UDC 517.9

S. Albeverio (Inst. Angew. Math. Univ. Bonn, Germany),
L. Nizhnik (Inst. Math. Nat. Acad. Sci. Ukraine, Kiev)

APPROXIMATION OF GENERAL ZERO-RANGE POTENTIALS

АПРОКСИМАЦІЯ ЗАГАЛЬНИХ ПОТЕНЦІАЛІВ НУЛЬОВОГО РАДІУСА

A norm resolvent convergence result is proved for approximations of general Schrödinger operators with zero-range potentials. An approximation of the δ'-interaction by non-local non-Hermitian potentials (without a renormalization of the coupling constant) is also constructed.

Наведено результати про апроксимацію загальних операторів Шредінгера з потенціалом нульового радіуса в сенсі резольвентної збіжності за нормою. Побудовано апроксимацію δ'-взаємодії за допомогою нелокальних неермітових потенціалів (без перенормування константи взаємодії).

1. Introduction. Zero-range potentials play an important role in solvable models of quantum mechanics [1]. Schrödinger operators with zero-range potentials are self-adjoint extensions of the free operator $-\Delta$ defined on the set $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. In the one-dimensional case, extensions form a four parameter family of operators $[1-5]$. The case of a δ-potential has been well studied. The case of δ'-potentials has also been considered in some papers [1, 6, 7]. In particular, there were studies on approximation (in the strong resp. norm resolvent sense) of Schrödinger operators with a zero-range potential using regular potentials [1, 4, 6, 8, 9]. In [10] new results were obtained concerning the possibility of approximating a δ'-interaction (and a general zero-range interaction) with a triple δ-functions that have appropriate strengths and approach each other. In this article we give a more general construction of approximations of δ'-interaction and general zero-range interaction.

2. Zero-range potentials. A one-dimensional Schrödinger operator corresponding to a zero-range potential at the point $x = 0$ can be given by the following expression:

$$\mathcal{L}_{\alpha,\beta,\gamma}u(x) = -\frac{d^2u}{dx^2} + \alpha\delta(x)u_{\text{reg}}(0) + \beta\delta'(x)u'_{\text{reg}}(0) + \\ + \gamma\delta(x)u'_{\text{reg}}(0) - \overline{\gamma}\delta'(x)u_{\text{reg}}(0). \tag{1}$$

Here $\alpha = \overline{\alpha}$, $\beta = \overline{\beta}$, and γ are given numbers, $\delta(x)$ and $\delta'(x)$ are the Dirac δ-function and its derivative (with support at 0), $u_{\text{reg}}(0) = \frac{1}{2}[u(+0) + u(-0)]$, $u'_{\text{reg}}(0) = \frac{1}{2}[u'(+0) + u'(-0)]$ are regularized values of the function $u(x)$ and its derivative at the point $x = 0$, the operator $\frac{d^2}{dx^2}$ is understood in the distribution sense.

Expression (1) defines an operator with the domain

$$W_2^2(-\infty, 0) \oplus W_2^2(0, +\infty) \subset L_2(-\infty, +\infty).$$

The values of $\mathcal{L}_{\alpha,\beta,\gamma}u$ belong to the space H_- which is the direct sum of the space L_2 and a two-dimensional space containing $\delta(x)$ and $\delta'(x)$.

Expression (1) defines a self-adjoint operator $\mathcal{L}_{\alpha,\beta,\gamma}$ in the space L_2 with the domain consisting of all u such that $\mathcal{L}_{\alpha,\beta,\gamma}u \in L_2$. The operator $\mathcal{L}_{\alpha,\beta,\gamma}$ acts on these functions as pure differentiation for $x \neq 0$, $\mathcal{L}_{\alpha,\beta,\gamma}u(x) = -u''(x)$.

Note that the domain of the self-adjoint operator $\mathcal{L}_{\alpha,\beta,\gamma}$ can be described in terms of boundary conditions at the point $x = 0$. These conditions can be obtained from (1) by using rules for generalized differentiation of piecewise discontinuous functions,

$$\begin{aligned} u(+0) - u(-0) &= \beta u'_{\mathrm{reg}}(0) - \bar{\gamma} u_{\mathrm{reg}}(0), \\ u'(+0) - u'(-0) &= \alpha u'_{\mathrm{reg}}(0) + \gamma u'_{\mathrm{reg}}(0). \end{aligned} \tag{2}$$

These conditions can also be written as

$$\begin{pmatrix} 1+\dfrac{\bar{\gamma}}{2} & -\dfrac{\beta}{2} \\ -\dfrac{\alpha}{2} & 1-\dfrac{\gamma}{2} \end{pmatrix} \begin{pmatrix} u(+0) \\ u'(+0) \end{pmatrix} = \begin{pmatrix} 1-\dfrac{\bar{\gamma}}{2} & \dfrac{\beta}{2} \\ \dfrac{\alpha}{2} & 1+\dfrac{\gamma}{2} \end{pmatrix} \begin{pmatrix} u(-0) \\ u'(-0) \end{pmatrix}. \tag{3}$$

We remark that (3) is the general form of self-adjoint conditions at the point 0 for the operator $-\dfrac{d^2}{dx^2}$ in the space $L_2(-\infty, +\infty)$.

In the particular case where $\beta = \gamma = 0$, the boundary conditions (3) are reduced to the form

$$\begin{aligned} u(+0) &= u(-0) = u_{\mathrm{reg}}(0), \\ u'(+0) - u'(-0) &= \alpha u_{\mathrm{reg}}(0), \end{aligned} \tag{4}$$

and correspond to a δ-interaction at the point $x = 0$ with coupling constant α.

In the case where $a = \gamma = 0$, the boundary conditions (3) become

$$\begin{aligned} u'(+0) &= u'(-0) = u'_{\mathrm{reg}}(0), \\ u(+0) - u(-0) &= \beta u'_{\mathrm{reg}}(0), \end{aligned} \tag{5}$$

and correspond to a δ'-interaction at the point $x = 0$ with coupling constant („strength") β.

Together with the operator $\mathcal{L}_{\alpha,\beta,\gamma}$ with zero-range potential, we consider a sequence of Schrödinger operators $-\dfrac{d^2}{dx^2} + V_n(x)$ with usual absolutely integrable potentials $V_n(x)$ having compact support such that the supports shrink to the point $x = 0$.

Definition 1. *We say that a sequence of regular potentials $V_n(x)$ approximates a zero-range potential with characteristics (α, β, γ) if $-\dfrac{d^2}{dx^2} + V_n(x) \to \mathcal{L}_{\alpha,\beta,\gamma}$ in the norm resolvent sense, i. e.*

$$\lim_{n\to\infty} \left\| \left(-\frac{d^2}{dx^2} + V_n - k^2 \right)^{-1} - \left(\mathcal{L}_{\alpha,\beta,\gamma} - k^2 \right)^{-1} \right\| = 0 \tag{6}$$

(where $\|\cdot\|$ is the operator norm in $L_2(-\infty, \infty)$).

Let $V_n(x)$ be a sequence which approximates a (α, β, γ)-zero range potential. Then for any $u \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\gamma})$ there exists a sequence $u_n \xrightarrow{L_2} u$ such that

$$-\frac{d^2}{dx^2}u_n \xrightarrow{W_2^{-2}} -\frac{d^2}{dx^2}u,$$

$$V_n u_n \xrightarrow{W_2^{-2}} \alpha\delta(x)u_r(0) + \beta\delta'(x)u_r'(0) + \gamma\delta(x)u_r'(0) - \bar{\gamma}\delta'(x)u_r(0), \tag{7}$$

$$\left(-\frac{d^2}{dx^2} + V_n\right)u_n = \mathcal{L}_{\alpha,\beta,\gamma}u = -u''(x) \quad (x \neq 0).$$

It is well known that if $V_n(x) \xrightarrow{W_2^{-1}} \alpha\delta(x)$ for $n \to \infty$, then the sequence $V_n(x)$ approximates a δ-potential with coupling constant α. In particular, if $\varphi(x) \in C_0^\infty$ and $\int \varphi(x)\,dx = 1$, then we can take $V_n(x)$ to be $\alpha\, n\varphi(nx)$.

To construct a sequence of regular potentials that approximates a δ'-potential, let us consider some auxiliary notions.

3. Oscillatory potentials.

Definition 2. *We say that a locally integrable function $V(x)$ is purely oscillatory on an interval $[a, b]$ if the Cauchy data for solutions of the equation*

$$-y''(x) + V(x)y(x) = 0 \tag{8}$$

on the interval $[a, b]$ satisfy the coinditions

$$\begin{pmatrix} y(b) \\ y'(b) \end{pmatrix} = \begin{pmatrix} 1 & b-a+\omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y(a) \\ y'(a) \end{pmatrix}. \tag{9}$$

The number $w = w[V;[a,b]]$ is called *oscillatory characteristic* of the function $V(x)$ on the interval $[a, b]$.

***Example* 1.** Consider a three step function

$$V_{\text{th}} = \begin{cases} V_1, & |x| \le a, \\ V_2, & a+l \le |x| \le b, \\ 0, & \text{other } x. \end{cases} \tag{10}$$

If $V_1 > 0$ and $V_2 < 0$, the condition for the function V_{th} to be purely oscillatory is

$$|V_2|^{1/2}\tan k_2 - |V_1|^{1/2}\tanh k_1 + l|V_1V_2|^{1/2}\tanh k_1 \tan k_2 = 0, \tag{11}$$

where $k_1 = |V_1|^{1/2}a$, $k_2 = |V_2|^{1/2}(b-a-l)$.

Here the oscillatory characteristic of the function V_{th} is determined from the formula

$$w[V_{\text{th}}] = |V_1|^{1/2}|V_2|^{-1/2}\frac{\sinh 2k_1}{\tan k_2}\left(|V_1|^{-1/2}\tanh k_1 + |V_2|^{-1/2}\tan k_2 + l\right) - 2b. \tag{12}$$

We can pass to the case of $V_1 < 0$ and $V_2 > 0$ by replacing $\tanh k_1$ by $-\tan k_1$ in formula (11) and hyperbolic functions by corresponding trigonometric ones in formula (12).

Property 1. If a purely oscillatory function $V(x)$ on an interval $[a, b]$ is extended with the zero value outside of $[a, b]$, then the resulting function $\tilde{V}(x)$ is purely oscillatory on any interval $[a', b'] \supset [a, b]$, and the oscillatory characteristic is preserved,

$$w\left[\tilde{V};[a',b']\right] = w[V;[a,b]].$$

This identity allows to regard purely oscillatory functions as compactly supported functions defined on the whole axis. The oscillatory characteristic of a function with compact support will be denoted by $w[V]$. Regardless the fact that purely oscillatory functions do not form a linear space, some of them can still be added. More precisely, we have the following Properties, which are easily proved.

Property 2. If the supports of two compactly supported purely oscillatory functions V_1 and V_2 belong to nonintersecting intervals, the sum $V_1 + V_2$ is a purely oscillatory function, and

$$w[V_1 + V_2] = w[V_1] + w[V_2].$$

Property 3. If $V(x)$ is a purely oscillatory function on intervals $[a, b]$ and $[a, c]$, where $b < c$, then $V(x)$ is a purely oscillatory function on the interval $[b, c]$, and

$$w[V;[b,c]] = w[V;[a,c] - w[V;[a,b]].$$

Definition 3. *Let $V(x)$ be a purely oscillatory function with compact support. We say that the function $V(x)$ admits a purely oscillatory restriction to the interval $[a, b]$, if $V(x)$ is purely oscillatory on the intervals $(-\infty, a)$, $(b, +\infty)$, and, hence, on the interval $[a, b]$.*

Since the oscillatory characteristic is an additive function of intervals on which the function admits a purely oscillatory restriction, one can introduce a local density of oscillatory characteristic.

Definition 4. *Let $V(x)$ be a purely oscillatory function with compact support. The function $\omega(x) = \omega(x; V)$ is called density of the oscillatory characteristic w of the function $V(x)$, if for any interval $[a, b]$ on which $V(x)$ admits a purely oscillatory restriction,*

$$w[V;[a,b]] = \int_a^b \omega(x;V)\,dx. \tag{13}$$

Lemma 1. *Let $V(x)$ be an absolutely integrable purely oscillatory function with compact support. Then its oscillatory characteristic admits a density $\omega(x; V)$.*

Proof. Consider a minimal algebra S of subsets containing all intervals $[a, b)$ on which $V(x)$ admits an oscillatory restriction. The values of the oscillatory characteristic $w[V;[a,b]]$ on intervals $[a,b) \in S$ define an additive set function on S. Since for small intervals one has $|w(V;[a,b])| \le c(b-a)^2$, this function is absolutely continuous with respect to the Lebesgue measure. By the Radon – Nikodym theorem, there exists a unique function $\omega(x) \in L_1(-\infty, +\infty; S; dx)$ such that (13) holds, i.e. $\omega(x)$ is the density of the oscillatory characteristic, $\omega(x) = \omega(x; V)$.

Very important is how the oscillatory characteristic depends on the scaling properties of the function.

Lemma 2. *Let $V(x)$ be a purely oscillatory function with compact support, and let $\omega(x; V)$ be the density of the oscillatory characteristic. Then the function $V_\varepsilon(x) = \varepsilon^{-2}V(x/\varepsilon)$ is also a purely oscillatory function, and*

$$\omega(x; V_\varepsilon) = \omega(x/\varepsilon, V). \tag{14}$$

Proof. If x is replaced with x/ε in equation (8), then we get

$$-y''\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon^2}V\left(\frac{x}{\varepsilon}\right)y\left(\frac{x}{\varepsilon}\right) = 0.$$

Hence, if $V(x)$ is purely oscillatory on the interval $[a, b]$, then $V_\varepsilon(x) = \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right)$ is purely oscillatory on the interval $[a\varepsilon, b\varepsilon]$. We also have that $w[V_\varepsilon;[a\varepsilon, b\varepsilon]] = \varepsilon\, w[V;[a, b]]$. Formula (14) now follows from (13) as seen from the change of variables in the integral,

$$w[V_\varepsilon;[a\varepsilon, b\varepsilon]] = \varepsilon w[V;[a,b]] = \varepsilon \int_a^b \omega(x, V)\,dx = \int_{a\varepsilon}^{b\varepsilon} \omega\left(\frac{x}{\varepsilon}; V\right) dx.$$

4. Approximation of δ'-interaction. We will be considering a Schrödinger operator $\mathcal{L} = -\frac{d^2}{dx^2} + V_n(x)$, where the real valued regular potential V has a sufficiently small support, $\operatorname{supp} V \subset [-\varepsilon, \varepsilon]$, $\varepsilon > 0$. Suppose that the relations for the Cauchy data are defined for the equation $\left[-\frac{d^2}{dx^2} + V_n(x)\right] y = 0$ at the points $x = -\varepsilon$ and $x = \varepsilon$ by a matrix $M = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$:

$$\begin{pmatrix} y(+\varepsilon) \\ y'(+\varepsilon) \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} y(-\varepsilon) \\ y'(-\varepsilon) \end{pmatrix}. \tag{15}$$

One can expect that, for small ε, the resolvent of the operator $\mathcal{L}$, will be close in norm to the resolvent of a Schrödinger operator with point interaction at the point $x = 0$ given by the boundary conditions at the point $x = 0$ in terms of the matrix M:

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix}. \tag{16}$$

In particular, one can expect that a sequence of purely oscillatory potentials $V_n(x)$ such that the corresponding sequence of densities of oscillatory characteristics $\omega(x, V_n(x))$ converges to the Dirac δ-function as $n \to \infty$ will approximate a δ'-interaction. To formulate and prove a precise statement, we will need the following notion.

Definition 5. *We will say that a family $\mathfrak{R}$ of locally integrable potentials $V(x)$ is uniformly regular if there exists a constant C such that, for any solution of the equation $-y'' + V(x)y - k^2 y = 0$ $\left(\left|k^2\right| \le 1\right)$ with a potential $V \in \mathfrak{R}$, the following estimate holds*:

$$\frac{1}{b-a} \int_a^b |y(x)|^2\,dx \le C\left[|y(a)|^2 + |y'(a)|^2 + |y(b)|^2 + |y'(b)|^2\right]. \tag{17}$$

The potentials considered in Example 1 for different values of the parameters V_1 and V_2 form uniformly regular families of potentials. This can be easily seen, since the solution y can be written in an explicit way. Note that these families contain purely oscillating potentials with arbitrarily large values of oscillatory characteristic.

Let us now formulate the main result.

Theorem 1. *Let a uniformly regular family of purely oscillatory potentials $V\left(x; \frac{\beta}{\varepsilon}\right)$, $\varepsilon > 0$, be given, and assume that th supports of the potentials lie in the* *finite interval* $[-1, 1]$ *and that the oscillatory characteristic eguals* $\frac{\beta}{\varepsilon}$, *where* $\beta = \bar{\beta}$ *is a fixed number,*

$$w\left[V\left(x; \frac{\beta}{\varepsilon}\right)\right] = \frac{\beta}{\varepsilon}.$$

Then the sequence of scaled potentials $V_\varepsilon(x) = \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}; \frac{\beta}{\varepsilon}\right)$ *approximates a* δ'-*interaction at the point* $x = 0$ *with coupling constant* β *as* $\varepsilon \to 0$.

Proof. The resolvent of the operator $\mathcal{L}_\varepsilon = -\frac{d^2}{dx^2} + V_\varepsilon(x)$ and the resolvent of the operator $\mathcal{L}_{0,\beta,0}$ with δ'-interaction that has coupling constant β are integral operators, the kernels of which are the corresponding Green's functions, which can be given in the form

$$G(x, y: k) = \frac{1}{W}\begin{cases} y_1(x)y_2(x), & x < y, \\ y_1(y)y_2(x), & x > y. \end{cases} \tag{18}$$

Here $y_1(x)$ and $y_2(x)$ are independent solutions of the equation

$$-y_i'' + V_\varepsilon(x)y_i - k^2 y_i = 0$$

in the case of the operator $\mathcal{L}_\varepsilon$, and solutions of the equation $-\tilde{y}_i'' - k^2\tilde{y}_i = 0$ subject to boundary conditions (5) at the point $x = 0$ in the case of a Schrödinger operator with δ'-interaction. The number W in (18) is the Wronskian of the corresponding solutions. Without loss of generality, we can assume that $y_1(x) = \tilde{y}_1(x) = e^{-ikx}$ ($\Im k > 0$) for $x < -\varepsilon$, and $y_2(x) = \tilde{y}_1(x) = e^{ikx}$ for $x > \varepsilon$. By using the conditions of the theorem one can show that $y_1(x)$ and $\tilde{y}_2(x)$ sufficiently close for $x > \varepsilon$, the same is true for $y_2(x)$ and $\tilde{y}_2(x)$ for $x < -\varepsilon$. Moreover, $\int_{-\varepsilon}^{+\varepsilon} |y_i(x)|^2\, dx \le C\varepsilon$, $i = 1, 2$, for some constant $C > 0$. This leads to the estimate for the resolvents:

$$\left\|\left(\mathcal{L}_\varepsilon - k^2\right)^{-1} - (\mathcal{L}_{0,\beta,0} - k^2)^{-1}\right\| \le C_1\varepsilon.$$

The proof follows from this inequality by making ε approach 0.

Remark. The condition of local integrability for purely oscillatory potentials in Theorem 1 can be weakened: An important example is the potential in the form of the sum of three Dirac δ-functions,

$$V_{\mathcal{D},\varepsilon}(x) = \frac{\beta}{\varepsilon^2}\delta(x) - \frac{1}{\varepsilon}\left(1 + \frac{2\varepsilon}{\beta}\right)^{-1}(\delta(x+\varepsilon) + \delta(x-\varepsilon)), \tag{19}$$

considered in [10] and approximating a δ'-interaction (as $\varepsilon \to 0$).

The sequence of potentials (19) approximates a δ'-interaction at the point $x = 0$ in the sense of Definition 1 with the coupling constant β for $\varepsilon > 0$. It is easy to check this directly, since the resolvent of a one-dimensional Schrödinger operator with potential (19), as it is the case for the resolvent of the Schrodinger operator with a δ'-interaction, has a simple explicit form [1].

We also remark that potentials (19) satisfy condition (9) for a purely oscillatory function with $a < -\varepsilon$, $b > \varepsilon$.

5. Approximation of δ'-interactions with nonlocal potentials. Consider a free Schrödinger operator perturbed by a one-dimensional operator,

$$\mathcal{L}u(x) = -\frac{d^2u}{dx^2} + V_1(x)(u, V_2). \tag{20}$$

If the perturbation is self-adjoint, i. e. $V_2 = -\beta V_1$, $\beta = \overline{\beta}$ is a real number, then there exists a sequence $V_n(x) \to \delta'(x)$ and a sequence of real numbers $\beta_n \to 0$ such that the sequence of operators $\mathcal{L}_\nu = -\frac{d^2}{dx^2} - \beta_n V_n(\cdot; V_n)$ converges in the strong resolvent sense to a Schrödinger operator with δ'-interaction [1, 3].

If the one-dimensional perturbation is not self-adjoint, then we can get rid of the renormalization condition imposed on the coupling constant, $\beta_n \to 0$, in the same way as it has been done in [11] for a interaction in the three-dimensional space.

Theorem 2. *Let two sequences of regular potentials* $V_n^{(j)}(x) \xrightarrow{W_2^{-2}} \delta'(x)$ *for* $n \to \infty$, $j = 1, 2$, *and let* $\operatorname{supp} V_n^{(1)}$ *lie to the left of* $\operatorname{supp} V_n^{(2)}$. *Then the seguence of Schrödinger operators* $\mathcal{L}_n$,

$$\mathcal{L}_n u = -\frac{d^2u}{dx^2} - \beta V_n^{(1)}(x)\left(u, V_n^{(2)}\right) \tag{21}$$

converges in the strong resolvent sense to a Schrödinger operator with δ'-*interaction at the point* $x = 0$ *with density* β.

Proof. The resolvent of the operator $\mathcal{L}_n$ can be represented in the form

$$\left(\mathcal{L}_n - k^2\right)^{-1} f = \mathcal{R}_k f - \beta\left[1 - \beta\left(\mathcal{R}_k V_n^{(1)}, V_n^{(2)}\right)\right]^{-1} \mathcal{R}_k V_n^{(1)}\left(\mathcal{R}_k f, V_n^{(2)}\right), \tag{22}$$

where $\mathcal{R}_k = \left(-\frac{d^2}{dx^2} - k^2\right)^{-1}$ is the resolvent of the free operator which is an integral operator with the kernel $G(x, y; k) = \frac{i}{2k} e^{ik|x-s|}$, $\Im k > 0$.

Consider the expression

$$J_n = \left(\mathcal{R}_k V_n^{(1)}, V_n^{(2)}\right) = \frac{i}{2k} \iint e^{ik|x-y|} V_n^{(1)}(y) \overline{V_n^{(2)}}(x)\, dx\, dy. \tag{23}$$

Since $\operatorname{supp} V_n^{(1)} < \operatorname{supp} V_n^{(2)}$ and $V_n^{(j)}(x) \to \delta'(x)$, we have that

$$J_n = \frac{i}{2k} \iint e^{ik(x-y)} V_n^{(1)}(y) \overline{V_n^{(2)}}(x)\, dx\, dy =$$

$$= \frac{i}{2k} \int e^{ikx} \overline{V_n^{(2)}}(x)\, dx \int e^{-ky} V_n^{(1)}(y)\, dy \to \frac{ik}{2}.$$

Because $R_k V_n^{(j)} \xrightarrow{L_2} -\frac{1}{2} \operatorname{sign} x e^{ik|x|}$, we have, due to (22), that $\left(\mathcal{L}_n - k^2\right)^{-1}$ converges, relatively to the operator norm, to an integral operator with the kernel

$$G_\beta(x, y; k) = \frac{i}{2k} e^{ik|x-y|} - \frac{\beta}{1 - ik\beta/2} \frac{1}{4} \operatorname{sign} x \operatorname{sign} y\, e^{ik(|x|+|y|)}. \tag{24}$$

This kernel is a Green's function for the Schrödinger operator with δ'-interaction with coupling constant β [1].

Example 2. As an example of functions $V_n^{(j)}(x)$, $j = 1, 2$, satisfying the conditions of Theorem 2, we can take the following:

$$V_n^{(1)}(x) = \begin{cases} n^2, & -\frac{1}{n} < x < 0, \\ -n^2, & 0 \le x \le \frac{1}{n}, \\ 0, & \text{other } x. \end{cases}$$

$$V_n^{(2)}(x) = V_n^{(1)}\left(x - \frac{2}{n}\right).$$

In this case, J_n from condition (23) can be explicitly calculated,

$$J_n = \frac{in^4}{2k^3}\left(1 - e^{-ik/n}\right)^4 \to \frac{ik}{2}$$

as $n \to \infty$.

1. *Albeverio S., Gesztesy F., Hoegh Krohn R., and Holden H.* Solvable models in quantum mechanics.– New Vork: Springer, 1988. – 568 p.
2. *Albeverio S, Gesztesy F., Hoegh Krohn R., and Kirsch W.* On point interactions in one dimension // J. Operator Theory. – 1984. – **12**. – P. 101 – 126.
3. *Šeba P.* The generalized point interaction in one dimension // Czechoslovak J. Phys. – 1986. – **36**. – P. 667 – 673.
4. *Šeba* P. A remark about the point interaction in one dimension // Ann. Phys. – 1987. – **44**. – P. 323 – 328.
5. *Albeverio S., Brzezniak Z., and Dabrowski L.* Fundamental solution of the heat and Schrödinger equations with point interaction // J. Funct. Anal. – 1995. – **128**. – P. 220 – 254.
6. *Šeba P.* Some remarks on the δ'-interaction in one dimension // Rept. Math. Phys. – 1986. – **24**. – P. 111 – 120.
7. *Chernoff P. R. and Hughes R. J.* A new class on point interaction in one dimension // J. Funct. Anal. – 1993. – **111**. – P. 99 – 117.
8. *Albeverio S., Dabrowski L., and Kurasov P.* Symmetries of Schrödinger operators with point interaction // Lett. Math. Phys. – 1998. – **45**. – P. 33 – 47.
9. *Albeverio S. and Koshmanenko V.* Form-sum approximation of singular perturbation of approximation of self-adjoint operators. – BiBoS-Preprint, 1997, Nr 771/4/97.
10. *Cheon T. and Shigehara T.* Realizing discontinuous wave functions with renormalized short-range potentials // Phys. Lett. – 1998. – **243**. – P. 111 – 116.
11. *Nizhnik L. P.* On point interaction in quantum mechanics // Ukr. Math. J. – 1997. – **49**, № 11. – P. 1557 – 1560.

Received 29.03.2000