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On fully wild categories of representations of posets

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Abstract. Assume that I is a finite partially ordered set and k is a field. We prove that if the category $\mathrm{prin}(kI)$ of prinjective modules over the incidence k-algebra kI of I is fully k-wild then the category $\mathbf{fpr}(I,k)$ of finite dimensional k-representations of I is also fully k-wild. A key argument is a construction of fully faithful exact endofunctors of the category of finite dimensional $k\langle x,y\rangle$-modules, with the image contained in certain subcategories.

1. Introduction

Throughout k is a field, I stands for a finite partially ordered set (poset).

Representations of posets have been successfully applied to investigate in particular: lattices over orders, Cohen-Macaulay modules, see [24], and abelian groups [1], see also [2], [3]. The present paper is motivated by the latter family of applications. More precisely, the motivation comes from the works of Arnold and Simson on realization of algebras as endomorphism algebras of so called filtered representations of posets (see [3]). This leads to the concept of k-endo-wildness [3], [28]. The main result of [3] is that k-endo wildness is equivalent to k-wildness for the category of k-representations of a poset I having a unique maximal element (for arbitrary, possibly finite, field). A case of our main result is used to obtain it in [3].

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Assume for a moment that k is algebraically closed. A combinatorial criterion for wildness of the matrix problem associated with I was given by Nazarova in [15], see [24, Theorem 15.3], in case when I has a unique maximal element. It has been extended partially to a class of posets with more maximal elements in [12], [13], [14]. Recall that the matrix problem associated with a poset I has an interpretation in terms of the category $\mathrm{prin}(kI)$ of prinjective modules or the category $\mathrm{mod}_{sp}(kI)$ of socle projective modules over the incidence algebra kI of I, see [25]. A consequence of the Nazarova's result and its proof is that the category of prinjective modules over kI is fully wild (in the sense of [26, Definition 2.4]) provided it is wild, when I has a unique maximal element. This is no longer true when we consider posets with more maximal elements (see [6, Remark 5.8]), but still remains true for a wide class of posets considered in [14].

For the concept of tame and wild representation types and a detailed discussion of their various aspects the reader is referred e.g. to [7], [24], [26].

The wildness of the category $\mathrm{prin}(kI)$ is equivalent to the wildness of $\mathrm{mod}_{sp}(kI)$ by the results of [25, Proposition 2.4], [27], [10]. It was not clear if this is true with respect to fully wildness as well.

The aim of this paper is to confirm that fully wildness of $\mathrm{prin}(kI)$ implies fully wildness of $\mathrm{mod}_{sp}(kI)$. To be precise, we work with the concept of fully k-wildness defined in [3], [28, Definition 2.4] over an arbitrary, not necessarily algebraically closed field k, see 2.5 below. Note that the results of [14] are valid over an arbitary field.

The paper is organized as follows. We collect basic concepts and formulate one of our main results, Theorem 2.9, in Section 2. For more information on posets and their representations the reader is referred to the monograph by Simson [24]. Sections 3 and 4, devoted to full endofunctors of the category of $k\langle x, y\rangle$-modules, can be read independently on the rest of the paper. Section 3 contains our second main result - Theorem 3.3. This is, together with Theorem 4.2, the main tool for the proof of Theorem 2.9, which is finished in Section 5. Throughout we formulate our considerations in terms of modules over the incidence algebra of a poset. The applications to filtered representations of posets are given in Section 6.

We use the following notation: $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, ...\}$. Given two indices i, j we put $\delta_{ij} = 1$ provided $i = j$ and $\delta_{ij} = 0$ otherwise. We denote by $\mathcal{W}$ the free associative k-algebra $k\langle x, y\rangle$ with two free noncommuting generators x and y. We often refer to the natural grading of $\mathcal{W}$ in which the generators x, y have degree 1. Given a k-algebra A let $\mathrm{mod}\,(A)$ be the category of right finitely generated

A-modules. The full subcategory of modules of finite k-dimension is denoted by $\mathrm{modf}(A)$. Given a ring S let $\mathbb{M}_{m\times n}(S)$ be the S-module of $m \times n$-matrices with coefficients in S. We often deal with block matrices; if the block partition of a matrix $\mathcal{A}$ is fixed then the (i,j)-th block of $\mathcal{A}$ is denoted usually by $\mathcal{A}_{ij}$. We put $\mathbb{M}_m(S)$ for $\mathbb{M}_{m\times m}(S)$ equipped with the natural ring structure. The identity matrix of size $m \times m$ is denoted by I_m.

2. Posets and their representations

2.1. Let $I = (I, \preceq)$ be a finite partially ordered set (poset) and denote by $\max I$ the set of its maximal elements. Assume that $I = \{1, ..., n, p_1, ..., p_r\}$, where $\max I = \{p_1, ..., p_r\}$. Denote $I \setminus \max I$ by I^-.

Let $v \in \mathbb{N}^I$. Following the idea from [16], [25] given a k-algebra S we define the variety of I-matrices of size v with coefficients in S as follows

$$\mathrm{Mat}_{I,v}(S) = \{\mathcal{A} = (\mathcal{A}_{pi}) \in \prod_{i\in I, p\in\max I} \mathbb{M}_{v(p)\times v(i)}(S) : \mathcal{A}_{pi} = 0 \ \text{ if } i \not\preceq p\}.$$

We have to admit "degenerated" matrices without rows or without columns, as in Chapter 2 of [24]. It is convenient to think about the elements of $\mathrm{Mat}_{I,v}(S)$ as block matrices with the horizontal blocks indexed by elements of $\max I$ and the vertical ones - by elements of I^-.

The reader is referred to [25] for the structure of a G-set on $\mathrm{Mat}_{I,v}(k)$ for a suitable algebraic group G.

2.2. Let kI denote the incidence algebra of I with coefficients in k, that is, the algebra formed by all $I\times I$-matrices $[\lambda_{ij}]_{i,j\in I}$ such that $\lambda_{ij} = 0$ provided $i \not\preceq j$ in I, [25]. For $i \preceq j$ let $e_{ij} \in kI$ be the elementary matrix with 1 at the (i,j)-position and zeros elsewhere. Let $e_i = e_{ii}$; e_i is the standard idempotent matrix corresponding to $i \in I$. The algebra kI can be viewed in a triangular matrix form

$$\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$$

where $A = kI^-$, B is the semisimple algebra $k(\max I)$ and M is the A-B-bimodule $\bigoplus_{i\prec p\in\max I} e_i kI e_p$. According to this presentation the right kI-modules can be treated as triples

$$(X'_A, X''_B, \phi : X'_A \otimes_A M \longrightarrow X''_B),$$

where X'_A is a right A-module, X''_B is a right B-module and ϕ is a B-homomorphisms. More precisely, one can define the category of such triples with morphisms defined in a usual way and observe that this category is equivalent to the category of right kI-modules. Dually, we can view kI-modules as triples

$$(X'_A, X''_B, \overline{\phi} : X'_A \longrightarrow \mathrm{Hom}_B(M, X''_B)).$$

If $\overline{\phi}$ is the homomorphism adjoint to ϕ then the two above triples represent isomorphic kI-modules. From now on we denote the functor $\mathrm{Hom}_B(M, -)$ by $|-|$. See [20] for details.

2.3. We recall from [25] the construction of the prinjective module associated to a block matrix $\mathcal{A} \in \mathrm{Mat}_{I,v}(k)$. Given $v \in \mathbb{N}^I$ let $P_A(v)$ be the right projective A-module $\bigoplus_{i\in I^-}(e_iA)^{v(i)}$ and let $Q_B(v) = \bigoplus_{p\in\max I}(e_pB)^{v(p)}$. If S is a k-algebra then we put $P^S_A(v) = S \otimes_k P_A(v)$ and $Q^S_B(v) = S \otimes_k Q_B(v)$.

Observe that

$$P_A(v)e_i \cong \bigoplus_{j\preceq i} k^{v(j)} \otimes_k ke_{ji}$$

and

$$|Q_B(v)|e_i \cong \bigoplus_{i\prec p\in\max I} k^{v(p)} \otimes_k ke_{ip}$$

as k-vector spaces for $i \in I^-$. We fix the isomorphisms and treat them as identities.

Denote by ξ_t the t-th standard basis element of the free S-module S^l, where $t \leq l$.

Now let $\mathcal{A} = (\mathcal{A}_{pi})_{i\in I^-, p\in\max I} \in \mathrm{Mat}_{I,v}(S)$. The matrix $\mathcal{A}$ defines an S-A-bimodule homomorphism

$$\phi_{\mathcal{A}} : P^S_A(v) \longrightarrow \mathrm{Hom}_B(M, Q^S_B(v))$$

such that the restriction of $\phi_{\mathcal{A}}$ to $P^S_A(v)e_i$ is defined by the block matrix

$$\Phi_{\mathcal{A}}[i] = \begin{bmatrix} \mathcal{A}_{q_1j_1} & \mathcal{A}_{q_1j_2} & \dots & \mathcal{A}_{q_1,j_t} \\ \mathcal{A}_{q_2j_1} & \mathcal{A}_{q_2j_2} & \dots & \mathcal{A}_{q_2,j_t} \\ \dots & & \dots & \\ \mathcal{A}_{q_sj_1} & \mathcal{A}_{q_sj_2} & \dots & \mathcal{A}_{q_s,j_t} \end{bmatrix}$$

with respect to the basis

$$\begin{gathered} \xi_1 \otimes e_{j_1,i}, \dots, \xi_{v(j_1)} \otimes e_{j_1,i}, \\ \xi_1 \otimes e_{j_2,i}, \dots, \xi_{v(j_2)} \otimes e_{j_2,i}, \\ \dots\dots\dots\dots\dots\dots \\ \xi_1 \otimes e_{j_t,i}, \dots, \xi_{v(j_t)} \otimes e_{j_t,i} \end{gathered}$$

of $P_A^S(v)e_i$ and

$$\begin{gathered}\xi_1 \otimes e_{i,q_1}, ..., \xi_{v(q_1)} \otimes e_{i,q_1},\\ \xi_1 \otimes e_{i,q_2}, ..., \xi_{v(q_2)} \otimes e_{i,q_2},\\ \dots\dots\dots\dots\dots\dots\dots\\ \xi_1 \otimes e_{i,q_s}, ..., \xi_{v(q_s)} \otimes e_{i,q_s}\end{gathered}$$

of $|Q_B^S(v)|e_i$, where $\{j_1, ..., j_t\} = \{j \in I : j \preceq i\}$, $\{q_1, ..., q_s\} = \{q \in \max I : i \prec q\}$.

The homomorphism $\phi_{\mathcal{A}}$ is represented by the block matrix

$$\begin{bmatrix} \Phi_{\mathcal{A}}[1] & 0 & ... & 0 \\ 0 & \Phi_{\mathcal{A}}[2] & ... & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & ... & \Phi_{\mathcal{A}}[n] \end{bmatrix}$$

in the suitable bases of $P_A^S(v)$ and $|Q_B^S(v)|$.

The S-kI-bimodule identified with the triple

$$(P_A^S(v), Q_B^S(v), \phi_{\mathcal{A}} : P_A^S(v) \longrightarrow \mathrm{Hom}_B(M, Q_B^S(v))$$

will be denoted by $\widehat{\mathcal{A}}^S$.

If $\mathcal{A} \in \mathrm{Mat}_{I,v}(k)$ then $\widehat{\mathcal{A}}^k$ is a finite dimensional kI-module which is **prinjective**, that is, its restriction to kI^- is a projective kI^--module. The category of all (finite dimensional) prinjective right kI-modules is denoted by $\mathrm{prin}(kI)$. We refer to [25], [18] for a detailed discussion of this category.

Every module X in $\mathrm{prin}(kI)$ is isomorphic to $\widehat{\mathcal{A}}^k$ for some $\mathcal{A} \in \mathrm{Mat}_{I,v}(k)$ and a uniquely determined $v \in \mathbb{N}^I$ by [25, Proposition 2.3]. Such v is called the **coordinate vector** of X and it is denoted by $\mathbf{cdn}(X)$. See [24], [25], [18] for a definition of $\mathbf{cdn}(X)$ expressed in terms of the module X. Introduce the following notation: $\mathbf{cdn}(X) = (\mathbf{cdn}'(X), \mathbf{cdn}''(X))$, where $\mathbf{cdn}'(X)$ and $\mathbf{cdn}''(X)$ are the projections of $\mathbf{cdn}(X)$ onto $\mathbb{N}^{I^-}$ and $\mathbb{N}^{\max I}$ respectively.

2.4. Lemma. *Let* $X = (X'_A, X''_B, \phi : X'_A \longrightarrow |X''_B|)$ *and* $Y = (Y'_A, Y''_B, \psi : Y'_A \longrightarrow |Y''_B|)$ *be prinjective* kI*-modules.*

(a) $\mathrm{Ext}^1_{kI}(X, Y) \cong \mathrm{Hom}_A(X', |Y''|)/\mathcal{B}(X, Y)$, *where*

$$\mathcal{B}(X, Y) = \{|r''|\phi - \psi r' : r'' \in \mathrm{Hom}_B(X'', Y''), r' \in \mathrm{Hom}_A(X', Y')\}.$$

(b) *There is a* k*-linear surjection*

$$\Xi_{X,Y} : \mathrm{Mat}_{I,v}(k) \longrightarrow \mathrm{Ext}^1_{kI}(X, Y)$$

where $v = (\mathbf{cdn}'(X), \mathbf{cdn}''(Y))$.

Proof. The proof of (a) is standard, whereas (b) follows from (a): we define the homomorphism

$$\mathrm{Mat}_{I,v}(k) \longrightarrow \mathrm{Hom}_A(X', |Y''|)$$

in the same way as the map $\mathcal{A} \mapsto \phi_{\mathcal{A}}$ in 2.3 and observe that it is surjective. □

2.5. Let $\mathcal{C}$ be a full exact additive subcategory of mod (kI). Following [26], [24, Section 14.2], [28] we say that the category $\mathcal{C}$ is of **fully** k**-wild** representation type if there exists a full, faithful and exact k-linear functor T : modf$(\mathcal{W}) \longrightarrow$ mod (kI) with the image contained in $\mathcal{C}$. The category $\mathcal{C}$ is k**-wild** if there exists an exact k-linear functor T : modf$(\mathcal{W}) \longrightarrow$ mod (kI) preserving indecomposability, respecting isomorphism classes and with the image contained in $\mathcal{C}$. It follows from the Wildness Correction Lemma in [26] that prin(kI) is of fully k-wild representation type if and only if it is of strictly wild representation type in the sense of [4] and [17], that is, there exists a $\mathcal{W}$-kI-bimodule ${}_{\mathcal{W}}N_{kI}$ which is a finitely generated free $\mathcal{W}$-module and induces a fully faithful exact functor $(-)\otimes_{\mathcal{W}} N_{kI}$: modf$(\mathcal{W}) \longrightarrow$ mod (kI) with the image in $\mathcal{C}$, [28, Lemma 2.5]. If k is algebraically closed then k-wildness is equivalent to wildness in the usual sense, see [7].

Our proofs of fully wildness are based on the idea from [19]: a suitable functor is determined by a pair of orthogonal "bricks" X, Y with at least 3-dimensional extension group $\mathrm{Ext}^1(X, Y)$.

2.6. Lemma. *Assume that the category* prin(kI) *is of fully* k*-wild prinjective type. Then there exists prinjective modules* X, Y *satisfying the conditions:*

1. $\mathrm{End}_{kI}(X) \cong \mathrm{End}_{kI}(Y) \cong k$,
2. $\mathrm{Hom}_{kI}(X, Y) = \mathrm{Hom}_{kI}(Y, X) = 0$,
3. $\dim_k \mathrm{Ext}^1_{kI}(X, Y) \geq 3$.

For the proof see e.g. Lemma 3.6 in [14] and its proof. □

We will show that if the category prin(kI) is fully k-wild then there is a bimodule ${}_{\mathcal{W}}N_{kI}$ defining its fully k-wildness and having a very special form.

2.7. Lemma. *Assume that the cateory* prin(kI) *is of fully* k*-wild prinjective type. Then there exists* $v \in \mathbb{N}^I$ *and* $\mathcal{N} \in \mathrm{Mat}_{I,v}(\mathcal{W})$ *such that* *the induced functor*

$$(-)\otimes_{\mathcal{W}}\widehat{\mathcal{N}}^{\mathcal{W}}:\mathrm{modf}(\mathcal{W})\longrightarrow\mathrm{prin}(kI)$$

is full, $\mathcal{N}$ has only two non-constant entries and they have degree 1.

Proof. Assume that $X=\widehat{\mathcal{A}}^k$ and $Y=\widehat{\mathcal{B}}^k$ are prinjective modules satisfying the conditions of Lemma 2.6. Let $v=(\mathbf{cdn}'(X),\mathbf{cdn}''(Y))$ and let $E^1,E^2,E^3\in\mathrm{Mat}_{I,v}(k)$ be elements such that

$$\Xi_{X,Y}(E^1),\Xi_{X,Y}(E^2),\Xi_{X,Y}(E^3)\in\mathrm{Ext}^1_{kI}(X,Y)$$

are linearly independent (see 2.4) and if $E^i=(E^i_{pj})_{j\in I^-,p\in\max I}$ then only one of the matrices E^i_{pj} is nonzero and it has only one nonzero entry, for $i=2,3$. Then it follows by Lemmas 1.5 and 8.6 [19], (see also [14]) that $\mathcal{N}=(\mathcal{N}_{pj})_{j\in I^-,p\in\max I}$, where each $\mathcal{N}_{pj}$ has the form

$$\mathcal{N}_{pj}=\begin{bmatrix}\mathcal{A}_{pj} & E^1_{pj}+xE^2_{pj}+yE^3_{pj}\\ 0 & \mathcal{B}_{pj}\end{bmatrix}$$

satisfies the required condition. □

2.8. Following [22], [20] a right kI-module is called **socle projective** if its right socle is a projective kI-module. A module X identified with a triple

$$(X'_A,X''_B,\overline{\phi}:X'_A\longrightarrow|X''_B|)$$

is socle projective if and only if the map $\overline{\phi}$ is injective. The category of socle projective kI-modules is denoted by $\mathrm{mod}_{sp}(kI)$.

There is a nice adjustment functor introduced in [20], [25], [18]

$$\mathbf{\Theta}_B:\mathrm{prin}(kI)\longrightarrow\mathrm{mod}_{sp}(kI)$$

defined as the restriction of the functor

$$\mathbf{\Theta}'_B:\quad\mathrm{mod}\ (kI)\longrightarrow\mathrm{mod}_{sp}(kI)$$

associating to a triple

$$(X'_A,X''_B,\overline{\phi}:X'_A\longrightarrow|X''_B|)$$

the triple $(\mathrm{Im}(\overline{\phi}),X''_B,u:\mathrm{Im}(\overline{\phi})\longrightarrow|X''_B|)$, where u is the identity embedding. More generally, given a ring S and an S-kI-bimodule X we denote by $\mathbf{\Theta}_B(X)$ the S-kI-bimodule defined in the same way.

Recall the relevant properties of the functor $\mathbf{\Theta}_B$.

Theorem [25, Lemma 2.1, Proposition 2.4]. *Let k be a field and I a finite poset.*

(a) *The functor* $\mathbf{\Theta}_B$ *is full, dense and the kernel of* $\mathbf{\Theta}_B$ *is the ideal in the category* $\operatorname{prin}(kI)$ *consisting of all homomorphism factorizing through modules of the form* $(X'_A, 0, 0)$.

(b) *The category* $\operatorname{prin}(kI)$ *is of finite representation type* (*that is, admits only finitely many isomorphism classes of indecomposable objects*) *if and only if* $\operatorname{mod}_{sp}(kI)$ *is of finite representation type.*

(c) *If* k *is algebraically closed then the category* $\operatorname{prin}(kI)$ *is of tame* (*resp. wild*) *representation type if and only if* $\operatorname{mod}_{sp}(kI)$ *is of tame* (*resp. wild*) *representation type.*

Proof. The assertions (a), (b) follow from Lemma 2.1(c) in [25] whereas (c) from Proposition 2.4 in [25], see also [27], [10]. □

2.9. One of the main results of this paper is the following theorem.

Theorem. *Let* k *be a field and* I *a finite poset. Then the category* $\operatorname{mod}_{sp}(kI)$ *is of fully* k*-wild representation type provided the following equivalent conditions hold.*

(a) *The category* $\operatorname{prin}(kI)$ *is of fully* k*-wild representation type.*

(b) *The integral Tits quadratic form* $q_I : \mathbb{Z}^I \to \mathbb{Z}$,

$$q_I(x) = \sum_{i \in I} x_i^2 + \sum_{i \prec j \in I^-} x_i x_j - \sum_{p \in \max I} (\sum_{i \prec p} x_i) x_p$$

is not weakly non-negative, that is, there exists a vector $v \in \mathbb{N}^I$ *such* $q_I(v) < 0$.

(c) *The poset* I *contains as a full peak subposet* (*see [25]*) *one of the hypercritical irreducible posets listed in Table 1 in [14], or a poset which is peak-reducible to any of the above ones* (*see Section 3 of [14]*) *in [14].*

The equivalence of (a), (b) and (c) is the main result of [14]. In 5.4 we prove that (a) implies fully k-wildness of $\operatorname{mod}_{sp}(kI)$.

Remark. We believe that the converse implication holds for every finite poset. It follows easily for classes of posets for which there are criteria for tameness in terms of weak nonnegativity of the Tits quadratic form. One-peak posets and thin two-peak posets (see [13]) form such classes thanks to Nazarova theorem [15], [24, Theorem 15.3] and the results of [13], [12]. The key argument is that wildness implies fully wildness of $\operatorname{prin}(kI)$ for such posets I, see [14]. We do not know a proof valid for arbitrary posets.

3. Full endofunctors of $\mathrm{modf}(k\langle x, y\rangle)$

In this section we prove the second of our main results - Theorem 3.3.

3.1. Given $\mathcal{A}, \mathcal{B} \in \mathbb{M}_m(\mathcal{W})$ denote by $M[\mathcal{A}, \mathcal{B}]$ the $\mathcal{W}$-$\mathcal{W}$-bimodule isomorphic to $\mathcal{W}^m$ as a left $\mathcal{W}$-module and with the right multiplication by x and y defined in the standard basis by the matrices $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\mathcal{C}, \mathcal{D} \in \mathbb{M}_n(\mathcal{W})$ then $\mathcal{A}(\mathcal{C}, \mathcal{D})$ denotes the matrix obtained from $\mathcal{A}$ by substituting x by $\mathcal{C}$ and y by $\mathcal{D}$. Clearly, a scalar entry λ of $\mathcal{A}$ is replaced by λI_n. Note that $\mathcal{A}(\mathcal{C}, \mathcal{D}) \in \mathbb{M}_{nm}(\mathcal{W})$ and $\mathcal{A}(\mathcal{C}, \mathcal{D}) \in \mathbb{M}_{nm}(k)$ if $\mathcal{C}, \mathcal{D} \in \mathbb{M}_n(k)$. Moreover $M[\mathcal{A}(\mathcal{C}, \mathcal{D}), \mathcal{B}(\mathcal{C}, \mathcal{D})] \cong M[\mathcal{C}, \mathcal{D}] \otimes_{\mathcal{W}} M[\mathcal{A}, \mathcal{B}]$ as $\mathcal{W}$-$\mathcal{W}$-bimodules.

Definition. *A pair $(\mathcal{A}, \mathcal{B})$ of square $\mathcal{W}$-matrices of size n is a* **full pair** *if the functor*

$$(-) \otimes_{\mathcal{W}} M[\mathcal{A}, \mathcal{B}] : \mathrm{modf}(\mathcal{W}) \longrightarrow \mathrm{modf}(\mathcal{W})$$

is full. □

Lemma. *Assume that $(\mathcal{A}, \mathcal{B})$ is a full pair of matrices of size n. Then*

(1) *$(a_{11}\mathcal{A} + a_{21}\mathcal{B} + \lambda I_n, a_{12}\mathcal{A} + a_{22}\mathcal{B} + \mu I_n)$ is a full pair provided $a_{11}, a_{12}, a_{21}, a_{22}, \lambda, \mu \in k$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$.*

(2) *if $(\mathcal{C}, \mathcal{D})$ is another full pair then $(\mathcal{A}(\mathcal{C}, \mathcal{D}), \mathcal{B}(\mathcal{C}, \mathcal{D}))$ is a full pair.*

The proof of (a) is straightforward, whereas (b) follows from the isomorphism $M[\mathcal{A}(\mathcal{C}, \mathcal{D}), \mathcal{B}(\mathcal{C}, \mathcal{D})] \cong M[\mathcal{C}, \mathcal{D}] \otimes_{\mathcal{W}} M[\mathcal{A}, \mathcal{B}]$. □

3.2. Introduce the following notation. Given $m \in \mathbb{N}$, $\mu \in k$ and $\rho = (\rho_0, \rho_1, ..., \rho_m) \in k^{m+1}$ let

$$\mathcal{X}_{m,\mu,\rho} = \begin{bmatrix} \mu & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & \mu & 0 & ... & 0 & 0 & 0 \\ \rho_0 x & 0 & \mu & ... & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ \rho_m x & 0 & 0 & ... & 0 & \mu & 0 \\ y & 0 & 0 & ... & 0 & 0 & \mu \end{bmatrix} \in \mathbb{M}_{m+4}(\mathcal{W})$$

$$\mathcal{Y}_m = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_{m+4}(\mathcal{W})$$

Lemma. *For every $m \in \mathbb{N}$, $\mu \in k$ and $\rho \neq 0$ the pair $(\mathcal{X}_{m,\mu,\rho}, \mathcal{Y}_m)$ is full.*

Proof. For simplicity we present the proof in the case $m = 0$, $\rho_0 = 1$, $\mu = 0$. The general proof does not differ essentially. Consider a block matrix $\mathcal{F} = [f_{ij}]_{i,j=1,\dots,4}$ with $f_{ij} \in \mathbb{M}_s(k)$ for some s and assume that $\mathcal{F}$ commutes with $\mathcal{X} = \mathcal{X}_{0,1,1}(X,Y)$ and $\mathcal{Y} = \mathcal{Y}_0(X,Y)$ for some matrices $X, Y \in \mathbb{M}_s(k)$. The latter commutativity implies the following conditions:

$$\begin{aligned}
&f_{ij} = 0, \ \ i > j,\\
&f_{11} = f_{22} = f_{33} = f_{44},\\
&f_{12} = f_{23} = f_{34},\\
&f_{13} = f_{24}.
\end{aligned}$$

Now the commutativity of $\mathcal{X}$ and $\mathcal{F}$ gives $f_{12} = f_{13} = f_{14} = 0$ and $f_{11}X = Xf_{11}$, $f_{11}Y = Yf_{11}$. □

□

From now on let $F_i = \alpha_i x + \beta_i y + \gamma_i$ be fixed nonzero polynomials of degree at most 1, $i = 1, \dots, m$. Without loss of generality we can assume that $F_i = x + \beta_i y + \gamma_i$ for $i = 1, \dots, m'$ and $F_i = y + \gamma_i$ for $i = m' + 1, \dots, m$ for some $m' \leq m$. If k is infinite we can assume $m' = m$ thanks to Lemma 3.2.

3.3. Theorem. *Under the notation above:*

(1) *There exists a full pair $(\mathcal{X}, \mathcal{Y})$ of size $m' + 4$ and invertible $\mathcal{W}$-matrices $C_i, D_i \in \mathbb{M}_{m'+4}(\mathcal{W})$, $i = 1, \dots, m$, such that $C_i F_i(\mathcal{X}, \mathcal{Y}) D_i \in \mathbb{M}_{m'+4}(k)$ or*

$$C_i F_i(\mathcal{X}, \mathcal{Y}) D_i = \begin{bmatrix} 0 & I_{m'+3} \\ \gamma_i' + \beta_i' y & 0 \end{bmatrix}$$

for some $\gamma_i', \beta_i' \in k$, $i = 1, \dots, m$. For the matrices $\mathcal{X}, \mathcal{Y}$ one can choose $\mathcal{X}_{m',\mu,\rho}$ and $\mathcal{Y}_{m'}$ for some $\mu \in k$, $\rho \in k^{m'+1}$.

(2) *There exists a full pair $(\mathcal{Z}, \mathcal{T})$ of size $4(m' + 4)$ and invertible $\mathcal{W}$-matrices $C_i', D_i' \in \mathbb{M}_{4(m'+4)}(\mathcal{W})$, $i = 1, \dots, m$, such that*

$$C_i' F_i(\mathcal{Z}, \mathcal{T}) D_i' \in \mathbb{M}_{4(m'+4)}(k)$$

for $i = 1, \dots, m$.

(3) *If k is infinite then there exists a full pair $(\mathcal{Z}, \mathcal{T})$ of size $4(m + 4)$ such that the matrices $F_i(\mathcal{Z}, \mathcal{T})$ are invertible for $i = 1, \dots, m$.*

The matrices $\mathcal{Z}, \mathcal{T}$ in (2) and (3) are: $\mathcal{Z} = \mathcal{X}(\mathcal{X}_{0,0,1}, \mathcal{Y}_0)$, $\mathcal{T} =$

$\mathcal{Y}(\mathcal{X}_{0,0,1}, \mathcal{Y}_0)$, where $\mathcal{X}, \mathcal{Y}$ are as in (1).

We precede the proof of the theorem by a series of lemmas.

3.4. First we observe that for an element of $\mathbb{M}_m(\mathcal{W})$ to be invertible it is enough to have one-sided inverse.

Lemma. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{M}_n(\mathcal{W})$ and $\mathcal{AB} = I_n$. Then $\mathcal{BA} = I_n$.*

Proof. For every $m \in \mathbb{N}$ and $a, b \in \mathbb{M}_m(k)$ we have $\mathcal{A}(a,b)\mathcal{B}(a,b) = I_{mn}$ and therefore $\mathcal{B}(a,b)\mathcal{A}(a,b) = I_{nm}$. This means that

$$\sum_{j=1}^{n} \mathcal{B}_{ij}(a,b)\mathcal{A}_{jl}(a,b) = \delta_{il} I_m$$

for every $m \in \mathbb{N}$, $i, l = 1, ..., n$, and $a, b \in \mathbb{M}_m(k)$. Recall that the k-algebra $\mathbb{M}_m(k)$ has no polynomial identity of degree less than $2m$, see e.g. [9, Lemma 6.3.1]. Taking m large enough we conclude that

$$\sum_{j=1}^{n} \mathcal{B}_{ij}\mathcal{A}_{jl} = \delta_{il}$$

for every i, l, that is, $\mathcal{BA} = I_m$. □

3.5. Recall that $F_i = x + \beta_i y + \gamma_i$ for $i = 1, ..., m'$.

Lemma. *The determinant of the matrix*

$$F_i(\mathcal{X}_{m',\mu,\rho}, \mathcal{Y}_{m'})$$

treated as a matrix with coefficients in $k[x,y]$, equals

$$\begin{aligned}&\lambda_i^{m'+4} - \beta_i \lambda_i^{m'+2} + \\ &+ [\rho_0 \beta_i^2 \lambda_i^{m'+1} + ... + (-1)^{m'} \rho_{m'} \beta_i^{m'+2} \lambda_i] x + (-1)^{m'+1} \beta_i^{m'+3} y,\end{aligned}$$

where $\lambda_i = \mu + \gamma_i$, for $i = 1, ... m'$.

Proof. Follows by direct calculation. □

3.6. Lemma. *Given $\mu \in k$ there exist $\rho \in k^{m'+1}$, $\rho \neq 0$, such that there are matrices $\mathcal{M}_i \in \mathbb{M}_{m'+4}(\mathcal{W})$ satisfying*

$$F_i(\mathcal{X}_{m',\mu,\rho}, \mathcal{Y}_{m'})\mathcal{M}_i = (\lambda_i^{m'+4} + (-1)^{m'+1} \beta_i^{m'+3} y) I_{m'+4},$$

where $\lambda_i = \mu + \gamma_i$, for $i = 1, ..., m'$.

Proof. Note that

$$\rho_0 \beta_i^2 \lambda_i^{m'+1} + ...(-1)^{m'} \rho_{m'} \beta_i^{m'+2} \lambda_i = 0, \ i = 1, ..., m'.$$

is a system of m' linear equations with $m'+1$ unknowns $\rho_0, ..., \rho_{m'}$, therefore it has a nonzero solution $\rho = (\rho_0, ..., \rho_{m'})$. When ρ is so then

$$\det(F_i(\mathcal{X}_{m',\mu,\rho}, \mathcal{Y}_{m'}) = \lambda_i^{m'+4} + (-1)^{m'} \beta_i^{m'+3} y$$

for $i = 1, ..., m'$ by 3.5.

Treat $F_i(\mathcal{X}_{m',\mu,\rho}, \mathcal{Y}_{m'})$ as a matrix over $k[x,y]$ and let $\mathcal{M}_i$ be the matrix adjoint to $F_i(\mathcal{X}_{m',\mu,\rho}, \mathcal{Y}_{m'})$. Then the required equality holds if we view the coefficients as elements of $k[x,y]$. But observe that every entry of $\mathcal{M}_i$ has degree at most 1 and the first row of $\mathcal{M}_i$ contains only constants. Therefore the equality is true also over $k\langle x, y\rangle$. □

3.7. Proof of Theorem 3.3. We keep the notation introduced in 3.5 and 3.6 above.

(1) First consider $i = 1, ..., m'$. Take $\mu \in k$ arbitrary and let ρ be as in 3.6. Let $\mathcal{X} = \mathcal{X}_{m',\mu,\rho}$ and $\mathcal{Y} = \mathcal{Y}_{m'}$. Then $F_i(\mathcal{X}, \mathcal{Y})$ is a square $\mathcal{W}$-matrix with $m'+4$ rows and columns and having nonconstant terms only in the first column. As before we set $\lambda_i = \mu + \gamma_i$.

If $\lambda_i = \beta_i = 0$ then $F_i(\mathcal{X}, \mathcal{Y})$ has only one nonzero column containing an entry 1 thus it can be reduced by elementary transformations on rows and columns to a matrix having only one nonzero entry equal 1.

Assume that $\lambda_i \neq 0$ or $\beta_i \neq 0$. The matrix obtained from $F_i(\mathcal{X}, \mathcal{Y})$ by deleting the first column has rank $m'+3$. After suitable elementary operations on rows and columns of $F_i(\mathcal{X}, \mathcal{Y})$ we can reduce it to the form

$$\begin{bmatrix} 0 & I_{m'+3} \\ G(x,y) & 0 \end{bmatrix}$$

where $G(x,y)$ is an element of $\mathcal{W}$ of degree one. Now

$$G(x,y) = (-1)^{m'+5}(\lambda_i^{m'+4} + (-1)^{m'} \beta_i^{m'+3} y)$$

by Lemma 3.5.

For $i > m'$ we note that $F_i(\mathcal{X}, \mathcal{Y}) = \mathcal{Y} + \gamma_i I_{m'+4}$ has all entries in k.

(2) Let $\mathcal{X}$, $\mathcal{Y}$ be as in (1) and

$$\mathcal{U} = \mathcal{X}_{0,0,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{V} = \mathcal{Y}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Set $\mathcal{Z} = \mathcal{X}(\mathcal{U}, \mathcal{V})$, $\mathcal{T} = \mathcal{Y}(\mathcal{U}, \mathcal{V})$. Then $(\mathcal{Z}, \mathcal{T})$ is a full pair by by Lemma 3.1 and satisfies the claim by (1).

(3) Since k is infinite we can choose $\mu \in k$ such that $\lambda_i = \mu + \gamma_i \neq 0$ for $i = 1, ..., m' = m$. By 3.6 for any $i = 1, ..., m$ there exists a matrix $\mathcal{M}_i \in \mathbb{M}_{m+4}(\mathcal{W})$ such that

$$F_i(\mathcal{X}, \mathcal{Y})\mathcal{M}_i = (\lambda_i^{m+4} + (-1)^m \beta_i^{m+3} y) I_{m+4}$$

Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Z}$, $\mathcal{T}$ be as in (2). Then

$$F_i(\mathcal{Z}, \mathcal{T})\mathcal{M}_i(\mathcal{Z}, \mathcal{T}) = diag(\lambda_i^{m+4} I_4 + (-1)^m \beta_i \mathcal{V}, m+4).$$

(Given a square matrix $\mathcal{A}$ we denote by $diag(\mathcal{A}, m)$ the block matrix with m blocks $\mathcal{A}$ at the diagonal and zeros outside.) This is an invertible matrix in $\mathbb{M}_{4(m+4)}(k)$, let $\mathcal{L}_i$ be its inverse. Then

$$F_i(\mathcal{Z}, \mathcal{T})\mathcal{M}_i(\mathcal{Z}, \mathcal{T})\mathcal{L}_i = I_{4(m+4)}$$

and $F_i(\mathcal{Z}, \mathcal{T})$ is invertible by 3.4. As above, $(\mathcal{Z}, \mathcal{T})$ is a full pair by Lemma 3.1. $\square$

Remarks. (a) We expect that Theorem 3.3 can be improved by skipping the assumption that k is infinite in (3).

(b) Let $\Sigma \subseteq \mathcal{W}$ be a finite set of nonzero elements of degree 1 and denote by modf($\mathcal{W}_\Sigma$) the full subcategory of modf($\mathcal{W}$) formed by all modules of the form $U(a, b)$ (see 5.1 below for the notation) such that $F(a, b)$ is an invertible matrix for every $F \in \Sigma$. Theorem 3.3 proves that the category modf($\mathcal{W}_\Sigma$) is fully k-wild provided k is an infinite field. It would be interesting to generalize this assertion to arbitrary finite set Σ of nonzero elements of $\mathcal{W}$. Note that if k is algebraically closed then wildness (not fully wildness) of modf($\mathcal{W}_\Sigma$) follows by Theorem 2 in [8], since modf($\mathcal{W}_\Sigma$) is an open subcategory of modf($\mathcal{W}$) in the sense of [8].

Example. We present a full pair $(\mathcal{X}, \mathcal{Y})$ such that the matrix $\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$ is invertible.

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

The above matrices were found with a help of a computer, partially by a random search. Unfortunately our methods do not suggest how to generalize Theorem 3.3 to arbitrary finite sets Σ of nonzero elements of $k\langle x, y\rangle$.

4. Pure $k\langle x,y\rangle$-matrices

4.1. Lemma. *Let F_0, F_1 be free left $\mathcal{W}$-modules of finite rank and assume that $f : F_0 \longrightarrow F_1$ is a $\mathcal{W}$-homomorphism. Let $\mathcal{A}$ be the matrix of f with respect to some bases of F_0 and F_1. Then*

(1) *The module* $\operatorname{Im} f$ *is free.*

(2) *The following conditions are equivalent:*

(a) $\operatorname{Im} f$ *is a direct summand of F_1 (that is,* $\operatorname{Im} f \hookrightarrow F_1$ *is a pure monomorphism),*

(b) *there exist invertible square $\mathcal{W}$-matrices B, C such that $B\mathcal{A}C$ is a block matrix of the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for some $r \in \mathbb{N}$,

(c) *the canonical homomorphism*

$$U \otimes_{\mathcal{W}} \operatorname{Im} f \longrightarrow \operatorname{Im}(U \otimes_{\mathcal{W}} f)$$

is an isomorphism for every right $\mathcal{W}$-module U.

If this is the case then r is the $\mathcal{W}$-rank of $\operatorname{Im} f$.

Proof. (1) follows since $\mathcal{W}$ is a free ideal ring [5, §2.4, Proposition 2.1]. An equivalence of (a) and (b) is easy thanks to (1), similarly as the implication $(a) \Rightarrow (c)$. In order to prove the converse implication assume that $F_1/\operatorname{Im} f$ is not projective. Then it is enough to take U such that $\operatorname{Tor}_1^{\mathcal{W}}(U, F_1/\operatorname{Im} f) \neq 0$. □

Definition. *A matrix $\mathcal{A} \in \mathbb{M}_{n\times m}(\mathcal{W})$ is* **pure** *if it satisfies the condition (2.b) of the lemma above.* □

4.2. Theorem. *Let $\mathcal{G}_i$ be $\mathcal{W}$-matrices (of arbitrary sizes), $i = 1, ..., m$. Assume that every entry of $\mathcal{G}_i$ has degree at most 1 and there is at most one column containing a non-constant entry for $i = 1, ..., m$. Then there exists a full pair $(\mathcal{Z}, \mathcal{T})$ such that the matrices $\mathcal{G}_i(\mathcal{Z}, \mathcal{T})$ are pure for $i = 1, ..., m$.*

Proof. Using elementary transformations on rows and columns we can reduce each $\mathcal{G}_i$ to a block matrix of the shape:

$$\begin{bmatrix} I_{r_i} & 0 & 0 \\ 0 & 0 & \mathbf{F}_i \end{bmatrix}$$

where $r_i \in \mathbb{N}$ and $\mathbf{F}_i$ is a column whose entries have degree at most 1. After applying suitable elementary operations on $\mathbf{F}_i$ we can assume that either $\mathbf{F}_i$ has at most one nonzero entry or it has exactly two nonzero entries of the form $x+a$, $y+b$ for some $a,b \in k$. Let $1 \le m_1 \le m_2 \le m$ be such that $\mathbf{F}_i$ has unique nonzero entry Φ_i ($\Phi_i \in \mathcal{W}$ is an element of degree at most 1) for $i = 1, ..., m_1$, $\mathbf{F}_i$ has two nonzero entries $x + a_i$, $y + b_i$ for $i = m_1 + 1, ..., m_2$ and $\mathbf{F}_i$ is a zero column for $i = m_2 + 1, ..., m$. Let $(\mathcal{Z}, \mathcal{T})$ be the full pair of matrices of size $s = 4(m_1 + 4)$ such that $C_i' \Phi_i(\mathcal{Z}, \mathcal{T}) D_i'$ has all the entries in k for some invertible $\mathcal{W}$-matrices C_i', D_i' for $i = 1, ..., m_1$, see Theorem 3.3 (2). Then the matrices $\mathcal{G}_i(\mathcal{Z}, \mathcal{T})$ are pure for $i = 1, ..., m_1$.

Observe that the remaining matrices $\mathcal{G}_i(\mathcal{Z}, \mathcal{T})$ are also pure. Indeed, analysis of the shapes of $\mathcal{Z}$ and $\mathcal{T}$ constructed in Theorem 3.3 shows that the block matrix

$$\begin{bmatrix} \mathcal{Z} + a_i I_s \\ \mathcal{T} + b_i I_s \end{bmatrix}$$

can be reduced by elementary transformations to

$$\begin{bmatrix} 0 \\ I_s \end{bmatrix}.$$

□

5. Proof of Theorem 2.9

5.1 We say that a $\mathcal{W}$-kI-bimodule $N = (N_A', N_B'', \phi : N_A' \longrightarrow |N_B''|)$, free as a left $\mathcal{W}$-module, is **purely defined** if ϕ satisfies the equivalent conditions in Lemma 4.1 (2), that is, it is defined by a pure matrix with respect to some/any bases.

Given $m \in \mathbb{N}$ and $a, b \in \mathbb{M}_m(k)$ let $U(a,b)$ be the right $\mathcal{W}$-module isomorphic to k^m as a k-module, equipped with the right action of x (resp. y) defined by the matrix a (resp. b) with respect to the standard basis of k^m. The following assertion is clear.

Lemma. *Let $\mathcal{C} \in \mathrm{Mat}_{I,v}(\mathcal{W})$. Then*

$$U(a,b) \otimes_{\mathcal{W}} \widehat{C}^{\mathcal{W}} \cong \widehat{\mathcal{C}(a,b)}^{k}.$$

□

5.2. Lemma. *Assume that $N = (N_A', N_B'', \phi : N_A' \longrightarrow |N_B''|)$ is a $\mathcal{W}$-kI-bimodule which is free as a $\mathcal{W}$-module and purely defined. Then $\mathbf{\Theta}_B(N)$ is a $\mathcal{W}$-kI-bimodule free as a $\mathcal{W}$-module and*

$$\mathbf{\Theta}_B(U \otimes_{\mathcal{W}} N) \cong U \otimes_{\mathcal{W}} \mathbf{\Theta}_B(N)$$

as kI-modules for every module U in $\mathrm{modf}(\mathcal{W})$.

Proof. First note that, since B is a semisimple k-algebra,

$$U \otimes_{\mathcal{W}} \mathrm{Hom}_B(M, N''_B) \cong \mathrm{Hom}_B(M, U \otimes_{\mathcal{W}} N''_B)$$

as right A-modules for every right $\mathcal{W}$-module U. Now the lemma follows by Lemma 4.1. (c) and the definition of $\mathbf{\Theta}_B$. □

5.3. Lemma. *Assume that $\mathcal{C} \in \mathrm{Mat}_{I,v}(\mathcal{W})$ has only two non-constant entries and they have degree 1. Then there exists a full pair $(\mathcal{X}, \mathcal{Y})$ such that the bimodule $\widehat{\mathcal{C}(\mathcal{X}, \mathcal{Y})}^{\mathcal{W}}$ is purely defined.*

Proof. After suitable linear change of variables we can assume that the non-constant entries are equal either x and $y + \gamma$ respectively or y and $y + \gamma$ for some $\gamma \in k$, see Lemma 3.1. Replacing $\mathcal{C}$ by $\mathcal{C}(\mathcal{U}, \mathcal{V})$, where

$$\mathcal{U} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{V} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can assume that x and y appear in the same column. By Theorem 4.2 we conclude that there exist a full pair $(\mathcal{Z}, \mathcal{T})$ such that all the matrices $\Phi_{\mathcal{C}}[i](\mathcal{Z}, \mathcal{T})$ (see 2.3 for the notation) are pure for $i \in I^-$. This means that the module $\widehat{\mathcal{C}(\mathcal{Z}, \mathcal{T})}^{\mathcal{W}}$ is purely defined. □

Corollary. *Assume that the category* $\mathrm{prin}(kI)$ *is fully k-wild. Then there exists a purely defined $\mathcal{W}$-kI-bimodule N such that the induced functor*

$$(-) \otimes_{\mathcal{W}} N_{kI} : \mathrm{modf}(\mathcal{W}) \longrightarrow \quad \mathrm{mod}\ (kI)$$

is full and its image is contained in $\mathrm{prin}(kI)$.

Proof. Apply the above lemma to the matrix $\mathcal{N}$ defined in Lemma 2.7. □

5.4. Proof of Theorem 2.9. Assume that the category $\mathrm{prin}(kI)$ is fully k-wild. By Corollary 5.3 there exists a purely defined $\mathcal{W}$-kI-bimodule N defining fully k-wildness of $\mathrm{prin}(kI)$.

We obtain the functor

$$\mathbf{\Theta}_B((-) \otimes_{\mathcal{W}} N) : \mathrm{modf}(\mathcal{W}) \longrightarrow \quad \mathrm{mod}\ (kI)$$

which is full (see 2.8), faithful and exact by 5.2 and its image is contained in $\mathrm{mod}_{sp}(kI)$. Therefore the category $\mathrm{mod}_{sp}(kI)$ is fully k-wild. □

5.5. Example. Let I be a poset with a unique maximal element p and containing five pairwise incomparable elements $i_1, ..., i_5$. Then the category prin(kI) is of fully k-wild representation type by Nazarova's Theorem (see [24, Theorem 15.3]) and [14]. A functor satisfying the conditions of the definition of fully k-wildness is also described in [24]: it is the functor determined by the matrix $\mathcal{N} \in \mathrm{Mat}_{I,v}(\mathcal{W})$, where $v(p) = 2$, $v(i_j) = 1$, $j = 1, ..., 5$ and $v(j) = 0$ for $j \notin \{p, i_1, ..., i_5\}$ and

$$[\mathcal{N}_{i_1p}|\mathcal{N}_{i_2p}|\mathcal{N}_{i_3p}|\mathcal{N}_{i_4p}|\mathcal{N}_{i_5p}] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & x & y \end{bmatrix}$$

The bimodule $\widehat{\mathcal{N}}$ is not necessarily purely defined, it depends on the position of $i_1, ..., i_5$ in I. For example, when $I \setminus \{p\}$ is the following:

$$\begin{array}{ccccccc} i_1 & i_2 & i_3 & & j & & \\ & & & \nearrow & & \nwarrow & \\ & & i_4 & & & & i_5 \end{array}$$

(an arrow $i \longrightarrow j$ indicates the relation $i \prec j$), then the matrix

$$\Phi_j(\mathcal{N}) = \begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}$$

is not pure.

In order to guarantee the pure definitness in this case it is enough to substitute (x, y) by a full pair $(\mathcal{X}, \mathcal{Y})$ turning the polynomial $x - y$ into a invertible matrix. Theorem 3.3 asserts that there exists such a pair of matrices of size 20×20. As usual, a particular case admits a simpler solution than the one suggested by a general theory: it is enough to take

$$\mathcal{X} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & y & 1 & 0 \end{bmatrix} \quad \mathcal{Y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Below we present the matrix inverse to $\mathcal{X} - \mathcal{Y}$:

$$\begin{bmatrix} y & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ y & 0 & 0 & 1 \\ -1 + xy & x & -1 & x \end{bmatrix}.$$

6. On filtered representations of posets

Let S be a ring. Following [3] (see [25]) $rep_{fg}(I, S)$ denotes the category of filtered finitely generated S-representations of I. The objects of this category are systems $U = (U_i)_{i\in I}$ of finitely generated left S-modules such that $U_i \subseteq \bigoplus_{i\preceq p\in\max I} U_p \subseteq \bigoplus_{p\in\max I} U_p$ for every $i \in I$ and $\pi_j(U_i) \subseteq U_j$ provided $i \preceq j$ in I, where π_i is the composition of the canonical homomorphisms

$$\bigoplus_{p\in\max I} U_p \longrightarrow \bigoplus_{i\preceq p\in\max I} U_p \longrightarrow \bigoplus_{p\in\max I} U_p$$

for $i \in I$.

A morphism from $U = (U_i)_{i\in I}$ to $V = (V_i)_{i\in I}$ in $rep_{fg}(I, S)$ is an S-module homomorphism $f : \bigoplus_{p\in\max I} U_p \longrightarrow \bigoplus_{p\in\max I} V_p$ such that $f(U_i) \subseteq V_i$ for every $i \in I$.

Further, $\mathbf{fspr}(I, S)$ (resp. $\mathbf{fpr}(I, S)$) denotes the full subcategory of $rep_{fg}(I, S)$ with objects $(U_i)_{i\in I}$ such that $\bigoplus_{p\in\max I} U_p$ is a projective S-module (resp. $\bigoplus_{p\in\max I} U_p$ and $\bigoplus_{p\in\max I} U_p/U_i$ are free S-modules for all $i \in I$).

When $S = k$ is a field the categories $rep_{fg}(I, S)$, $\mathbf{fspr}(I, S)$ and $\mathbf{fpr}(I, S)$ coincide with the category $I - spr$ of peak I-spaces defined in [25].

Recall that

$$\mathbf{\Theta}'_B : \quad \mathrm{mod}\ (kI) \longrightarrow \mathrm{mod}_{sp}(kI)$$

is the functor, whose restriction to $\mathrm{prin}(kI)$ is the adjustment functor $\mathbf{\Theta}_B$ (see 2.8)

Recall from [25] the definition of the functor

$$\mathbf{\Theta}_I : \quad \mathrm{mod}\ (kI) \longrightarrow rep_{fg}(I, k).$$

Given a kI-module X identified with the triple $(X'_A, X''_B, \phi : X'_A \to |X''_B|)$ the representation $\mathbf{\Theta}_I(X)$ is defined as $(\overline{X}_i)_{i\in I}$ where $\overline{X}_p = X''_B e_p$ for $p \in \max I$ and $\overline{X}_i = \phi(X'_A e_i)$ for $i \in I^-$. This correspondence extends to a functor in a natural way.

It is proved in [25] that the restriction of this functor to $\mathrm{mod}_{sp}(kI)$ yields an equivalence of the categories $\mathrm{mod}_{sp}(kI)$ and $rep_{fg}(I, k)$. The inverse functor

$$\rho : rep_{fg}(I, k) \longrightarrow \mathrm{mod}_{sp}(kI)$$

sends a representation $(X_i)_{i\in I}$ to a kI-module X isomorphic to $\bigoplus_{i\in I} X_i$ with the right multiplication given by $x_i e_{jl} = \pi_l(x_i)$ when $i = j$ and $x_i e_{jl} = 0$ when $i \neq j$ for $j \preceq l$ and $x_i \in X_i$.

Given a ring S we extend the above definitions to S-representations of I: $\mathbf{\Theta}_I$ sends S-kI-bimodules to objects of $rep_{fg}(I, S)$ and ρ acts in a inverse direction.

The relevant properties of these correspondences are listed in the following lemma.

6.1. Lemma *Let N be a $\mathcal{W}$-kI-bimodule free finitely generated as a left $\mathcal{W}$-module and purely defined. Then*

(a) $\mathbf{\Theta}_I(N)$ *is an object of* $\mathbf{fpr}(I, \mathcal{W})$.

(b) *There is a natural isomorphism of k-representations of I:*

$$U \otimes_{\mathcal{W}} \mathbf{\Theta}_I(N) \cong \mathbf{\Theta}_I(U \otimes_{\mathcal{W}} N)$$

for any $U \in \mathrm{modf}(\mathcal{W})$ (the tensor product at the left hand side is defined in a natural way, see [3, Sect. 3]).

(c) *There is a natural isomorphism of kI-modules:*

$$\rho(U \otimes_{\mathcal{W}} \mathbf{\Theta}_I(N)) \cong U \otimes_{\mathcal{W}} \mathbf{\Theta}_B(N)$$

for any $U \in \mathrm{modf}(\mathcal{W})$.

Proof. The assertion (a) is a direct consequence of the definition of pure definitness, whereas (b) follows by 4.1, as in the proof of Lemma 5.2. In order to prove (c) observe that the functors $\rho \circ \mathbf{\Theta}_I$ and $\mathbf{\Theta}'_B$ are naturally equivalent and the proof follows by Lemma 5.2 and (b). □

Now we can formulate a version of the main statement of Theorem 2.9 in terms of the category $\mathbf{fpr}(I, k)$.

Corollary (cf. [3]). *Let I be a poset such that the category* $\mathrm{prin}(kI)$ *is fully k-wild. There exists an object N of* $\mathbf{fpr}(I, \mathcal{W})$ *such that the functor*

$$(-) \otimes_{\mathcal{W}} N : \mathrm{modf}(\mathcal{W}) \longrightarrow \mathbf{fpr}(I, k)$$

is exact, full and faithful. Therefore the category $\mathbf{fpr}(I, k)$ *is fully k-wild in the sense of [3].*

Proof. Let M be a purely defined $\mathcal{W}$-kI-bimodule defining the fully k-wildness of $\mathrm{prin}(kI)$. Such a bimodule exists by Corollary 5.3. Then the representation $N = \mathbf{\Theta}_I(M)$ satisfies the conditions of the corollary thanks to the lemma above, see the proof of Theorem 2.9. □

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