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ON THE MAXIMUM-MINIMUM PRINCIPLE FOR ADVECTION-DIFFUSION EQUATIONS

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The air pollution transport model is generally solved with the so-called operator splitting technique. The original problem is split into several subproblems and the solution of the model is obtained by solving the subproblems cyclically. In this paper, we analyze the advection, diffusion and emission subproblems. These subproblems have to possess certain qualitative properties that follow from physical considerations: nonnegativity preservation, maximum-minimum principle and maximum norm contractivity. We show that these properties are valid for the subproblems, and we shad light on their relations.

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Introduction

Nowadays, more and more stress is put on environment protection. In order to understand how air pollutants or radioactive dust-clouds move in the air, or how unhealthy materials seep into the ground, we generally set up mathematical models based on physical or chemical considerations. The solutions of these models help us to intervene in harmful processes. One of these models is the air pollution transport model ([Zlatev, 1995],[Csomós, 2006])

$$\frac{\partial v_l}{\partial t} = -\nabla(\mathbf{u}v_l) + \nabla(K\nabla v_l) + R_l + E - \sigma v_l \quad (l = 1,\ldots,r), \tag{1}$$

which, after prescribing the initial and boundary conditions, forecasts the concentration of the air pollutants as a function of time *t*. Here the unknown function $v_l = v_l(\mathbf{x},t)$ is the concentration of the *l*th pollutant, the function $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ describes the wind velocity, $K = K(\mathbf{x},t)$ is the diffusion coefficient, $R_l = R_l(\mathbf{x},t,\mathbf{v})$ describes the chemical reactions between the investigated pollutants, $E = E(\mathbf{x},t)$ is the emission function and $\sigma = \sigma(\mathbf{x},t)$ describes the deposition. Because of its complexity, system (1) is generally solved applying the so-called operator splitting technique. The system is split into several subproblems according to the physical and chemical processes involved in the model: advection, diffusion, chemical reaction, emission and deposition. These subproblems are solved cyclically with some appropriate methods. Then, the solution of the model can be obtained using the solutions of the subproblems. Naturally, the properties of the solution of the air pollution model are determined by the properties of the methods that are applied for the subproblems. In this paper three remarkable qualitative properties – the nonnegativity preservation, the maximum-minimum principle and the maximum norm contractivity – will be defined and investigated for certain subproblems of (1).

Let Ω and $\partial\Omega$ denote, respectively, a bounded domain in $\mathbb{R}^d$ $(d \in \mathbb{N}^+)$ and its boundary and we introduce the sets

$$Q_\tau = \Omega\times(0,\tau), \qquad Q_{\bar\tau} = \Omega\times(0,\tau], \qquad \Gamma_\tau = (\partial\Omega\times[0,\tau])\cup(\Omega\times\{0\})$$

for any arbitrary positive number τ. For some fixed number $T > 0$, we consider the initial-boundary value problem

$$\frac{\partial v}{\partial t} - \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial v}{\partial x_i} = f \text{ in } Q_T, \tag{2}$$

$$v|_{\Gamma_T} = g, \tag{3}$$

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where $g:\Gamma_T \to I\!R$ is a given continuous function and $f:Q_T \to I\!R$ is bounded in Q_T. The linear partial differential operators in (2) have bounded coefficient functions defined in Q_T. Moreover, the coefficient functions a_{ij} fulfill the property $a_{ij} = a_{ji}$ and the inequality

$$\sum_{i,j=1}^{d} a_{ij} x_i x_j > 0$$

is valid for all vectors $\mathbf{0} \neq \mathbf{x} \in I\!R^d$. We say that a continuous function $v:\overline{Q_T} \to I\!R$ solves the problem (2)-(3) if its derivatives in (2) are bounded, and *v* satisfies the equality (2) and the condition (3).

Remark 1.1 *If* $\partial u_i / \partial x_i = 0 \quad (i = 1,\dots,d)$, *then the problem (2)-(3) involves the advection, diffusion and emission subproblems of system (1). The function v plays the role of the concentration of one of the pollutants. We do not investigate the chemical reaction subproblem, which are generally described by nonlinear functions, and the deposition subproblem.*

Under the natural assumption that the initial and boundary conditions for the concentration are nonnegative, the concentration must be nonnegative in any point and at any time instant. This property is called nonnegativity preservation and it must hold for the solution of the system (2)-(3) too. The nonnegativity preservation property is a direct corollary of another property: the maximum-minimum principle, which says that – under certain conditions – the solution of (2)-(3) can be estimated from below and from above by the values of the functions *g* and *f*. As a special case, it follows from Fick's laws that if there is no emission source present in the computational space, then the concentration takes its maximum and minimum values in the initial state or on the boundary. The maximum norm contractivity property holds when for arbitrary two initial functions the maximum norm of the difference of the solutions at every time level is not greater than the maximum norm of the difference of the initial functions.

In paper [Faragó & Horváth, 2006], we considered the problem

$$\frac{\partial v}{\partial t} - \nabla(K\nabla v) = f, \qquad v|_{\Gamma_T} = g$$

and we showed that the validity of the maximum-minimum principle is a sufficient condition of the maximum norm contractivity and it is equivalent to the nonnegativity preservation property. In this paper, we will prove the above statement for the more general problem (2)-(3).

For more details regarding maximum principles consult [Protter and Weinberger, 1967]. For Readers who are interested or involved in scientific computations we remark, that the subproblems of (1) are generally solved numerically. It is a natural requirement of an adequate numerical method for the air pollution transport model that it has to possess the discrete equivalents of the qualitative properties listed in the previous paragraph. The discrete maximum principle is generally guaranteed by some geometrical conditions for the meshes ([Borisov & Sorek, 2004], [Faragó at al, 2005], [Faragó & Horváth, 2006], [Fujii, 1973]). The conditions of the discrete nonnegativity preservation was discussed e.g. in [Faragó & Horváth, 2001]. The discrete maximum norm contractivity was analyzed for one-dimensional parabolic problems in [Horváth, 1999] and in [Kraaijevanger, 1992].

1 Maximum-Minimum Principle and the Nonnegativity Preservation

In this section, we will define the maximum-minimum principle and the nonnegativity preservation property and we show their validity for the problem (2)-(3). We show the equivalence of the two properties.

DEFINITION 2.1. We say that the problem (2)-(3) satisfies the maximum-minimum principle if for any fixed functions *g* and *f* the solution *v* satisfies the inequality

$$\min_{\mathbf{\Gamma}_{t_1}} g + t_1 \cdot \min\left\{0, \inf_{Q_{t_1}} f\right\} \le v(\mathbf{x}, t_1) \le \max_{\mathbf{\Gamma}_{t_1}} g + t_1 \cdot \max\left\{0, \sup_{Q_{t_1}} f\right\}$$

$$\text{for all } x \in \Omega,\ 0 < t_1 < T.$$

The maximum-minimum principle guarantees the uniqueness of the solution of problem (2)-(3). We consider the function $\bar{v} = v^* - v^{**}$ with two different solutions v^* and v^{**}. The function $\bar{v}$ is a solution of the problem (2)-(3) with the choice *f=0* and *g=0*. Thus, based on the maximum-minimum principle we have $\bar{v}(\mathbf{x}, t_1) = 0$ for all $x \in \Omega$ and $t_1 \in (0,T)$. This implies the uniqueness of the solution.

DEFINITION 2.2. The problem (2)-(3) is called nonnegativity preserving if for any fixed functions *g* and *f* with $g|_{\Gamma_{t_1}} \geq 0$ and $f|_{Q_{\bar{t}_1}} \geq 0$ $(0 < t_1 < T)$ the solution *v* is nonnegative in $Q_{\bar{t}_1}$.

Theorem 2.3 The problem (2)-(3) satisfies the maximum-minimum principle if and only if it preserves the nonnegativity.

PROOF. The necessity of the condition is trivial. To show the sufficiency, let us fix the functions *g* and *f*. Then, we define the function

$$\overline{v} = v - \min_{\mathbf{\Gamma}_{t_1}} g - t \cdot \min\{0, \inf_{Q_{\bar{t}_1}} f\}$$

with the solution *v*. Clearly, $\overline{v}$ is a solution of the problem

$$\frac{\partial \overline{v}}{\partial t} - \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 \overline{v}}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial \overline{v}}{\partial x_i} = f - \min\{0, \inf_{Q_{\bar{t}_1}} f\},$$

$$\overline{v}|_{\mathbf{\Gamma}_T} = g - \min_{\mathbf{\Gamma}_{t_1}} g - t \cdot \min\{0, \inf_{Q_{\bar{t}_1}} f\} .$$

Naturally,

$$(f - \min\{0, \inf_{Q_{\bar{t}_1}} f\})|_{Q_{\bar{t}_1}} \geq 0$$

and

$$(g - \min_{\mathbf{\Gamma}_{t_1}} g - t \cdot \min\{0, \inf_{Q_{\bar{t}_1}} f\})|_{\mathbf{\Gamma}_{t_1}} \geq 0 ,$$

and these relations imply that $\overline{v}$ is nonnegative on $Q_{\bar{t}_1}$ by virtue of the nonnegativity preservation assumption. Thus the lower estimation

$$\min_{\mathbf{\Gamma}_{t_1}} g + t_1 \cdot \min\left\{0, \inf_{Q_{\bar{t}_1}} f\right\} \leq v(\mathbf{x}, t_1)$$

is satisfied. By choosing

$$\overline{v} = \max_{\mathbf{\Gamma}_{t_1}} g - v + t \cdot \max\{0, \sup_{Q_{\bar{t}_1}} f\} ,$$

the upper bound is proved similarly. This completes the proof.

Theorem 2.4 Let *g* and *f* be two fixed functions. Then, the solution *v* of the problem (2)-(3) satisfies the inequality

$$\sup_{\lambda>0}\left(e^{\lambda t_1} \min\left\{ \min_{\mathbf{\Gamma}_{t_1}} g e^{-\lambda t}, \frac{1}{\lambda} \inf_{Q_{\bar{t}_1}} f e^{-\lambda t} \right\}\right) \leq$$

$$\leq v(\mathbf{x}, t) \leq \inf_{\lambda>0}\left(e^{\lambda t_1} \max\left\{ \max_{\mathbf{\Gamma}_{t_1}} g e^{-\lambda t}, \frac{1}{\lambda} \sup_{Q_{\bar{t}_1}} f e^{-\lambda t} \right\}\right) \tag{4}$$

for any $t_1 \in (0, T)$ and $\mathbf{x} \in \mathbf{\Omega}$.

PROOF. For any arbitrary number $\lambda > 0$ we define the function $\hat{v}(\mathbf{x}, t) = v(\mathbf{x}, t) e^{-\lambda t}$. It can be seen easily that $\hat{v}$ is a solution of the problem

$$\frac{\partial \hat{v}}{\partial t} - \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 \hat{v}}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial \hat{v}}{\partial x_i} + \hat{v} \lambda e^{\lambda t} = f , \tag{5}$$

$$\hat{v}|_{\boldsymbol{\Gamma}_T} = ge^{-\lambda t}.$$

As $\hat{v}$ is continuous on $\overline{Q}_{t_1}$, it takes its maximum either on the boundary $\boldsymbol{\Gamma}_{t_1}$ or in $Q_{\bar{t}_1}$ at some point $(\mathbf{x}^0, t^0)$. In the first case we trivially have

$$\sup_{Q_{t_1}} \hat{v} \le \max_{\boldsymbol{\Gamma}_{t_1}} ge^{-\lambda t}. \tag{6}$$

In the second case, the relations

$$\sup_{Q_{t_1}} \hat{v} \le \hat{v}(\mathbf{x}^0, t^0), \ \frac{\partial \hat{v}}{\partial t}(\mathbf{x}^0, t^0) \ge 0, \ \frac{\partial \hat{v}}{\partial x_i}(\mathbf{x}^0, t^0) = 0 \quad (i = 1, \dots, d),$$

$$-\sum_{i,j=1}^{d} a_{ij}(\mathbf{x}^0, t^0) \frac{\partial^2 \hat{v}}{\partial x_i \partial x_j}(\mathbf{x}^0, t^0) \ge 0$$

hold. The last two relations and equation (5) imply that

$$f(\mathbf{x}^0, t^0) - \hat{v}(\mathbf{x}^0, t^0)\lambda e^{\lambda t^0} \ge 0,$$

that is

$$\hat{v}(\mathbf{x}^0, t^0) \le \frac{f(\mathbf{x}^0, t^0)e^{-\lambda t^0}}{\lambda} \le \sup_{Q_{\bar{t}_1}} \frac{fe^{-\lambda t}}{\lambda}. \tag{7}$$

Thus, in general case, using the upper bounds (6) and (7) we obtain the estimation

$$\hat{v}(\mathbf{x}, t_1) \le \sup_{Q_{t_1}} \hat{v} \le \max\left\{ \max_{\boldsymbol{\Gamma}_{t_1}} ge^{-\lambda t}, \sup_{Q_{t_1}} \frac{fe^{-\lambda t}}{\lambda} \right\}.$$

Multiplying both sides by $e^{\lambda t_1}$ and taking into account that the relation is true for all positive numbers $\lambda > 0$, we obtain the inequality on the right-hand side of (4). The lower bound can be proved similarly.

Theorem 2.5. The problem (2)-(3) satisfies the maximum-minimum principle and the nonnegativity preservation property.

PROOF. Because of Theorem 2.3, it is enough to show that the problem (2)-(3) preserves the nonnegativity. Let $t_1 \in (0, T)$ be an arbitrary number and *f* and *g* two fixed functions with the properties $f|_{Q_{\bar{t}_1}} \ge 0$ and $g|_{\boldsymbol{\Gamma}_{t_1}} \ge 0$. Then, for any $t_0 \in (0, t_1]$, we have $f|_{Q_{t_0}} \ge 0$ and $g|_{\boldsymbol{\Gamma}_{t_0}} \ge 0$, which result in $0 \le v(\mathbf{x}, t_0)$ in view of (4). That is *v* is nonnegative in $Q_{\bar{t}_1}$.

2. Maximum Norm Contractivity

In this section, we define the maximum norm contractivity property and prove that the problem (2)-(3) possesses this property.

DEFINITION 3.1. The problem (2)-(3) is called contractive in maximum norm when for all arbitrary three functions *f*, $g = g^*$ and $g = g^{**}$ with the property $(g^* - g^{**})|_{\partial\boldsymbol{\Omega}\times[0,t_1]} = 0$ $(0 < t_1 < T)$ the solutions v^* and v^{**} of the problem (2)-(3) satisfy the relation

$$\max_{\mathbf{x}\in\boldsymbol{\Omega}} |v^*(\mathbf{x}, t_1) - v^{**}(\mathbf{x}, t_1)| \le \max_{\mathbf{x}\in\boldsymbol{\Omega}} |g^*(\mathbf{x}, 0) - g^{**}(\mathbf{x}, 0)|.$$

Theorem 3.2 The problem (2)-(3) is contractive in maximum norm.

PROOF. Let *f*, $g = g^*$ and $g = g^{**}$ be three arbitrary functions with the property $(g^* - g^{**})|_{\partial\boldsymbol{\Omega}\times[0,t_1]} = 0$. Let v^* and v^{**} the solutions of the problem (2)-(3). We consider the functions $v_\pm = \zeta \pm v^* - v^{**}$ with $\zeta = \max_{\mathbf{x}\in\boldsymbol{\Omega}} |g^*(\mathbf{x}, 0) - g^{**}(\mathbf{x}, 0)|$. These functions solve the problem

$$\frac{\partial v_\pm}{\partial t} - \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 v_\pm}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial v_\pm}{\partial x_i} = 0, \tag{5}$$

$$(v_\pm)|_{\Gamma_T} = \zeta \pm (g^* - g^{**}).$$

It is easy to see that $(\zeta \pm (g^* - g^{**}))|_{\Gamma_{t_1}} \geq 0$. Due to Theorem 2.5, the problem (2)-(3) preserves the nonnegativity. That is, $v_\pm = \zeta \pm v^* - v^{**}$ is nonnegative in Q_{t_1}, thus we have

$$\max_{\mathbf{x} \in \mathbf{\Omega}} | v^*(\mathbf{x}, t_1) - v^{**}(\mathbf{x}, t_1) | \leq \max_{\mathbf{x} \in \mathbf{\Omega}} | g^*(\mathbf{x}, 0) - g^{**}(\mathbf{x}, 0) | .$$

This completes the proof.

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