),$$

and for $|z| < 1/2$,

$$|G_{\setminus\varepsilon}(X_m|Y_m)| \le \sum_{j>0} \frac{(2z)^j}{j^{3/2}} \frac{a^+ - a^-}{(2\pi\beta)^{1/2}}.$$

If the above differences do not belong to the lattice, then the RDMs are zero in the thermodynamic limit.

3. Proofs. Our proofs are based on the transition to the loop ensemble introduced by Ginibre [8, 9]. In order to derive the results, one needs to make twice a resummation (equations (14), (17)) and use relation (16). For the convenience of the reader, we follow all the steps of Ginibre's method. The proof of equation (14) can be found in [8]. We start from the obvious equality

$$(\exp\{-\beta H_{n,\Lambda}\})(X_n; Y_n) =$$

$$= \exp\{i[U(X_n) - U(Y_n)]\} \int P^{\beta}_{X_n,Y_n}(d\omega_n)\,\chi_\Lambda(\omega_{(n)}),$$

where

$$P^{\beta}_{X_n,Y_n}(d\omega_{(n)}) = \prod_{j=1}^{n} P^{\beta}_{x_j,y_j}(d\omega_j), \qquad \omega_{(n)} = (\omega_1, \dots, \omega_n),$$

$$\chi_\Lambda(\omega_{(n)}) = \prod_{j=1}^{n} \chi_\Lambda(\omega_j).$$

Substituting this equality into (7), we obtain

$$\rho^\Lambda_{\setminus\varepsilon}(X_m, Y_m) = \Xi^{-1}_{\setminus\varepsilon,\Lambda} \sum_{n\geq 0} \frac{z^{n+m}}{n!} \int_{\Lambda^n} dX'_n \int \exp\{i\beta[U(X_m, X'_n) - U(Y_m, X'_n)]\} \times$$

$$\times \sum_{\pi\in S_{m+n}} e^{|\pi|} P^\beta_{(X_m, X'_n), \Pi(Y_m, X'_n)}(d\omega_{(m)}\, d\omega'_{(n)})\, \chi_\Lambda(\omega_{(m)}, \omega'_{(n)}). \tag{13}$$

In order to redefine the right-hand side of (13) we have to use the combinatorial formula proved by Ginibre ([5], Lemma 2.1)

$$\int dX'_n \int \sum_{\pi\in S_{n+m}} e^{|\pi|} P^\beta_{(X_m, X'_n), \pi(Y_m, X'_n)}(d\omega_{(m)}\, d\omega'_{(n)}) =$$

$$= \int dX'_r \sum_{r=0}^{n} C^n_r \sum_{|J_m|-m=n-r} (n-r)!\, e^{|J_m|-m} \times$$

$$\times \int \sum_{\pi\in S_m} e^{|\pi|} P^{J_m\beta}_{X_m, \pi Y_m}(d\omega_{(m)}) \sum_{\pi\in S_r} P^\beta_{X'_r, \pi X'_r}(d\omega'_{(r)}), \tag{14}$$

where

$$J_m = (j_1, \ldots, j_m), \quad |J_m| = \sum_{k=1}^{m} j_k.$$

This relation yields

$$\rho^\Lambda_{\setminus\varepsilon}(X_m | Y_m) = \sum_{\pi\in S_m} e^{|\pi|} \sum_{J_m\in 0^m} e^{|J_m|-m} P^{J_m\beta}_{X_m, \pi Y_m}(d\omega_{(m)})\, \rho^\Lambda_\varepsilon(\omega_{(m)}), \tag{15}$$

where ($W(X_m | X'_n) = U(X_m, X'_n) - U(X'_n)$)

$$\rho^\Lambda_{\setminus\varepsilon}(\omega_{(m)}) = \Xi^{-1}_{\setminus\varepsilon,\Lambda} \exp\{i[U(X_m) - U(Y_m)]\}\, \chi_\Lambda(\omega_{(m)}) \times$$

$$\times \sum_{n\geq 0} \frac{z^{n+m}}{n!} \int_{\Lambda^n} dX'_n \exp\{i[W(X_m | X'_n) - W(Y_m | X'_n)]\} \times$$

$$\times \int \sum_{\pi\in S_n} e^{|\pi|} P^\beta_{X'_n, \pi X'_n}(d\omega'_{(n)})\, \chi_\Lambda(\omega'_{(n)}).$$

From the definition of the Wiener measure, the following equality is obtained for a symmetrical function F:

$$\int_{\Lambda^n} dX'_n\, F(X'_n) \prod_{j=2}^{n} P^\beta_\Lambda(x'_j; x'_{j-1})\, P^\beta_\Lambda(x'_n; x'_1) =$$

$$= \int_\Lambda dx \int P^{n\beta}_{x,x}(d\omega)\, F(\omega(\beta), \ldots, \omega(n\beta))\, \chi_\Lambda(\omega). \tag{16}$$

Every permutation in S_n is a product of cycles. Let δ_j be the number of cycles of length j, i.e., $\sum j\delta_j = n$. One can decompose S_n into nonintersecting subsets (conjugate classes) described by different $\{\delta\}_j,\ j = 1, \dots, n$. The number of elements in the class denoted by δ is given by the number $h(\delta, n)$,

$$h(\delta, n) = n! \prod_{j=1}^{n} j^{\delta_j} \delta_j!, \quad \delta = (\delta_{j_1}, \dots, \delta_{j_r}),$$

and all components of the sequence δ are nonzero. From the last equality and equation (16), representing the sum over S_n as a sum over cycles, we obtain

$$\rho^{\Lambda}_{\setminus\varepsilon}(\omega_{(m)}) = \Xi^{-1}_{\setminus\varepsilon,\Lambda} \exp\{i[U(X_m) - U(Y_m)]\} \times$$

$$\times \sum_{\{\delta_j\}} z^{\sum j_k + |J_m|} \prod_j (\delta_j!\, j^{\delta_j})^{-1} \int_{\Lambda^r} dX'_r \int P^{J_r\beta}_{X'_r, X'_r}(d\omega'_{(r)'})\, e^{\sum j_k - \sum \delta_j} \times$$

$$\times \prod_{s=1}^{n} \exp\{i[W(X_m|\tilde{\omega}'_{(s)}(\beta)) - W(Y_m|\tilde{\omega}'_{(s)}(\beta))]\}\, \chi_\Lambda(\omega_{(m)}, \omega'_{(n)}), \tag{17}$$

where

$$\tilde{\omega}_{(s)} = (\omega_s(\beta), \dots, \omega(j_s\beta)), \quad \sum \delta_j = r, \quad \sum_{k=1}^{m} j_k = \sum j\delta_j = n,$$

and the summation is performed over all finite families of integers δ such that δ_j is not zero only for finitely many values of j and all n (in the sequence $(j_1\beta, \dots, j_n\beta)$, the value j occurs δ_j times). We now pass to the summation over $(j_1, \dots, j_n)$. It is clear that

$$\prod_j j^{-\delta_j} = \prod_{s=1}^{r} j_s^{-1}, \quad \sum_{k=1}^{r} j_k - \sum \delta_j = \sum_{k=1}^{r} (j_k - 1).$$

Since the number of decompositions of the set $(1, \dots, n)$ into k subsets with $n_1, \dots, n_k$ elements is equal to $\dfrac{n!}{n_1! \dots n_k!}$, it follows from the symmetricity of the function F that

$$\sum_{n \ge 0} \sum_{j_1, \dots, j_n \ge 0} F(j_1, \dots, j_n) = \sum_{\{\delta_j\}} \prod_j (\delta_j!)^{-1} F(j_1, \dots, j_n),$$

where j_n and δ_n are related as above. The resummation yields

$$\rho^{\Lambda}_{\setminus\varepsilon}(\omega_{(m)}) = z^{|J_m|} \chi_\Lambda(\omega_{(m)}) \exp\{i[U(X_m) - U(Y_m)]\}\, \Xi^{-1}_{\setminus\varepsilon,\Lambda} \times$$

$$\times \sum_{n \ge 0} \sum_{j_1, \dots, j_n \ge 0} \prod_{l=1}^{n} \frac{z^{j_l} e^{j_l - 1}}{j_l} \int_{\Lambda^n} dX'_n \int P^{J_n\beta}_{X'_n, X'_n}(d\omega'_{(n)})\, \chi_\Lambda(\omega_{(n)}) \times$$

$$\times \prod_{s=1}^{n} \exp\{i[W(X_m|\tilde{\omega}'_{(s)}) - W(Y_m|\tilde{\omega}'_{(s)})]\} =$$

$$= z^{|J_m|} \chi_\Lambda(\omega_{(m)}) \exp\{i[U(X_m) - U(Y_m)]\}\, \Xi^{-1}_{\setminus\varepsilon,\Lambda} \times$$

$$\times \exp\left\{ \sum_{j \ge 1} e^{j-1} z^j \int_\Lambda dx \int P^{j\beta}_{x,x}(d\omega) \exp i[W(X_m|\tilde{\omega}_j(\beta)) - \right.$$

$$- W(Y_m|\tilde{\omega}_j(\beta))]\Bigg\}\chi_\Lambda(\omega).$$

The lemma is proved because, by the same arguments, we derive

$$\Xi_{\setminus\varepsilon,\Lambda} = \exp\left\{\sum_{j\geq 1}\frac{e^{j-1}z^j}{j}\int\limits_\Lambda dx\int P^{j\beta}_{x,x}(d\omega)\,\chi_\Lambda(\omega)\right\}.$$

Proof *of the Theorem* 1. It easily follows from the lemma, the estimate $|\exp\{i\varphi(x)\}-1|\leq|\varphi(x)|$, and the equality

$$\int dx\int P^{j\beta}_{x,x}(d\omega)\,|\varphi(x-\omega(s\beta))| = |\varphi|_{L^1}(2\pi\beta)^\nu.$$

Proof *of the Theorem* 2. Let $\Lambda=[-L,L]$, $\chi_{[a,b]}(x)$ be the characteristic function of the interval $[a,b]$. Then the following equality is obvious for $x\in[-L,L]$, $[a,b]\subset[-L,L]$:

$$1 = \prod_{s=1}^{j}\chi_{[-l,l]\setminus[a,b]}(x_s) + \left(1-\prod_{s=1}^{j}(1-\chi_{[a,b]}(x_s))\right) =$$

$$= \prod_{s=1}^{j}\chi_{[-L,L]\setminus[a,b]}(x_s) - \sum_{k=1}^{j}\sum_{(l_1,\ldots,l_k)\in(1,\ldots,j)}\prod_{s\in(l_1,\ldots,l_k)}\chi_{[a,b]}(x_s).$$

As a result, for $a^+=\max(X_m,Y_m)$, $a^-=\min(X_m,Y_m)$, we have

$$G^{[-L,L]}_{\setminus\varepsilon}(X_m|Y_m) = \sum_{j>0}\frac{e^{j-1}z^j}{j}\int\limits_{-L}^{L}dx\int P^{j\beta}_{x,x}(d\omega)\,\chi_{[-L,L]}(\omega)\times$$

$$\times\sum_{k=1}^{j}\sum_{(l_1,\ldots,l_k)\subset(1,\ldots,j)}\prod_{s\in(l_1,\ldots,l_k)}\chi_{[a^+,a^-]}(\omega(s\beta))\times$$

$$\times\left[1-\exp\left\{i\alpha\sum_{j=1}^{m}\sum_{l=1}^{j}(|x_j-\omega(\beta l)|-|y_j-\omega(\beta l)|)\right\}\right]+$$

$$+\sum_{k=0}^{j}\sum_{(l_1,\ldots,l_k)\subset(1,\ldots,j)}\Bigg[\exp\Bigg\{i\alpha\Bigg(\sum_{l\in(l_1,\ldots,l_k)}(x_l-y_l)+$$

$$+\sum_{l'\in(1,\ldots,j)\setminus(l_1,\ldots,l_k)}(y_{l'}-x_{l'})\Bigg)\Bigg\}-1\Bigg]\times$$

$$\times\int\limits_{-L}^{L}dx\int P^{j\beta}_{x,x}(d\omega)\,\chi_{[-L,L]}(\omega)\times$$

$$\times\prod_{s\in(l_1,\ldots,l_k)}\chi_{[-L,a^-]}(\omega(\beta s))\prod_{s'\in(1,\ldots,j)\setminus(l_1,\ldots,l_k)}\chi_{[a^+,L]}(\omega(\beta s')).$$

Here, we have used the equalities

$$|x_j-\omega(\beta s)|-|y_j-\omega(\beta s)| = x_j-y_j,\qquad \omega(\beta s)\in[-L,a^-],$$

$$|x_j - \omega(\beta s)| - |y_j - \omega(\beta s)| = y_j - x_j, \quad \omega(\beta s) \in [a^+, L].$$

The last integral tends to $-\infty$ if $x_j - y_j$ does not belong to $2\pi\alpha^{-1}\mathbb{Z}$ and $\rho_\varepsilon^{[-L,L]}$ converges to zero if L tends to ∞. If

$$x_j - y_j \in 2\pi\alpha^{-1}\mathbb{Z},$$

then

$$\lim_{L\to\infty} G_{\backslash\varepsilon}^{[-L,L]}(X_m, Y_m) = G_{\backslash\varepsilon}(X_m | Y_m).$$

From the bound

$$\sum_{k=1}^{j} \sum_{(l_1,\ldots,l_k)\in(1,\ldots,j)} \prod_{s\in(l_1,\ldots,l_k)} \chi_{[a^+,a^-]}(\omega(\beta s)) \le$$

$$\le 2^j \sum_{s=1}^{j} \chi_{[a^-,a^+]}(\omega(\beta s)),$$

and the equality

$$\int dx \int P_{x,x}^{j\beta} \chi_{[a^-,a^+]}(\omega(\beta s)) =$$

$$= \int dx \int P^{\beta s}(|x-y|)\, P^{\beta(j-s)}(|x-y|)\, \chi_{[a^-,a^+]}(y)\, dy =$$

$$= |a^+ - a^-| \int P^{\beta s}(|x|)\, P^{\beta(j-s)}(|x|)\, dx = |a^+ - a^-|\, P^{\beta j}(0),$$

the needed estimate is obtained. In deriving the last inequality, we changed the order of integration and replaced $x-y$ by x. Theorem 2 is proved.

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