ON RANDOMLY PERTURBED LINEAR OSCILLATING MECHANICAL SYSTEMS

1. Introduction. A linear oscillating system in $R^m$ is a system with the potential energy of the form

$$U(x) = \frac{1}{2} \langle \Lambda x, x \rangle, \quad x \in R^m,$$

where $\Lambda$ is a non-negative symmetric matrix. The kinetic energy of the system is

$$T(v) = \frac{1}{2} \langle v, v \rangle, \quad v \in R^m.$$

The motion of the system is determined by the system of differential equations

$$\frac{d}{dt} x = v,$$

$$\frac{d}{dt} v = -\Lambda x.$$

Let $\{e_1, \ldots, e_m\}$ be the basis formed by eigenvectors of the matrix $\Lambda$. Set

$$x_k = (x, e_k), \quad \lambda_k^2 = \langle \Lambda e_k, e_k \rangle, \quad k = 1, \ldots, m.$$

Then the system (2) can be rewritten in the form

$$\frac{d^2}{dt^2} x_k(t) + \lambda_k^2 x_k(t) = 0, \quad k = 1, \ldots, m.$$

So

$$x_k(t) = a_k \sin \lambda_k(t + \varphi_k),$$

$$v_k(t) = \lambda_k a_k \cos \lambda_k(t + \varphi_k), \quad k = 1, \ldots, m,$$

where $a_k, \varphi_k, k = 1, \ldots, m$, are determined by initial values $x(0), v(0)$. The functions represented by formulas (5) are called the eigen oscillations of the system.

A randomly perturbed linear oscillating system is defined as the solution to the system of differential equations

$$\frac{d}{dt} x_e(t) = v_e(t),$$

$$\frac{d}{dt} v_e(t) = -\Lambda x_e(t) + F(e, t, x_e(t), v_e(t), \omega),$$

where
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\[ F : R_+ \times R_+ \times R^m \times R^m \times \Omega \to R^m. \]

We assume that random perturbations are defined on a probability space \( \{ \Omega , \mathcal{F}, P \} \) and that

\[ \lim_{\varepsilon \to 0} \int_0^t F(\varepsilon, s, x, u, \omega) \, dt = 0 \]

and probability for all \( t > 0 \). We consider two particular cases.

A. Fast Markov perturbation. We assume that

\[ F(\varepsilon, t, x, u, \omega) = f(x, u, y(t/\varepsilon), \omega), \]

where \( f : R^m \times R^m \times Y \to R^m \) and \( (Y, C) \) is a measurable space, \( y(t, \omega) \) is a homogeneous Markov process in \( (Y, C) \), this process is ergodic with an ergodic distribution \( \rho(dy) \), satisfying the following strong mixing condition:

\[ \sup_y \int \text{var} (P(t, y, \cdot) - \rho(\cdot)) \, dt < \infty, \]

where \( P(t, y, \cdot) \) is the transition probability of the Markov process, and \( \text{var}(\cdot) \) is a variation of the signed measure under consideration. We suppose that the function \( f(x, u, y) \) is bounded, measurable in \( y \), twice differentiable in \( x, u \) with bounded derivatives, and the relation

\[ \int f(x, u, y) \rho(dy) = 0, \quad x \in R^m, \quad u \in R^m, \quad (7) \]

is fulfilled.

B. Small Wiener perturbation. We assume that

\[ F(\varepsilon, t, x, u, \omega) = \sqrt{\varepsilon} F(x, u) \frac{d}{dt} \omega(t), \]

here \( F(x, u) \) is a twice differentiable \( L(R^m) \)-valued function which is bounded with its derivatives, and \( \omega(t) \) is the Wiener process in \( R^m \). In this case the second equation of system (5) should be rewritten as a stochastic differential equation.

Differential equations with random functions containing a small parameter were studied first by R. Z. Khasminskii [1–3]. The problems considered in the article are related to diffusion approximation for randomly perturbed differential equations. Under various conditions the problems of such a kind were studied by R. Z. Khasminskii [3], G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan [4], A. V. Skorokhod [5], M. I. Freidlin and A. D. Wentzell [6].

2. Asymptotic properties of unperturbed systems. We need results concerning the behaviour of averaged values of functions of phase variables along the trajectories of the system. Let \( x(t), \ y(t) \) be a solution to system (3). For a function \( \Phi \in C(R^m \times R^m) \) denote:

\[ A_T(\Phi ; x(0), y(0)) = \frac{1}{T} \int_0^T \Phi(x(t), y(t)) \, dt. \]

Theorem 1. A limit exists

\[ \lim_{T \to \infty} A_T(\Phi ; x(0), y(0)) = A(\Phi ; x(0), y(0)), \]

where the function \( A : C(R^m \times R^m) \times R^m \times R^m \to R \) is a non-negative linear function in \( \Phi \), and it is determined by the relation

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\[ A(\Phi; (a_1 \cos \theta_1; \ldots; a_m \cos \theta_m), (-\lambda_1 a_1 \sin \theta_1; \ldots; -\lambda_m a_m \sin \theta_m)) = \]
\[= \frac{\delta_1 \ldots \delta_r}{(2\pi)^r} \int_0^{2\pi} \ldots \int_0^{2\pi} \Phi(X(\delta_1, s_1, \ldots, \delta_r, s_r), V(\delta_1, s_1, \ldots, \delta_r, s_r)) \, ds_1 \ldots ds_r, \]

where the vectors \( X, Y \) are determined by their coordinates:
\[ X_k(\delta_1, s_1, \ldots, \delta_r, s_r) = a_k \cos \left( \sum_{j=1}^{r} n_{kj} \delta_j s_j + \theta_k \right), \]
\[ V_k(\delta_1, s_1, \ldots, \delta_r, s_r) = -\lambda_k a_k \sin \left( \sum_{j=1}^{r} n_{kj} \delta_j s_j + \theta_k \right), \]

here \( r \) is the dimension of the linear span \( L(\lambda_1, \ldots, \lambda_m) \) of \( \lambda_k, k = 1, \ldots, m, \) over the ring \( \mathbb{Z} \), the positive numbers \( \delta_j, j = 1, \ldots, r, \) are formed a basis in \( L(\lambda_1, \ldots, \lambda_m) \), and
\[ \lambda_k = \sum_{j=1}^{r} n_{kj} \delta_j, \quad n_{kj} \in \mathbb{Z}, \quad k = 1, \ldots, m, \quad j = 1, \ldots, r. \]

The proof of the theorem can be obtained from formula (5).

Remark 1. It is easy to see that \( \theta_k = \lambda_k \phi_k \). Formula (10) implies that
\[ A(\Phi; x(0), v(0)) = \hat{A}(\Phi; a_1, \ldots, a_m, \phi_2 - \phi_1, \ldots, \phi_m - \phi_1), \]

where \( x(0), v(0) \) are determined by formula (5) with \( t = 0 \), the function \( \hat{A}(\Phi; \cdot) \) from \( R_+^m \times [-2\pi, 2\pi]^{m-1} \) into \( R \) is expressed through \( A(\Phi; \cdot) \) in a natural way.

Remark 2. Let
\[ \Phi(x, v) = \Phi_1(r_1, \ldots, r_m) \Phi_2(\psi_1, \ldots, \psi_m), \]
where
\[ x_k = r_k \cos \psi_k, \quad v_k = -\lambda_k r_k \sin \psi_k, \quad k = 1, \ldots, m, \]
and
\[ r_k \in R_+, \quad \psi_k \in [0, 2\pi), \quad k = 1, \ldots, m. \]
Then
\[ \hat{A}(\Phi; a_1, \ldots, a_m, \phi_2 - \phi_1, \ldots, \phi_m - \phi_1) = \]
\[ = \Phi(\eta_1, \ldots, \eta_m) \hat{A}(1, \ldots, 1, \phi_2 - \phi_1, \ldots, \phi_m - \phi_1). \]

3. Fast Markov perturbations. We consider the stochastic process \( (x_\varepsilon(t); v_\varepsilon(t)) \) for which \( x_\varepsilon(t) \) and \( v_\varepsilon(t) \) satisfy the system of differential equations
\[ \frac{d}{dt} x_\varepsilon(t) = v_\varepsilon(t), \]
\[ \frac{d}{dt} v_\varepsilon(t) = -\Lambda x_\varepsilon(t) + f(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t)), \]
and \( y_\varepsilon(t) = y \left( \frac{t}{\varepsilon} \right) \), where the stochastic process \( y(t) \) satisfies condition \( A \) of Section 1. We assume that \( x_\varepsilon(0) = x(0), \ v_\varepsilon(0) = v(0) \) are non-random. We will use some results related to the Markov process \( y(t) \) and the solutions to system (12). Denote
\[ R(y, C) = \int_{0}^{\infty} (P(t, y, C) - \rho(C)) \, dt \]  
(13)

and set

\[ Rg(y) = \int g(y') R(y, dy') \]  
(14)

for any measurable bounded function \( g : Y \to R \).

**Lemma 1.** Let \( A \) be the generator of the process \( \gamma(t) \):

\[ A g(y) = \lim_{h \downarrow 0} \frac{1}{h} (E_{y} g(y(h)) - g(y)) \]  
(15)

which is defined on all measurable bounded function \( g(y) \) for which

\[ \frac{1}{h} (E_{y} g(y(h)) - g(y)) \]

is bounded and the limit in the right-hand side of relation (15) exists, \( E_{y} \) is the conditional expectation under the condition \( y(0) = y \).

Then for any measurable bounded function \( g(y) \) satisfying the condition

\[ \int g(y) \rho(dy) = 0 \]

we have

\[ ARg(y) = -g(y). \]  
(16)

The proof is obtained by calculation.

**Lemma 2.** Let a measurable bounded function \( \phi(y) \) satisfies the condition

\[ \int \phi(y) \rho(dy) = 0. \]

Then the stochastic process

\[ \xi_{T}(t) = \frac{1}{\sqrt{T}} \int_{0}^{T} \phi(y(t)) \, dt \]  
(17)

converges weakly to the Wiener process \( \xi(t) \) for which

\[ E\xi(t) = 0, \quad E\xi^{2}(t) = 2t \int \phi(y) R \phi(y) \rho(dy). \]

The proof can be derived from the general theorem on convergence to a diffusion process ([4, p. 78], theorem 1).

**Corollary 1.** The stochastic process \( \gamma(t) \) satisfying condition \( A \) of Section 1 is uniformly ergodic, i.e. for any measurable bounded function \( g(y) \) the following relation is fulfilled

\[ \lim_{T \to \infty} \sup_{y} E_{y} \left( \frac{1}{T} \int_{0}^{T} g(y(t)) \, dt - \int g(y) \rho(dy) \right)^{2} = 0. \]  
(18)

In the next theorem the results on averaging and normal deviations which can be derived from [1, 3, 4], Sec. 2.5, are formulated for system (12).

**Theorem 2.** Let \( (x_{e}(t), v_{e}(t)) \) be the solution to system (12), the function \( f(x, v, y) \) is bounded continuous in \( x, v \) and had bounded continuous in \( x, v \) derivatives

\[ f_{x}(x, v, y), \quad f_{y}(x, v, y), \]

\[ f_{xx}(x, v, y), \quad f_{yx}(x, v, y), \quad f_{vv}(x, v, y), \]

and let \( (x(t), v(t)) \) be the solution to system (3) satisfying the same initial conditions. Then
(i) for any $T > 0$ with probability 1 the relation

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} (|x_\varepsilon(t) - x(t)| + |v_\varepsilon(t) - v(t)|) = 0$$

(19)
is fulfilled;

(ii) set

$$\hat{x}_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} (x_\varepsilon(t) - x(t)),$$

$$\hat{v}_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} (v_\varepsilon(t) - v(t)),$$

(20)
as $\varepsilon \to 0$ the stochastic process $(\hat{x}_\varepsilon(t); \hat{v}_\varepsilon(t))$ converges weakly to the stochastic process $(\hat{x}(t); \hat{v}(t))$ satisfying the system of stochastic differential equations

$$d\hat{x}(t) = \hat{v}(t) \, dt,$$

$$d\hat{v}(t) = -A \hat{x}(t) \, dt + dz(t)$$

(21)

with the initial condition $\hat{x}(0) = \hat{v}(0) = 0$, where $z(t)$ is the Gaussian process with independent increments with $EZ(t) = 0$, and

$$E(z(t), u)^2 = \int_0^t \int (f(x(s), v(s), y), u) (f(x(s), v(s), y'), u) R(y, dy') \rho(dy) \, ds$$

for all $u \in R^m$.

Now we consider the composite stochastic process

$$X_\varepsilon(t) = (x_\varepsilon(t); v_\varepsilon(t); y_\varepsilon(t))$$

in the space $(R^m)^2 \times Y$, here $(x_\varepsilon(t); v_\varepsilon(t))$ is the solution to system (12). It is easy to see that $X_\varepsilon(t)$ is a homogeneous Markov process, and its generator is of the form

$$H_\varepsilon g(x, v, y) = H^0 g(x, v, y) + \frac{1}{\varepsilon} A g(x, v, y),$$

(22)

where

$$H^0 g(x, v, y) = (v, g_x(x, v, y)) - (Ax, g_x(x, v, y)) + (f(x, v, y), g_v(x, v, y))$$

(23)

and $A$ is the generator of the process $y(t)$ which is acting on $g$ as a function of $y$.

The operator $H_\varepsilon$ is defined on the functions $g : (R^m)^2 \times Y \to R$ satisfying the condition $(\mathcal{H})$:

a) $g(x, v, y)$, $g_x(x, v, y)$, $g_v(x, v, y)$ are measurable bounded function continuous in $x$, $v$ uniformly with respect to $y$,

b) the limit

$$\lim_{h \downarrow 0} \frac{1}{h} (E_y g(x, v, y(h)) - g(x, v, y))$$

exists locally uniformly in $x$, $v$.

Denote by $E_{x,v,y}$ the conditional expectation under the condition

$$x_\varepsilon(0) = x, \quad v_\varepsilon(0) = v, \quad y_\varepsilon(0) = y.$$ 

For any function $g$ satisfying condition $(\mathcal{H})$ the following formula is valid

$$E_{x,v,y} g(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t)) - g(x, v, y) = E_{x,v,y} \int_0^t \frac{1}{\varepsilon} A g(x(s), v(s), y(s)) \, ds.$$ 

(24)
Denote by $\mathcal{F}_t^e$ the $\sigma$-algebra generated by $\{X_e(s), s \leq t\}$.

**Lemma 3.** Let $g$ satisfy condition $(\mathcal{H})$ and $\int g(x, u, y) \rho(dy) = 0$. Set

$$G(x, u, y) = \int g(x, u, y') R(y, dy').$$

(25)

Then for $t_1 \leq t_2$ the relation

$$E \left( \int_{t_1}^{t_2} g(x_e(s), u_e(t), y_e(s)) \, ds \bigg| \mathcal{F}_{t_1}^e \right) =$$

$$= \varepsilon E \left( G(x_e(t_1), u_e(t_1), y_e(t_1)) - G(x_e(t_2), u_e(t_2), y_e(t_2)) + \right.$$  

$$\left. + \int_{t_1}^{t_2} H^0 G(x_e(s), u_e(s), y_e(s)) \, ds \bigg| \mathcal{F}_{t_1}^e \right)$$

(26)

is valid.

The proof follows from formulas (16), (22), (24).

**Corollary 2.** Let $g$ satisfy condition $(\mathcal{H})$. Then for $t_1 < t_2$ we have

$$E \left( \int_{t_1}^{t_2} g(x_e(s), u_e(s), y_e(s)) \, ds \bigg| \mathcal{F}_{t_1}^e \right) =$$

$$= E \left( \int_{t_1}^{t_2} \int g(x_e(s), u_e(s), y) \rho(dy) \, ds \bigg| \mathcal{F}_{t_1}^e \right) + O(\varepsilon(1 + (t_2 - t_1))).$$

(27)

To prove this we apply Lemma 3 to the function

$$\hat{g}(x, u, y) = g(x, u, y) - \int g(x, u, y) \rho(dy).$$

Denote by

$$\{x_{ek}, k = 1, \ldots, m\}, \quad \{v_{ek}, k = 1, \ldots, m\}$$

the coordinates of the vectors $x_e, u_e$. Set

$$z^e_k(t) = \lambda^2_k x^2_{ek}(t) + v^2_{ek}(t).$$

(28)

Let $\{\hat{\theta}^e_k, k = 1, \ldots, m\}$ be determined by relation

$$x^e_{ek}(t) = (\lambda_k)^{-1} \sqrt{z^e_k(t)} \cos \lambda_k \hat{\theta}^e_k(t),$$

$$v^e_{ek}(t) = -\sqrt{z^e_k(t)} \sin \lambda_k \hat{\theta}^e_k(t).$$

(29)

Set

$$\theta^e_k(t) = \hat{\theta}^e_k(t) - \hat{\theta}^e_k(t), \quad k = 2, \ldots, m.$$  

(30)

**Lemma 4.** The stochastic process $z^e_k(t), k = 1, \ldots, m,$ and $\theta^e_k(t), k = 2, \ldots, m,$ satisfy the system of differential equations

$$\frac{d}{dt} z^e_k(t) = f_k(x^e(t), u^e(t), y^e(t)), \quad k = 1, \ldots, m,$$

(31)

$$\frac{d}{dt} \theta^e_k(t) = \frac{x^e_k(s) f_k(x^e(t), u^e(t), y^e(t))}{2 z^e_k(t)} -$$

$$- \frac{x^e_{ek}(s) f_k(x^e(t), u^e(t), y^e(t))}{2 z^e_k(t)}, \quad k = 2, \ldots, m.$$  

(32)
The proof follows from formulas (28) – (30). Consider the compound stochastic process

\[ (\tilde{x}^\varepsilon(t); \tilde{\theta}^\varepsilon(t)) = (\tilde{x}^\varepsilon \left( \frac{t}{\varepsilon} \right); \tilde{\theta}^\varepsilon \left( \frac{t}{\varepsilon} \right)) \]  

(33)

in the space \( R^m \times R^{m-1} \), where

\[ \tilde{x}^\varepsilon(t) = (x_1^\varepsilon(t), \ldots, x_m^\varepsilon(t)), \]
\[ \tilde{\theta}^\varepsilon(t) = (\theta_1^\varepsilon(t), \ldots, \theta_m^\varepsilon(t)). \]

We will prove that the stochastic process given by formula (33) converges weakly in \( C \) to a diffusion process. For the description of this process and the proof of the statement we need some notation. Let

\[ x \in R^m, \ v \in R^m, \ x = (x_1, \ldots, x_m), \ v = (v_1, \ldots, v_m). \]

We introduce new variables

\[ z_k = \lambda_k^2 x_k^2 + v_k^2, \quad k = 1, \ldots, m, \]
and

\[ \theta_k = \theta_k - \theta_1, \quad k = 2, \ldots, m, \]

where

\[ x_k = \lambda_k^{-1} \sqrt{z_k} \cos \lambda_k \theta_k, \]
\[ v_k = -\sqrt{z_k} \sin \lambda_k \theta_k. \]

Denote by \( B(x, v) \) a \((m-1) \times m\) matrix with elements which are determined by the relations:

\[ b_{ij}(x, v) = \frac{1}{2} \left( \frac{x_i}{z_i} 1_{j=i} - \frac{x_j}{z_j} 1_{j=i} \right), \quad i = 1, \ldots, m, \quad j = 2, \ldots, m. \]

Let

\[ \hat{f}(x, v) = \int \int f_j(x, v, y) f_j(x, v, y') R(y, dy) \rho(dy'). \]

(34)

Denote

\[ a(x, v) = \int \int f_j(x, v, y') f(x, v, y) R(y, dy) \rho(dy'), \]

and let the vector \( b(x, v) \) be determined by its coordinates

\[ b_k(x, v) = \frac{x_k v_k}{z_k} \hat{f}_{jk}(x, v) - \frac{x_i v_i}{z_i} \hat{f}_{jk}(x, v), \quad k = 2, \ldots, m. \]

(36)

Note that the following formulas are valid for the Jacobians:

\[ \frac{Dz}{Dv} = 2 V, \quad \frac{D\theta}{Dv} = 2 B(x, v), \]

where the elements of the matrix \( V \) are given by the relation \( v_{ij} = v_i 1_{(i=j)} \).

Introduce the matrices \( \hat{F}(x, v) \) with elements \( \hat{f}_{ij}(x, v) \) and

\[ C^{zz}(x, v) = 2V \hat{F}^*(x, v), \quad C^{z\theta}(x, v) = 2B(x, v) \hat{F}^*(x, v), \]
\[ C^{\theta z}(x, v) = 2V \hat{F}^*(x, v) B^*(x, v), \quad C^{\theta\theta}(x, v) = B(x, v) \hat{F}^*(x, v) B^*(x, v). \]

Let the vectors \( \hat{a}(z, \theta) \), \( \hat{b}(z, \theta) \) and matrices
\( \hat{C}^{zz}(z, \theta), \hat{C}^{z\theta}(z, \theta), \hat{C}^{\theta z}(z, \theta), \hat{C}^{\theta\theta}(z, \theta) \)

are \( \hat{A} \)-transformations of the vectors \( a(x, v), b(x, v) \) and matrices

\( C^{zz}(x, v), C^{z\theta}(x, v), C^{\theta z}(x, v), C^{\theta\theta}(x, v), \)

for example

\[ \hat{a}_l(z, \theta) = \hat{A}(a_l; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m), \]

where the function \( \hat{A} \) was introduced in Remark 1. Denote by \( L^{z\theta} \) the differential operator which is determined for \( \Phi \in C^{(2)}(R^m \times R^{m-1}) \) by the relation

\[
L^{z\theta} \Phi(z, \theta) = \left( \Phi_z(z, \theta), \hat{a}(z, \theta) \right) + \left( \Phi_{\theta}(z, \theta), \hat{b}(z, \theta) \right) + \\
+ \text{Tr} \Phi_{zz}(z, \theta) \hat{C}^{zz}(z, \theta) + \text{Tr} \Phi_{z\theta}(z, \theta) \left( \hat{C}^{z\theta}(z, \theta) \right)^* + \\
+ \text{Tr} \Phi_{\theta z}(z, \theta) \left( \hat{C}^{\theta z}(z, \theta) \right)^* + \text{Tr} \Phi_{\theta\theta}(z, \theta) \hat{C}^{\theta\theta}(z, \theta) \right). \quad (37)
\]

**Theorem 3.** The compound stochastic process \( (\hat{z}^\varepsilon(t); \hat{\theta}^\varepsilon(t)) \) converges weakly in \( C \) as \( \varepsilon \to 0 \) to the diffusion process \( (\hat{z}(t); \hat{\theta}(t)) \) in the same space with the initial value \( (z^0; \theta^0) \), where

\[ z_k^0 = E_k(x_k(0), v_k(0)), \quad k = 1, \ldots, m, \]
\[ \theta_k^0 = \varphi_k^0 - \varphi_1^0, \quad k = 2, \ldots, m, \]

and the generator \( L^{z\theta} \) which is determined by formula \( (37) \).

**Proof.** We will use Theorem 1 on [4, p. 78]. We have to prove the relation

\[
\lim_{\varepsilon \to 0} E \left| E \left( \Phi(z^\varepsilon(e^{-1}t_1), \theta^\varepsilon(e^{-1}t_2)) - \Phi(z^\varepsilon(e^{-1}t_1), \theta^\varepsilon(e^{-1}t_1)) \right) - \\
- \int_{t_1}^{t_2} L^{z\theta}(\hat{z}^\varepsilon(s), \hat{\theta}^\varepsilon(s)) \, ds \bigg| \mathcal{F}_{e^{-1}t_1} \right| = 0 \quad (38)
\]

for \( t_1 < t_2 \). To prove this we use the following sequence of relations

\[
E \left( \Phi(z^\varepsilon(t_1), \theta^\varepsilon(t_1)) \right) - \Phi(z^\varepsilon(t_1), \theta^\varepsilon(t_1)) \bigg| \mathcal{F}_{t_1} \right) = \\
= E \left( \int_{t_1}^{t_2} \left[ \Phi_z(z^\varepsilon(s), \theta^\varepsilon(s)), \frac{dz^\varepsilon(s)}{ds} \right] + \left[ \Phi_\theta(z^\varepsilon(s), \theta^\varepsilon(s)), \frac{dz^\varepsilon(s)}{ds} \right] \right) \, ds \bigg| \mathcal{F}_{t_1} \right) = \\
= E \left( \int_{t_1}^{t_2} \Phi_z(z^\varepsilon(s), \theta^\varepsilon(s)) + \\
+ B^*(x_e(s), v_e(s)) \Phi_\theta(z^\varepsilon(s), \theta^\varepsilon(s)), f_e(x_e(s), v_e(s), y_e(s)) \right) \, ds \bigg| \mathcal{F}_{t_1} \right) .
\]

Applying to the last integral Lemmas 4, 3, and Corollary 2 we can obtain the realtion

\[
E \left( \Phi(z^\varepsilon(t_1), \theta^\varepsilon(t_2)) - \Phi(z^\varepsilon(t_1), \theta^\varepsilon(t_1)) \right) \bigg| \mathcal{F}_{t_1} \right) = \\
= O(e^2(t_2 - t_1 + 1)) + \varepsilon E \left( \int_{t_1}^{t_2} \left[ \Phi_z, a(x_e(s), v_e(s)) \right] + \\
+ B^*(x_e(s), v_e(s)) \Phi_\theta(z^\varepsilon(s), \theta^\varepsilon(s)), f_e(x_e(s), v_e(s), y_e(s)) \right) \, ds \bigg| \mathcal{F}_{t_1} \right) .
\]
\[
+ (\Phi_0, b(x_\varepsilon(s), v_\varepsilon(s))) + \text{Tr} \Phi_{zz} \left( C^{zz}(x_\varepsilon(s), v_\varepsilon(s)) \right)^* + \\
+ \text{Tr} \Phi_{z} \left( C^{z}(x_\varepsilon(s), v_\varepsilon(s)) \right)^* + \text{Tr} \Phi_{zz} \left( C^{zz}(x_\varepsilon(s), v_\varepsilon(s)) \right)^* + \\
+ \text{Tr} \Phi_{zz} \left( C^{zz}(x_\varepsilon(s), v_\varepsilon(s)) \right)^* \right] ds \left| \mathcal{T}^e_t \right.
\]

(39)

In this formula the derivatives of the function \( \Phi \) have as their arguments the functions \( x_\varepsilon(s), \theta_\varepsilon(s) \). It follows from statement (i) of Theorem 2 that for any continuous bounded functions \( G : R^m \times R^m \to R \) and \( \Psi : R^m \times R^{m-1} \to R \) the formula is fulfilled:

\[
\lim_{\varepsilon \to 0} \varepsilon E \left[ \int_{s_1}^{s_2} \Psi(x_\varepsilon(s), \theta_\varepsilon(s)) \left| ds \right| \mathcal{T}^e_{s_1} \right] = 0, \quad (40)
\]

where

\[
\hat{G}(x, \theta) = \hat{A}(G; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m).
\]

Formulas (39) and (40) implies the relation

\[
E \left[ \phi \left( \hat{x}_\varepsilon(t_2), \hat{x}_\varepsilon(t_2) \right) - \phi \left( \hat{x}_\varepsilon(t_1), \hat{x}_\varepsilon(t_1) \right) - \int_{t_1}^{t_2} L^z \phi \left( \hat{x}_\varepsilon(s), \hat{x}_\varepsilon(s) \right) ds \right] = O(\varepsilon) + \alpha(\varepsilon), \quad (41)
\]

where

\[
\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0.
\]

Formula (41) implies formula (38), so the theorem is proved.

4. Wiener perturbations. We consider the functions \( x_\varepsilon(t), v_\varepsilon(t) \) satisfying the system of stochastic differential equations:

\[
dx_\varepsilon(t) = v_\varepsilon(t) dt, \quad (42)
\]

\[
dv_\varepsilon(t) = -\Lambda x_\varepsilon(t) + \sqrt{\varepsilon} F(x_\varepsilon(t), v_\varepsilon(t)) dw(t),
\]

where the function

\[
F : R^m \times R^m \to L(R^m)
\]

is bounded and smooth enough, and \( w(t) \) is \( R^m \)-valued Wiener process. Let the stochastic processes \( x_\varepsilon(s), \theta_\varepsilon(s) \) are determined by formulas (28)–(30), where \( x_\varepsilon(t), v_\varepsilon(t) \) satisfy the system (42).

**Lemma 5.** The functions

\[
x_\varepsilon^k(t), \quad k = 1, \ldots, m,
\]

\[
\theta_\varepsilon^i(t), \quad i = 2, \ldots, m,
\]

satisfy the system of stochastic differential equations.
\[
\begin{align*}
\dot{z}_k^\varepsilon(t) &= \sqrt{\varepsilon} \sum_j \alpha_{kj}(x_k(t), v_k(t)) \, dw_j(t) + \varepsilon \beta_k(x_k(t), v_k(t)) \, dt, \\
\dot{\theta}_k^\varepsilon(t) &= \sqrt{\varepsilon} \sum_j \gamma_{kj}(x_k(t), v_k(t)) \, dw_j(t) + \varepsilon \delta_k(x_k(t), v_k(t)) \, dt,
\end{align*}
\]
where
\[
\alpha_{kj}(x, v) = 2v_k F_{kj}(x, v), \quad \beta_k(x, v) = 2v_k \sum_j F_{kj}^2(x, v)
\]
and
\[
\gamma_{kj}(x, v) = \frac{x_i}{z_i} F_{kj}(x, v) - \frac{x_i}{z_i} F_{kj}(x, v),
\]
\[
\delta_k(x, v) = \sum_j \left( \frac{x_i v_j}{z_i} F_{kj}^2(x, v) - \frac{x_i v_j}{z_i} F_{kj}^2(x, v) \right)
\]
and \(F_{ij}\) are the elements of the matrix \(F\).

The proof is obtained by calculation.

**Theorem 4.** The compound stochastic process
\[
(z^\varepsilon(t); \theta^\varepsilon(t)) = \left( z^\varepsilon \left( \frac{t}{\varepsilon} \right); \theta^\varepsilon \left( \frac{t}{\varepsilon} \right) \right)
\]
converges weakly in \(C\) as \(\varepsilon \to 0\) to the same diffusion process \((z(t); \theta(t))\) as in Theorem 3 for which
\[
\hat{\alpha}_k(z, \theta) = \hat{A} (\beta_k; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m),
\]
\[
\hat{\beta}_i(z, \theta) = \hat{A} (\delta_i; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m),
\]
\[
\hat{G}^{\varepsilon z}_{ki}(z, \theta) = \hat{A} \left( \sum_j \alpha_{kj} \gamma_{ij}; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m \right),
\]
\[
\hat{G}^{\varepsilon \theta}_{ki}(z, \theta) = \hat{G}^{\theta z}_{ki}(z, \theta) = \hat{A} \left( \sum_j \alpha_{kj} \gamma_{ij}; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m \right),
\]
\[
\hat{G}^{\theta \theta}_{ij}(z, \theta) = \hat{A} \left( \sum_k \gamma_{ik} z_1, \ldots, z_m, \theta_2, \ldots, \theta_m \right).
\]

The proof of the theorem follows from the Itô's formula and the Theorem 1.


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