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RESEARCH ARTICLE

Weakly P-small not P-small subsets in Abelian groups

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Abstract. Answering a question of D. Dikranjan and I. Protasov we prove that each infinite Abelian group contains a weakly P-small subset that is not P-small.

We recall the basic definitions. A subset A of an Abelian group G is called

- *large* if there exists a finite subset $F \subset G$ such that $F + A = G$;
- *small* if for every finite subset $F \subset G$ the subset $G \setminus (F + A)$ is large;
- *small in sense of Prodanov* (*briefly, P-small*) if there exists an infinite subset B of G such that $(b + A) \cap (b' + A) = \emptyset$ for all distinct $b, b' \in B$;
- *weakly P-small* if for every natural number n there exists a subset $B_n \subset G$ of size $|B_n| = n$ such that $(b + A) \cap (b' + A) = \emptyset$ for all distinct $b, b' \in B_n$.

Obviously, each P-small subset is weakly P-small. By Theorem 4.2 of [3], every P-small subset of an Abelian group G is small. Looking at the proof of this theorem one can notice that it gives a little bit more, namely

Proposition 1. *Each weakly P-small subset of an Abelian group is small.*

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Therefore for a subset of an Abelian group we get implications

$$\text{(P-small)} \Rightarrow \text{(weakly P-small)} \Rightarrow \text{(small)}.$$

A small subset need not be weakly P-small. The simplest example is the set $A = \{a_n : n \in \omega\}$ of $\mathbb{Z}$ where $a_0 = 0$ and $a_{n+1} = a_n + n$ for $n \geq 1$, see Example 4.15 of [3]. In fact, each countable group contains a small subset which is not P-small [4].

In [1] D.Dikranjan and I.Protasov posed the following question: *Is each weakly P-small set P-small?* We answer this question in negative.

Main Theorem. *Each infinite Abelian group G contains a weakly P-small subset $A \subset G$ which is not P-small.*

Proof of the Main Theorem relies on the existence of a subset $B = -B$ of G with the following properties:

(1) for every $n \in \mathbb{N}$ there is a subset B_n of size $|B_n| = n$ such that $B_n - B_n \subset B$;

(2) $B_\infty - B_\infty \not\subset B$ for any infinite subset B_∞;

(3) $F + B \neq G$ for any subset $F \subset G$ of size $|F| < |G|$.

By $|A|$ we denote the cardinality of a set A.

Assuming for a moment that such a set B exists we shall construct a weak P-small set $A \subset G$ which is not P-small.

Let $B^\circ = B \setminus \{0\}$. We shall construct a subset A such that $(A + B^\circ) \bigcap A = \emptyset$ and $G \setminus B^\circ \subset A - A$, and then show that A satisfies the conclusion of the theorem.

Let $\kappa = |G \setminus B^\circ|$ and $G \setminus B^\circ = \{g_\alpha : \alpha < \kappa\}$ be an enumeration of $G \setminus B^\circ$ by ordinals $\alpha < \kappa$.

By induction, we define a sequence (a_α) in G such that for any ordinal $\alpha < \kappa$

$$a_\alpha \notin \bigcup_{\beta<\alpha} (a_\beta + B) \cup (a_\beta + g_\beta + B) \cup (a_\beta - g_\alpha + B) \cup (a_\beta + g_\beta - g_\alpha + B).$$

We start with $a_0 = 0$.

Assuming that for some α the points $a_\beta, \beta < \alpha$, have been constructed, pick any point $a_\alpha \in G$ with

$$a_\alpha \notin \bigcup_{\beta<\alpha} (a_\beta + B) \cup (a_\beta + g_\beta + B) \cup (a_\beta - g_\alpha + B) \cup (a_\beta + g_\beta - g_\alpha + B).$$

Such a point a_α exists because of the property (3) of the set B.

Finally, put $A = \bigcup_{\alpha<\kappa}\{a_\alpha, a_\alpha + g_\alpha\}$. It is easily seen that $G\backslash B^\circ \subset A - A$. From the definition of a_α it follows that $(A + B^\circ)\bigcap A = \emptyset$.

It remains to show that A is weakly P-small not P-small.

It follows from $(A + B^\circ) \cap A = \emptyset$ and property (2) of the set B that for every natural number n there exists a subset B_n of size n such that $(b - b^{'} + A) \cap A = \emptyset$ for all distinct $b, b^{'} \in B_n$. Thus the family $\{b + A : b \in B_n\}$ is disjoint. So A is weakly P-small.

Let us show that A is not P-small. According to the property (1) for every infinite B_∞ there are $b, b^{'} \in B_\infty$ such that $b - b^{'} \notin B$. Then $b - b^{'} \in G\backslash B^\circ \subset A - A$ and $b + A \bigcap b^{'} + A \neq \emptyset$ which completes the proof.

It remains to construct a subset B of G with properties (1)—(3).

For this we recall some facts from the theory of Abelian groups. From now on talking about groups we shall have in mind Abelian groups.

Obviously, if a group G contains an element g of infinite order, then there is a subgroup B of G which is isomorphic to the group of integer numbers $\mathbb{Z}$. Otherwise, if each element of G has finite order, then G is called a *periodic* group. By [2,Theorem 8.4], each periodic group G can be presented as the direct sum $G = \oplus_p A_p$ of p-groups A_p. Next, we note that if A_p contains an element of infinite height then there exists a subgroup B of A_p isomorphic to the quasicyclic group

$$\mathbb{Z}(p^\infty) = \{z \in \mathbb{C} : z^{p^k} = 1 \text{ for some } k \in \mathbb{N}\}.$$

Otherwise, when each element of a countable p-group A_p has finite height, then by [2,Theorem 17.3] A_p is the direct sum of cyclic groups

$$A_p = \oplus\langle g_i\rangle.$$

Hence we get that each group G contains a subgroup H isomorphic to $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ for some prime p or to the direct sum of cyclic groups $\oplus_{i\in\omega}\langle g_i\rangle$.

It is easy to see that each subset B with the properties (1)-(3) in a subgroup $H \subset G$ has these properties in the whole group G. So the problem reduces to constructing a set B in the groups $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ and $\oplus_i\langle g_i\rangle$. This will be done separately in the following three lemmas.

Lemma 1. *The group $\mathbb{Z}$ contains a subset B with the properties* (1)–(3).

Proof. We start with the definition of the subsets B_n for all n. We put $B_n = \{d_n \cdot i : 1 \leq i \leq n\}$ where $d_0 = 1$ and $d_n = 3nd_{n-1}$. Then the set

$$B = \bigcup_{n=1}^{\infty}(B_n - B_n)$$

satisfies condition (1).

Next, we show that for any infinite subset B_∞ of $\mathbb{Z}$ there are $b, b^{'} \in B_\infty$ such that $b - b^{'} \notin B$. Suppose, contrary to our claim, that there is an infinite subset B_∞ satisfying $B_\infty - B_\infty \subset B$. Let b_0 belong to B_∞ and $b_0 \neq 0$. Then

$$B_\infty - b_0 \subset B_\infty - B_\infty \subset B = \bigcup_{n=1}^{\infty}(B_n - B_n).$$

The infinity of the set $B_\infty - b_0$ and the finity of the sets $B_n - B_n$ ensure the existence of points $b, b^{'} \in B_\infty \backslash \{b_0\}$ such that $b - b_0 = b_n \in B_n - B_n; b^{'} - b_0 = b_m \in B_m - B_m$, where $n \neq m$. Hence $b - b^{'} = b_n - b_m$. And if $b_n - b_m \notin B$ then $b - b^{'} \notin B$. This will contradict the inclusion $B_\infty - B_\infty \subset B$. So it is enough to prove that $b_n - b_m \notin B$ for any non-zero $b_n \in B_n - B_n$ and $b_m \in B_m - B_m$.

Without loss of generality we can assume that $n > m$.

Recall that

$$B = \bigcup_{n=1}^{\infty}(B_n - B_n) = \bigcup_{n=1}^{\infty}\{i \cdot d_n : |i| < n\}.$$

The choice of the sequence (d_n) ensures that $d_n > 2(n-1)d_{n-1}$ for all $n > 1$.

It is clear that

$$d_n > jd_m + ld_k$$

for all $m, k < n$ and $|j| < m$, $|l| < k$. This inequality implies

$$id_n \neq jd_m + ld_k$$

for all $i \neq 0$, $m, k < n$ and $|j| < m$, $|l| < k$.

Therefore $b_n - b_m = id_n - jd_m \notin B_k - B_k = \{ld_k : |l| < k\}$ and hence the set B has the property (2).

Next we show that B has property (3). We have to prove that $F + B \neq \mathbb{Z}$ for any finite subset $F \subset \mathbb{Z}$.

Find $k \in \mathbb{Z}$ with $F \subset [-k, k]$ and $n \in \mathbb{N}$ such that $d_n > 3k$. Then there exists x such that $d_n + k < x < 2d_n - k$. It follows from the definition of the set B that $x \notin F + B$ and hence $F + B \neq G$.

$\square$

Lemma 2. *The quasicyclic group $\mathbb{Z}(p^\infty)$ contains a set B with properties* (1)—(3).

Proof. As in the proof of Lemma 1 we start with the definition of the subsets B_n. Let

$$B_n = \{e^{i\varphi} : \varphi = \frac{2\pi}{p^{d_n+j}},\ 1 \le j \le n\}$$

where $d_1 = 1$ and we chose d_n so that the following inequality holds:

$$\frac{2\pi}{p^{d_n}} < \min_{k,l<n}\{|\varphi_k - \varphi_l| : e^{i\varphi_l} \in B_l - B_l, e^{i\varphi_k} \in B_k - B_k, \varphi_k \neq \varphi_l\}.$$

Next we put

$$B = \bigcup_{n=1}^{\infty}(B_n - B_n).$$

It is clear that B has property (1). Analogously as in Lemma 1, one can prove that B has property (2). So it remains to show that B is satisfies property (3).

This follows from the fact that the space $\mathbb{Z}(p^\infty)$ considered as a subset of the circle has no isolated point while any finite shift $F + B$ of B has only finitely many non-isolated points.

□

Lemma 3. *The infinite direct sum $G = \oplus\langle g_i\rangle$ of cyclic groups contains a set B with properties* (1)—(3).

Proof. First of all we define subsets B_n for all n. We can think of the cyclic groups $\langle g_i\rangle$ as subgroups of the group G. We put

$$B_n = \{g_i : \frac{n(n-1)}{2} + 1 \le i \le \frac{n(n-1)}{2} + n\}$$

and

$$B = \bigcup_{n=1}^{\infty}(B_n - B_n).$$

It is easy to check that the set B has properties (1)–(3). □

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