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Bimodule problems and cell complexes

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Abstract. We investigate the geometrical properties of the universal covering $\widehat{\mathcal{A}}$ of a bimodule problem $\mathcal{A}$.

Introduction

The paper deals with a study within the framework of the representation theory of bimodule problems ([6], [10]). The class Q_r of bimodule problems over $\Bbbk[[t]]$ introduced in [10] is considered. Any bimodule problem $\mathcal{A} \in Q_r$ is endowed with the standard multiplicative basis. It allows us to associate a two-dimensional cell complex $\mathfrak{L}$ with the problem $\mathcal{A}$ and to construct the Poincare groupoid and the universal covering bimodule problem $\widehat{\mathcal{A}}$ of $\mathcal{A}$ ([10], [11]). To investigate the representation type of $\widehat{\mathcal{A}}$ we use the geometrical technique of diagrams, contracting closed walks and quadratic form theory ([1]). The geometrical part of this technique has originally been developed as part of the geometrical group theory ([3], [4]). It turns out that some geometrical properties of $\mathfrak{L}$ imply some properties of the bimodule problem $\mathcal{A}$, in particular of its Tits quadratic form. It gives us a geometrical proof of a criterion of absence of minimal non-simply connected subproblems in $\mathcal{A}$.

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1. Basic notions

1.1. Bimodule problem and bigraph

We refer to [10], [11] for a detailed exposition. Let us fix an algebraically closed field $\Bbbk$. We say that a pair $\mathcal{A} = (\mathcal{C}, \mathcal{M})$ is a $\Bbbk$-bimodule problem, if $\mathcal{C}$ is a $\Bbbk$-category and $\mathcal{M}$ is a $\mathcal{C}$-bimodule. Besides, let us assume that $\mathcal{C}$ is local, and that both $\mathcal{C}$ and $\mathcal{M}$ are locally finite dimensional. For a bimodule problem $\mathcal{A} = (\mathcal{C}, \mathcal{M})$ the greatest ideal $\mathcal{I} \subset \operatorname{Rad}\mathcal{C}$ such that $\mathcal{I}\mathcal{M} = \mathcal{M}\mathcal{I} = 0$ is called the *annihilator* of $\mathcal{M}$ and is denoted by $\operatorname{Ann}_{\mathcal{C}}\mathcal{M}$.

Let us consider the category $\mathcal{C}_2 = \Bbbk[[t]] \oplus \Bbbk[[t]]$ such that $\operatorname{Ob}\mathcal{C}_2 = \{\mathtt{1}, \mathtt{2}\}$, $\mathcal{C}_2(\mathtt{1},\mathtt{1}) = \Bbbk[[t]]\cdot 1_{\mathtt{1}}$, $\mathcal{C}_2(\mathtt{2},\mathtt{2}) = \Bbbk[[t]]\cdot 1_{\mathtt{2}}$, $\mathcal{C}_2(\mathtt{1},\mathtt{2}) = 0$ and $\mathcal{C}_2(\mathtt{2},\mathtt{1}) = 0$, where $\Bbbk[[t]]$ is the power series ring in one variable over $\Bbbk$. For integers $n_1 > 0$, $n_2 > 0$, $n \geqslant 0$ let us consider the bimodule $\mathcal{M}_{n_1,n,n_2}$ over the category $\mathcal{C}_2$ such that $\mathcal{M}_{n_1,n,n_2}(\mathtt{1},\mathtt{2})$ is the vector space over the field $\Bbbk$ with a basis $v_1, \ldots, v_n$, and $\mathcal{M}_{n_1,n,n_2}(\mathtt{1},\mathtt{1}) = \mathcal{M}_{n_1,n,n_2}(\mathtt{2},\mathtt{1}) = \mathcal{M}_{n_1,n,n_2}(\mathtt{2},\mathtt{2}) = 0$, and for any $i = 1, 2, \ldots, n$

$$v_i(1_{\mathtt{1}}t) = \begin{cases} v_{i+n_1}, & \text{if } i + n_1 \leqslant n, \\ 0 & \text{otherwise,} \end{cases} \qquad (t1_{\mathtt{2}})v_i = \begin{cases} v_{i+n_2}, & \text{if } i + n_2 \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

The bimodule problem $\mathcal{A}_{n_1,n,n_2} = (\mathcal{C}_2/(\operatorname{Ann}_{\mathcal{C}_2}\mathcal{M}_{n_1,n,n_2}), \mathcal{M}_{n_1,n,n_2})$, where $n_1 = 1$ or $n_2 = 1$, is called the *standard uniserial bimodule problem* and is depicted by the oriented marked graph (diagram) $\underset{n_1}{\circ}\xrightarrow{\ \ n\ \ }\underset{n_2}{\circ}$. In the case when $n = 0$ we set $n_1 = n_2 = 0$ and depict the correspondent bimodule problem $\mathcal{A}_{0,0,0}$ by two disjoint vertices.

Let $\mathcal{C}_m$ be the category such that $\operatorname{Ob}\mathcal{C}_m = \{\mathtt{1}, \ldots, \mathtt{m}\}$, $\mathcal{C}_m(\mathtt{i},\mathtt{j}) = 0$, $\mathtt{i} \neq \mathtt{j}$, $\mathcal{C}_m(\mathtt{i},\mathtt{i}) = \Bbbk[[t]]\cdot 1_{\mathtt{i}}$, $\mathtt{i},\mathtt{j} \in \operatorname{Ob}\mathcal{C}_m$. Consider the bimodule problem $\mathcal{A} = (\mathcal{C}, \mathcal{M})$, where $\mathcal{M}$ is a $\mathcal{C}_m$-bimodule and $\mathcal{C} = \mathcal{C}_m/\operatorname{Ann}_{\mathcal{C}_m}\mathcal{M}$, such that for any $\mathtt{i},\mathtt{j} \in \operatorname{Ob}\mathcal{C}$, $\mathtt{i} \neq \mathtt{j}$, the restriction $\mathcal{A}_{\mathtt{i},\mathtt{j}}$ of bimodule problem $\mathcal{A}$ to these objects is equivalent to the standard uniserial bimodule problem $\mathcal{A}_{n_1,n,n_2}$. Such a bimodule problem can be decoded by the oriented marked graph $\Delta(\mathcal{A}) = (\Delta_0, \Delta_1)$, where the set of vertices is $\Delta_0 = \operatorname{Ob}\mathcal{C}$, and for any $\mathtt{i},\mathtt{j} \in \Delta_0$, $\mathtt{i} \neq \mathtt{j}$, the full subbigraph on these vertices is precisely the oriented marked graph for $\mathcal{A}_{\mathtt{i},\mathtt{j}}$. So given $\mathcal{A}_{\mathtt{i},\mathtt{j}} \sim \mathcal{A}_{n_1,n,n_2}$ we have $\Delta_1(\mathtt{i},\mathtt{j}) = \{a_{\mathtt{i},\mathtt{j}}\}$ if $n > 0$, where $a_{\mathtt{i},\mathtt{j}}$ is an unique arrow from $\mathtt{i}$ to $\mathtt{j}$, and $\Delta_1(\mathtt{i},\mathtt{j}) = \varnothing$ if $n = 0$. We say that an arrow $a_{\mathtt{i},\mathtt{j}} \in \Delta_1$ has *weight* $w(a_{\mathtt{i},\mathtt{j}}) = n \in \mathbb{Z}$ if $\mathcal{A}_{\mathtt{i},\mathtt{j}} \sim \mathcal{A}_{n_1,n,n_2}$ for some integers n_1, n_2.

Let us denote by Q_r the class of such bimodule problems $\mathcal{A}$ that have a connected tree-like graph $\Delta(\mathcal{A})$.

A *bigraph* $\Gamma = (\Gamma_0, \Gamma_1 = \Gamma_1^0 \cup \Gamma_1^1, s, e)$ consists of a set of *vertices* Γ_0, sets of *solid* and *dotted arrows* Γ_1^0 and Γ_1^1 ($\Gamma_1^0 \cap \Gamma_1^1 = \varnothing$) and a pair of maps $s, e : \Gamma_1 \to \Gamma_0$ that take an arrow x to its *starting* vertex $s(x)$ and its *ending* vertex $e(x)$ respectively. Let us denote the sets of all arrows x, solid arrows x and dotted arrows x, satisfying $s(x) = \mathtt{i}$ and $e(x) = \mathtt{j}$, by $\Gamma_1(\mathtt{i},\mathtt{j})$, $\Gamma_1^0(\mathtt{i},\mathtt{j})$ and $\Gamma_1^1(\mathtt{i},\mathtt{j})$ respectively.

The bigraph $\Gamma' = (\Gamma_0', \Gamma_1', s', e')$ is called the *subbigraph* of the bigraph $\Gamma = (\Gamma_0, \Gamma_1, s, e)$ if $\Gamma_0' \subset \Gamma_0$, $\Gamma_1' \subset \Gamma_1$, $s|_{\Gamma'} = s'$ and $e|_{\Gamma'} = e'$. A subbigraph Γ' of Γ is called *full* if $\Gamma_1'(\mathtt{i},\mathtt{j}) = \Gamma_1(\mathtt{i},\mathtt{j})$ for all $\mathtt{i},\mathtt{j} \in \Gamma_0'$. Every subbigraph $\Gamma' \subset \Gamma$ is contained in the unique full subbigraph $\Gamma'' \subset \Gamma$ such that $\Gamma_0' = \Gamma_0''$.

A *solid path* σ on Γ is defined as a sequence $\sigma = a_1 \dots a_k$ of solid arrows $a_1, \dots, a_k \in \Gamma_1^0$ such that $e(a_i) = s(a_{i+1})$ for $i = 1, \dots, k-1$, $k \in \mathbb{N}$. Let $s(\sigma) = s(a_1)$, $e(\sigma) = e(a_k)$. Given some $a \in \Gamma_1^0$ let us denote by a^{-1} the opposite arrow such that $s(a^{-1}) = e(a)$ and $e(a^{-1}) = s(a)$. Let $\widetilde{\Gamma}_1^0 = \Gamma_1^0 \cup \{a^{-1} \mid a \in \Gamma_1^0\}$. A *solid walk* σ on Γ is a path $\sigma = a_1 \dots a_k$ with $a_1, \dots, a_k \in \widetilde{\Gamma}_1^0$. A solid walk σ is called *closed* if $s(\sigma) = e(\sigma)$.

A *basis* of a bimodule problem $\mathcal{A} = (\mathcal{C}, \mathcal{M})$ is a bigraph $\Gamma\,(= \Gamma(\mathcal{A}))$ such that $\Gamma_0 = \operatorname{Ob}\mathcal{C}$, $\Gamma_1^0(\mathtt{i},\mathtt{j})$ is a basis in $\mathcal{M}(\mathtt{i},\mathtt{j})$, $\Gamma_1^1(\mathtt{i},\mathtt{j})$ is a basis in $\mathcal{C}(\mathtt{i},\mathtt{j})$ for $\mathtt{i} \neq \mathtt{j}$, and $\Gamma_1^1(\mathtt{i},\mathtt{i})$ is a basis in $\operatorname{Rad}\mathcal{C}(\mathtt{i},\mathtt{i})$, $\mathtt{i},\mathtt{j} \in \Gamma_0$. Note that $\Gamma_1^0(\mathtt{i},\mathtt{j}) \neq \varnothing$ if and only if $\Delta_1(\mathtt{i},\mathtt{j}) \neq \varnothing$. Then we can define the identification maps $\lambda : \Gamma_0(\mathcal{A}) \to \Delta_0(\mathcal{A})$ and $\lambda : \Gamma_1^0(\mathcal{A}) \to \Delta_1(\mathcal{A})$ by setting $\lambda(\mathtt{i}) = \mathtt{i}$ and $\lambda(x) = a_{\mathtt{i},\mathtt{j}}$ for any $x \in \Gamma_1^0(\mathtt{i},\mathtt{j})$, $\mathtt{i},\mathtt{j} \in \Gamma_0 = \Delta_0$.

A basis Γ is called *multiplicative* provided the composition of any two composable arrows is either 0 or an arrow in Γ. Any bimodule problem $\mathcal{A} \in Q_r$ is endowed with the standard multiplicative basis Γ ([10]).

Let us denote by $\operatorname{Rep}\mathcal{A}$ the category of representations of $\mathcal{A}$ ([10]).

1.2. Cell complex over the bimodule problem

We will use the following definition of a cell complex (see [7], chapter 5). Let X be a topological space. Also let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathfrak{L}^d = \{\mathrm{C}_\alpha^d \subset X \mid \alpha \in J_d\}$, where J_d is an index set, be a family of sets from X, $d \in \mathbb{N}_0$. Let us denote $\bigsqcup\limits_{d\in\mathbb{N}_0} \mathfrak{L}^d$ by $\mathfrak{L}$. We will call the set $\mathfrak{L}^{\leqslant d} = \bigsqcup\limits_{t \leqslant d} \mathfrak{L}^t$ the *d-skeleton* of $\mathfrak{L}$, $d \in \mathbb{N}_0$, and $X^d = \bigcup\limits_{\substack{\alpha\in J_t\\ t\leqslant d}} \mathrm{C}_\alpha^t$. For any $\mathrm{C}_\alpha^d \in \mathfrak{L}$ the set $\mathrm{C}_\alpha^{d^\bullet} = \mathrm{C}_\alpha^d \cap X^{d-1}$ is called the *boundary* of C_α^d. The set $\mathrm{C}_\alpha^{d^\circ} = \mathrm{C}_\alpha^d \setminus \mathrm{C}_\alpha^{d^\bullet}$ is called the *interior* of C_α^d.

The family $\mathfrak{L}$ is called the *cell complex* on X provided:

1. $X = \bigcup\limits_{\mathrm{C}_\alpha^d \in \mathfrak{L}} \mathrm{C}_\alpha^d$;

2. $\mathrm{C}_\alpha^{d^\circ} \cap \mathrm{C}_\beta^{d'^\circ} \neq \varnothing$ implies that $d = d'$, $\alpha = \beta$;

3. for any $\mathrm{C}^d_\alpha \in \mathfrak{L}$ there exists a surjective map of pairs

$$f^d_\alpha : (D^d, S^{d-1}) \to (\mathrm{C}^d_\alpha, \mathrm{C}^{d\bullet}_\alpha)$$

such that f^d_α induces a homeomorphism $\operatorname{Int} D^d \to \mathrm{C}^{d\circ}_\alpha$, where D^d is the d-dimensional disk in $\mathbb{R}^d$, $\operatorname{Int} D^d$ is its interior and the sphere S^{d-1} is its boundary.

A set C^d_α is called the *d-cell* or the *d-dimensional cell*. The map f^d_α is called the *characteristic mapping* of the cell C^d_α.

A cell complex $\mathfrak{L}$ is called a *d-complex* or a complex of *dimension* $d = \dim \mathfrak{L}$ if $\mathfrak{L}^k = \varnothing$ for all $k > d$ but $\mathfrak{L}^d \neq \varnothing$. The cell complex structure $\mathfrak{L}$ on X induces the structure of cell complexes $\mathfrak{L}^{\leqslant k} = \mathfrak{L}^{\leqslant k}(X)$ on X^k, $k \in \mathbb{N}_0$.

If X, Y are topological spaces with cell complex structures $\mathfrak{L}(X)$, $\mathfrak{L}(Y)$ respectively, then a continuous map $f : X \to Y$ is called a *cellular map*, provided it maps the k-th skeleton $\mathfrak{L}(X)^{\leqslant k}$ to the k-th skeleton $\mathfrak{L}(Y)^{\leqslant k}$ for all $0 \leqslant k \leqslant \dim \mathfrak{L}$.

The cell $\mathrm{C}^{d'}_{\alpha'}$ is called the *face* of the cell C^d_α if $\mathrm{C}^{d'\circ}_{\alpha'} \subset \mathrm{C}^d_\alpha$. The cell spaces under consideration satisfy the condition of a CW-complex (see [7], chapter 5). Namely, every cell has a finite number of faces and the space is endowed with a weak topology. We consider here only the so called *combinatorial* cellular maps (see [5]), i. e. for any cell C of $\mathfrak{L}(X)$ the map f induces a homeomorphism of C° onto C'° for some cell $\mathrm{C}' \in \mathfrak{L}(Y)$.

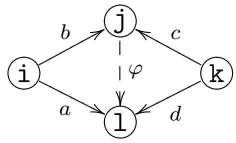
Given a complex $\mathfrak{L}$ and some $\mathrm{C}^0 \in \mathfrak{L}^0$ let us denote by $\mathfrak{L}_{\mathrm{C}^0}$ the subcomplex of $\mathfrak{L}$ that contains C^0 and all the cells $\mathrm{C} \in \mathfrak{L}$ such that $\mathrm{C}^0 \in \mathrm{C}^\bullet$. The complex $\mathfrak{L}_{\mathrm{C}^0}$ is called the *star* of the 0-cell C^0 in $\mathfrak{L}$.

For $n > 0$ define the *cycle* $\mathfrak{C}(n)$ of length n as the 1-complex with $\mathfrak{C}(n)^0 = \{\mathtt{1}, \ldots, \mathtt{n}\}$, $\mathfrak{C}(n)^1 = \{x_1, \ldots, x_n\}$, $\mathfrak{C}(n)^k = \varnothing$ for $k \geqslant 2$, such that $x_i{}^\bullet = \{\mathtt{i}, \mathtt{i+1}\}$ if $i < n$ and $x_n{}^\bullet = \{\mathtt{n}, \mathtt{1}\}$.

Let the bigraph Γ be a multiplicaive basis of the bimodule problem $\mathcal{A}$. Let us define the 2-complex $\mathfrak{L}(\mathcal{A})$. Let us set $\mathfrak{L}^0 = \Gamma_0 \times \{D^0\}$, $\mathfrak{L}^1 = \Gamma^0_1 \times \{D^1\}$, and let us denote by $\mathrm{C}^0_{\mathtt{i}} = (\mathtt{i}, D^0) \in \mathfrak{L}^0$ for any $\mathtt{i} \in \Gamma_0$ and $\mathrm{C}^1_a = (a, D^1) \in \mathfrak{L}^1$ for any $a \in \Gamma^0_1$. Then $\mathrm{C}^{1\bullet}_a = \{\mathrm{C}^0_{s(a)}, \mathrm{C}^0_{e(a)}\}$ for each $a \in \Gamma^0_1$. The characteristic mappings are now defined in a straightforward fashion. The 1-dimensional complex $\mathfrak{L}^{\leqslant 1}(\mathcal{A})$ we have obtained is the 1-skeleton of $\mathfrak{L}(\mathcal{A})$. As a topological space, $\mathfrak{L}^{\leqslant 1}(\mathcal{A})$ is homeomorphic to the subbigraph in Γ formed by the solid arrows.

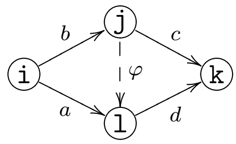
The structure of 2-cells is defined by multiplication in $\mathcal{A}$. The 2-cell on $\mathcal{A}$ corresponds to the family (a, b, c, d, φ) for the first three cases below and to the family $(a, b, c, d, \varphi, \psi)$ for the fourth case:

1) $\mathrm{C}^2_{a,b,c,d,\varphi}$:



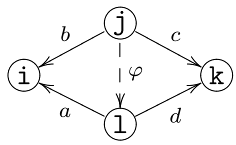
$\varphi b = a, \qquad \varphi c = d;$

2) $\mathrm{C}^2_{a,b,c,d,\varphi}$:



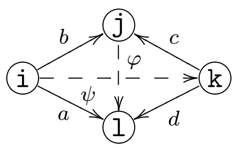
$\varphi b = a, \qquad d\varphi = c;$

3) $\mathrm{C}^2_{a,b,c,d,\varphi}$:



$a\varphi = b, \qquad d\varphi = c;$

4) $\mathrm{C}^2_{a,b,c,d,\varphi,\psi}$:



$\varphi b = a, \qquad \varphi c = d;$
$c\psi = b; \qquad d\psi = a,$

where $a,b,c,d \in \Gamma_1^0$, $\varphi,\psi \in \Gamma_1^1$. Namely, $\mathfrak{L}^2 = \{(a,b,c,d,\varphi,D^2) \mid \varphi b = a, \varphi c = d\} \cup \{(a,b,c,d,\varphi,D^2) \mid \varphi b = a, d\varphi = c\} \cup \{(a,b,c,d,\varphi,D^2) \mid a\varphi = b, d\varphi = c\} \cup \{(a,b,c,d,\varphi,\psi) \mid \varphi b = a, \varphi c = d, c\psi = b, d\psi = a\}$. The labels on the arrows and the vertices denote the images of the identification map, restricted to the boundary of corresponding cell. We denote the 2-cells by $\mathrm{C}^2_{a,b,c,d,\varphi}$ for the first three cases and $\mathrm{C}^2_{a,b,c,d,\varphi,\psi}$ for the last one.

Note that the cells defined in the cases 1)-4) above are endowed with an extra structure, namely, the inner oriented paths of these cells correspond to the dotted arrows $\varphi,\psi \in \Gamma_1^1$.

Let $\sigma = a_1 \dots a_n$ be a closed walk on Γ. Then a *contracting diagram* for σ is defined as a structure of the 2-complex $\mathfrak{L} = \mathfrak{L}(\sigma)$ on a contractible subspace $X \subset \mathbb{R}^2$ with the following properties.

1. The interior ${\mathrm{C}^1}^\circ$ belongs to at most two 2-cells in $\mathfrak{L}$ for any $\mathrm{C}^1 \in \mathfrak{L}^1$.
2. There is a cellular map $\imath_X : \mathfrak{C}(n) \to \mathfrak{L}$, $n > 0$, such that a unique unbounded connected component of $\mathbb{R}^2 \backslash \mathrm{Im}\, \imath_X$ coincides with $\mathbb{R}^2 \backslash X$ and the pre-image $\imath_X^{-1}({\mathrm{C}^1}^\circ)$ of the interior of C^1 belongs to at most two 1-cells in $\mathfrak{C}(n)$ for any $\mathrm{C}^1 \in \mathfrak{L}^1$.
3. There is a *marking* cellular map $\ell : \mathfrak{L} \to \mathfrak{L}(\mathcal{A})$, such that $\ell\imath_X(x_i) = a_i$ for all $i = 1, \dots, n$.

In this case the 1-complex $\mathfrak{B} = \imath_X(\mathfrak{C}(n))$ is called the *outer boundary* of the contractible 2-complex $\mathfrak{L}$. Note that a contracting diagram for σ is by no means unique.

Given $\mathfrak{L}(\sigma)$ we divide $\mathfrak{L}^0 = \operatorname{out}\mathfrak{L} \cup \operatorname{inn}\mathfrak{L}$ into two disjoint subsets of the boundary 0-cells, i. e. the ones belonging to $\mathfrak{B}(\sigma)$, and the inner ones. Two different 2-cells $\mathrm{C}_1^2, \mathrm{C}_2^2 \in \mathfrak{L}^2$ are called *neighbour* if the intersection $\mathrm{C}_1^{2\bullet} \cap \mathrm{C}_2^{2\bullet}$ contains at least one 1-cell.

A contracting diagram $\mathfrak{L}(\sigma)$ is called *minimal* if every other contracting diagram for the same closed walk σ contains either at least as many 2-cells or, if the number of 2-cells is the same, at least as many inner vertices.

We assume σ to be a reduced solid closed walk, in the sense that it does not contain any subwalks of the form aa^{-1}, $a^{-1}a$, $a \in \Gamma_1^0$, and without self intersections. The contracting diagram $\mathfrak{L}(\sigma)$ is called *reduced* provided $\ell(\mathrm{C}_1^2) \neq \ell(\mathrm{C}_2^2)$ for any neighbour cells $\mathrm{C}_1^2, \mathrm{C}_2^2 \in \mathfrak{L}^2$. A minimal contracting diagram is reduced.

1.3. Quadratic form and universal covering

An integer unit quadratic form in n variables is a polynomial

$$q(X) = \sum_{i=1}^{n} X_i^2 + \sum_{1\leqslant i<j\leqslant n} q_{ij}X_iX_j, \quad q_{ij} \in \mathbb{Z},\ X = (X_1, \ldots, X_n).$$

A root $x \in \mathbb{Z}^n$ of the equation $q(X) = 1$ is called a *positive root* of $q(X)$ if $x > 0$, i. e. $x \neq 0$ and all $x_i \geqslant 0$. Let us denote by $\mathbb{E}_q^+ \subset \mathbb{Z}^n$ the set of all positive roots of $q(X)$. The standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ are called the *simple* roots of $q(X)$. Let us denote by $(\ ,\ )_q$ the symmetrical bilinear form associated with $q(X)$. The linear map $w_i : \mathbb{Q}^n \to \mathbb{Q}^n$, $x \mapsto w_i(x) = x - (x, e_i)_q e_i$, is called the i-th *reflection map*.

A quadratic form $q(X)$ is called *weakly positive* if $q(x) > 0$ for all $x \in \mathbb{Q}^n$, $x > 0$. $q(X)$ is weakly positive if and only if $|\mathbb{E}_q^+| < \infty$ (see [1]).

The *quadratic form (Tits form)* of a bimodule problem $\mathcal{A}$ with a basis Γ is defined by

$$\chi_{\mathcal{A}}(X) = \sum_{\mathtt{i}\in\Gamma_0} X_{\mathtt{i}}^2 + \sum_{\mathtt{i},\mathtt{j}\in\Gamma_0} (|\Gamma_1^1(\mathtt{i},\mathtt{j})| - |\Gamma_1^0(\mathtt{i},\mathtt{j})|)X_{\mathtt{i}}X_{\mathtt{j}}.$$

If $\chi_{\mathcal{A}}(X)$ is not weakly positive, then $\mathcal{A}$ is of a strictly unbounded type [2], [6].

Let $\mathcal{A} \in Q_r$. We can construct, in a standard way, the universal covering bimodule problem $\widehat{\mathcal{A}}$ and the covering morphism $\pi : \widehat{\mathcal{A}} \to \mathcal{A}$ associated with the multiplicative basis Γ of the bimodule problem $\mathcal{A}$ (see [10]). There always exists a basis $\widehat{\Gamma} = \widehat{\Gamma}(\widehat{\mathcal{A}})$ of $\widehat{\mathcal{A}}$ such that $\pi(\widehat{\Gamma}) = \Gamma$. This basis $\widehat{\Gamma}$ does not contain any loops or parallel arrows. Let us denote by $\widehat{\chi} = \chi_{\widehat{\mathcal{A}}}$ the Tits form of the covering bimodule problem $\widehat{\mathcal{A}}$.

Now let us introduce the identification map $\widehat{\lambda} : \widehat{\Gamma}_1^0(\widehat{\mathcal{A}}) \to \Delta_1(\mathcal{A})$ as the composition $\widehat{\lambda} = \lambda\pi$. For any $x \in \widehat{\Gamma}_1^0(\widehat{\mathcal{A}})$ the element $\widehat{\lambda}(x)$ is called the *label* of x.

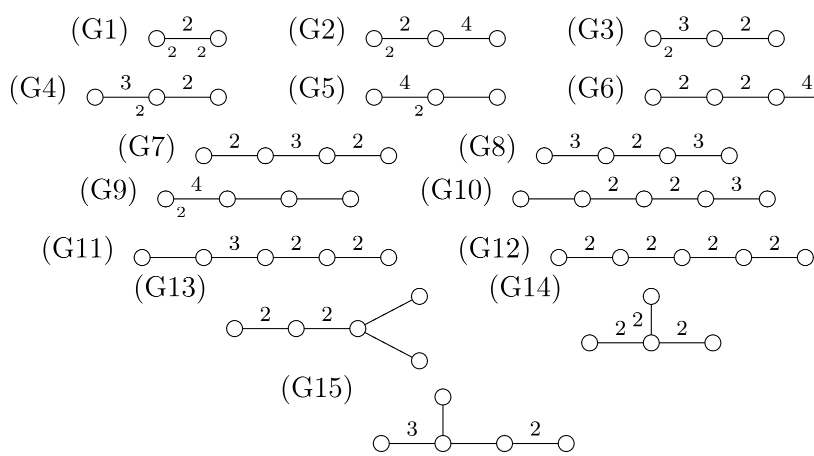
The bimodule problem $\mathcal{A}$ is called *simply connected* with respect to the basis Γ if $\widehat{\mathcal{A}} = \mathcal{A}$.

The bimodule problem $\mathcal{A}$ is called *absolutely simply connected* with respect to the basis Γ if for any indecomposable representation $M \in \operatorname{Rep}\mathcal{A}$ the subproblem $\mathcal{A}_{\operatorname{supp} M}$ is simply connected with respect to the correspondent subbasis of Γ.

If $\operatorname{Ann}_{\mathcal{C}} \mathcal{M} \neq 0$ then the bimodule problem $\mathcal{A}$ is not simply connected. The bimodule problem $\mathcal{A} = (\mathcal{C}, \mathcal{M})$ is called *trivially non simply connected* provided $\operatorname{Ann}_{\mathcal{C}} \mathcal{M} \neq 0$ and $\mathcal{A}' = (\mathcal{C}/\operatorname{Ann}_{\mathcal{C}} \mathcal{M}, \mathcal{M})$ is an absolutely simply connected bimodule problem, and $\mathcal{A}$ is called *minimal trivially non simply connected* if in addition to the above each proper sincere subproblem of $\mathcal{A}$ is absolutely simply connected.

Let $\mathcal{A} = (\mathcal{C}, \mathcal{M})$ be a bimodule problem with a basis Γ and a weakly positive Tits form χ. A vertex $\mathtt{i} \in \Gamma_0$ is called *special* for a root $x \in \mathbb{E}_\chi^+$ if $x_{\mathtt{i}} = 1$ and $w_{\mathtt{i}}(x) = x - e_{\mathtt{i}}$. A root $x \in \mathbb{E}_\chi^+$ is called *special* if x has two special vertices $\mathtt{i}, \mathtt{j} \in \Gamma_0$, $\mathtt{i} \neq \mathtt{j}$, and $w_{\mathtt{k}}(x) = x$ for any $\mathtt{k} \in \Gamma_0 \setminus \{\mathtt{i}, \mathtt{j}\}$. If $x \in \mathbb{E}_\chi^+$ is the smallest non-special root, then x has at least 3 special vertices. It follows immediately that a minimal trivially non simply connected bimodule problem $\mathcal{A}$ with $|\Gamma_0| \geqslant 3$ has some vertices $\mathtt{i}, \mathtt{j} \in \Gamma_0$ such that $(\operatorname{Ann}_{\mathcal{C}} \mathcal{M})|_{\Gamma_0 \setminus \{\mathtt{i}\}} = 0$ and $(\operatorname{Ann}_{\mathcal{C}} \mathcal{M})|_{\Gamma_0 \setminus \{\mathtt{j}\}} = 0$. Moreover, each sincere positive root of χ is special with the special vertices $\mathtt{i}, \mathtt{j}$.

Theorem 1 ([10])**.** *For the bimodule problem $\mathcal{A} \in Q_r$ containing as a subproblem one of the bimodule problems*



the quadratic Tits form $\widehat{\chi}$ of the universal covering $\widehat{\mathcal{A}}$ with respect to the standard multiplicative basis is not weakly positive.

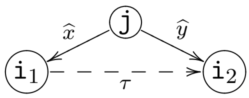
We exclude from further considerations the bimodule problems that contain critical bimodule problems from the list above.

2. Contracting diagram

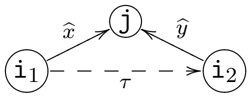
Let $\widehat{\mathcal{A}}$ be the universal covering for a bimodule problem $\mathcal{A} \in Q_r$ associated with multiplicative basis Γ, $\pi : \widehat{\mathcal{A}} \to \mathcal{A}$ be the covering morphism, and let $\widehat{\Gamma}$ be a multiplicative basis of $\widehat{\mathcal{A}}$ such that $\pi(\widehat{\Gamma}) = \Gamma$. Let us denote by the $\mathfrak{L}(\widehat{\mathcal{A}}) = (\mathfrak{L}^0, \mathfrak{L}^1, \mathfrak{L}^2)$ the 2-dimensional cell complex over $\widehat{\mathcal{A}}$.

Each of the following full subbigraphs of $\widehat{\Gamma}$ is called a *triangle*:

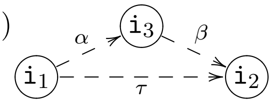
(T1)



(T2)



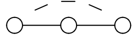
(T3)

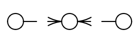
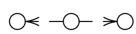


Here, for the first two cases, $\widehat{\lambda}(\widehat{x}) = \widehat{\lambda}(\widehat{y}) \in \Delta_1$, $\pi(\mathtt{i}_1) = \pi(\mathtt{i}_2)$, $\pi(\tau) \in \Gamma_1^1(\pi(\mathtt{i}_1), \pi(\mathtt{i}_1))$, and $\pi(\tau)\pi(\widehat{x}) = \pi(\widehat{y})$ or $\pi(\widehat{y})\pi(\tau) = \pi(\widehat{x})$ for (T1) and (T2) respectively. For the third case $\pi(\mathtt{i}_1) = \pi(\mathtt{i}_2) = \pi(\mathtt{i}_3)$ and $\pi(\beta)\pi(\alpha) = \pi(\tau)$.

The following structure lemma follows from the construction of $\widehat{\mathcal{A}}$.

Lemma 1. *1.* $|\widehat{\Gamma}_1(\mathtt{i}, \mathtt{j}) \cup \widehat{\Gamma}_1(\mathtt{j}, \mathtt{i})| \leqslant 1$ *for all* $\mathtt{i}, \mathtt{j} \in \widehat{\Gamma}_0$, $\mathtt{i} \neq \mathtt{j}$.

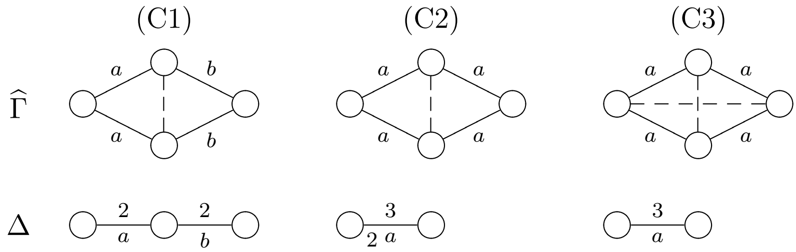
2. Each subbigraph of the form  *on* $\widehat{\Gamma}$ *is* (T1) *or* (T2).

3. For each subbigraph on $\widehat{\Gamma}$ *of the form*  *or*  *the full completed subbigraph is* (T3).

4. There are no oriented cycles on $\widehat{\Gamma}$.

On the diagrams below we will attach to any solid edge $\widehat{x} \in \widehat{\Gamma}_1$ its label $x = \widehat{\lambda}(\widehat{x}) \in \Delta_1$. Sometimes we will omit the orientation of edges on the diagrams and assume it to be suitable.

The 2-cell from $\mathfrak{L}^2(\widehat{\mathcal{A}})$ is a gluing of two triangles of the type (T1) or (T2) along the common dotted edge. We have the following cases (with the suitable orientation of edges) on $\widehat{\Gamma}$ (and on Δ):



Here and below we write a instead of C^1_a, $\mathtt{i}$ instead of $\mathrm{C}^0_{\mathtt{i}}$ etc. The edges marked with the same label have the same images under the map λ. The second line of diagrams denotes the underlying graph Δ.

Lemma 2. *Let the Tits form $\widehat{\chi}(X)$ of $\widehat{\mathcal{A}}$ be weakly positive. Then each solid closed quadrangle on $\widehat{\Gamma}$ is of a type* (C1)*,* (C2) *or* (C3)*.*

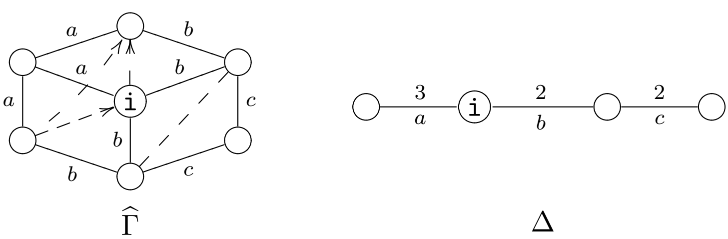
The proof follows from lemma 1 and the weak positivity condition.

Lemma 3. *There exists the contracting diagram $\mathfrak{L}(\widehat{\sigma})$ for any solid closed walk $\widehat{\sigma}$ on $\widehat{\Gamma}$.*

Proof. Since $\widehat{\Gamma}$ is simply connected, any solid closed walk $\widehat{\sigma}$ on $\widehat{\Gamma}$ can be presented as the triangle contracting diagram [3]. Since the annihilator of $\widehat{\mathcal{A}}$ is trivial, each triangle of type (T3) is a gluing of three triangles of type (T1) or (T2). Hence the triangle contracting diagram can be modified in a straightforward fashion into the contracting diagram $\mathfrak{L}(\widehat{\sigma})$ with 2-cells of the form (C1), (C2), (C3). □

Lemma 4. *Let $\mathfrak{L}(\widehat{\sigma})$ be the minimal reduced contracting diagram of a solid closed walk $\widehat{\sigma}$ on $\widehat{\Gamma}$. Given an inner* 0*-cell $\mathrm{C}^0_{\mathtt{i}} \in \operatorname{inn}\mathfrak{L}$ one of the following holds:*

1. *The initial bimodule problem has one of the critical ones* (G1)*,* (G3)*,* (G4)*,* (G6)*,* (G7)*,* (G8)*,* (G10)*,* (G11)*,* (G12)*,* (G13)*,* (G14)*,* (G15) *as a subproblem.*
2. *The star $\mathfrak{L}_{\mathrm{C}^0_{\mathtt{i}}}$ is of the following type:*



(1)

Proof. If the 1-cell $\mathrm{C}^1_a \in \mathfrak{L}^1$ is a direct face of $\mathrm{C} \in \mathfrak{L}^2$, then the weight of $\pi(a)$ is at least 2. Therefore, due to the exception of the problem (G14), the 0-cell $\mathrm{C}^0_{\mathtt{i}}$ is a direct face of 1-cells corresponding to the arrows having at most two different labels. The rest of the proof uses some combinatorial technique, the associativity condition for multiplication in $\mathcal{A}$ and the minimality assumption. $\square$

Note that if the bimodule problem $\mathcal{A}$ has a subproblem (1) then it does not have any additional edge in $\Delta(\mathcal{A})$, since otherwise $\mathcal{A}$ would have one of the critical subproblems (G10), (G11), (G9), (G15). Moreover, the edges a, b, c can not have the greater weight since otherwise $\mathcal{A}$ would have one of the critical problems (G6), (G7), (G8).

3. Main result

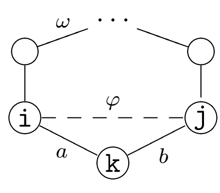
Theorem 2. *Let the bimodule problem $\mathcal{A} \in Q_r$ contain no critical subproblem* (G1)–(G15) *and let the Tits form $\chi_{\widehat{\mathcal{A}}}(X)$ of the universal covering $\widehat{\mathcal{A}}$ be weakly positive. Then $\widehat{\mathcal{A}}$ has no minimal trivially non simply connected bimodule subproblem.*

Proof. The proof is carried out in the following 10 steps.

1. Theorem 2 holds for the subbigraph (1) from lemma 4. The proof for this case may be given directly. Now we can assume that the minimal reduced contracting diagram does not contain any inner 0-cell.

2. Suppose there exists a minimal trivially non simply connected bimodule subproblem $\widehat{\mathcal{A}}_S = (\widehat{\mathcal{C}}_S, \widehat{\mathcal{M}}_S)$ on $S \subset \widehat{\Gamma}_0$. Then there are two special vertices $\mathtt{i}, \mathtt{j} \in S$ and $\varphi \in \widehat{\Gamma}^1_1(\mathtt{i}, \mathtt{j})$ such that $\mathrm{Ann}_{\widehat{\mathcal{C}}_S} \widehat{\mathcal{M}}_S = \{\varphi\}$ (by statement 1 of lemma 1). Let us consider the absolutely simply connected bimodule problem $\widehat{\mathcal{A}}'_S = (\widehat{\mathcal{C}}_S/\mathrm{Ann}_{\widehat{\mathcal{C}}_S} \widehat{\mathcal{M}}_S, \widehat{\mathcal{M}}_S)$. Let x be the minimal sincere positive root of $\widehat{\chi}' = \chi_{\widehat{\mathcal{A}}'_S}$ with two special vertices $\mathtt{i}, \mathtt{j}$.

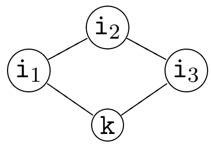
3. The vertices $\mathtt{i}, \mathtt{j}$ are connected with the solid walk $\omega : \mathtt{i} \to \mathtt{j}$ on the bigraph $\widehat{\Gamma}_S$ since $\widehat{\chi}'$ is a sincere form and the bigraph $\widehat{\Gamma}_S$ is connected by solid edges. Using the fact that the global annihilator $\mathrm{Ann}_{\widehat{\mathcal{C}}} \widehat{\mathcal{M}}$ is trivial we obtain the existence of a $\mathtt{k} \in \widehat{\Gamma}_0 \setminus S$ such that the subbigraph $\widehat{\Gamma}_{S \cup \{\mathtt{k}\}}$ contains a triangle of the type either (T1) or (T2) with dotted arrow φ. Hence we obtain the following subbigraph



where $a, b \in \widehat{\Gamma}_1^0$, and either $s(a) = s(b) = \mathtt{k}$ or $e(a) = e(b) = \mathtt{k}$.

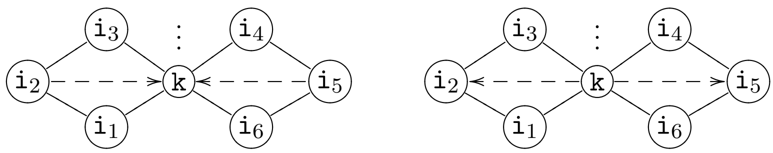
4. By lemma 3 there exists the minimal contracting diagram $\mathfrak{L}$ of the solid closed walk $\widehat{\sigma} = \omega b^\beta a^\alpha : \mathtt{i} \to \mathtt{i}$ on $\widehat{\Gamma}$ for the suitable $\alpha, \beta = \pm 1$. In addition, among the solid walks $\omega : \mathtt{i} \to \mathtt{j}$ on $\widehat{\Gamma}_S$ we choose the one with a minimal contracting diagram $\mathfrak{L}$.

Then any 2-cell from $\mathfrak{L}$ is of the type



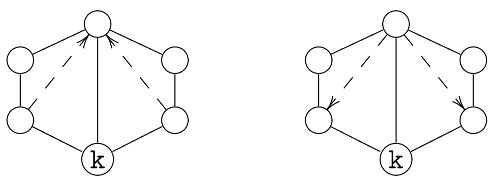
with $\mathtt{i}_1$, $\mathtt{i}_2$, $\mathtt{i}_3$ the consecutive vertices of ω and $\mathtt{i}_1$, $\mathtt{i}_3$ or (and) $\mathtt{i}_2$, $\mathtt{k}$ connected with a dotted arrow.

5. The cases



are impossible on $\mathfrak{L}$. Indeed, if first case were possible, statement 3 of lemma 1 would imply the existence of a dotted arrow between the vertices $\mathtt{i}_2$ and $\mathtt{i}_5$. Then, by the associativity of multiplication, there would exist a solid edge either between $\mathtt{i}_1$, $\mathtt{i}_4$ or between $\mathtt{i}_3$, $\mathtt{i}_6$, which would contradict the minimality of $\mathfrak{L}$. The impossibility of the second case may be shown in a similar way.

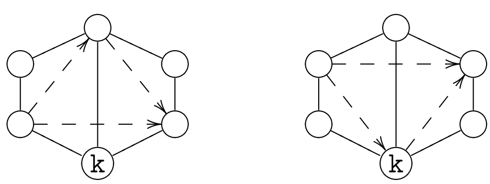
6. The cases



are impossible on $\mathfrak{L}$ for similar reasons.

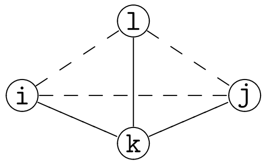
7. Assume that $\mathfrak{L}$ contains a cell of the type (C2). Then, excluding the critical problems (G3), (G4), we obtain that all 2-cells from $\mathfrak{L}$ are of the type (C2) or (C3) (with the same label a).

8. Given one of the cases



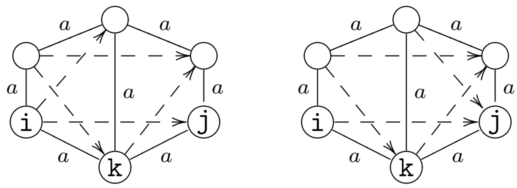
and excluding (G3), (G4), (G14), we conclude that at least one of the pictured 2-cells is of the type (C3).

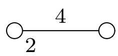
9. The cell complex $\mathfrak{L}$ does not have the fragment



with $\mathtt{l} \in S$, since otherwise, given the root $x' = w_{\mathtt{j}} w_{\mathtt{i}}(x) \in \mathbb{E}^{+}_{\widehat{\mathcal{A}}'_S}$, the $\mathtt{l}$-th coordinate of $w_{\mathtt{l}}(x') - x'$ would be at least 2, thereby contradicting the weak positivity of $\widehat{\chi}'$.

10. We conclude that there exist just two 2-cells from $\mathfrak{L}$ and at least one of them is of the type (C3). Therefore we have one of the following fragments:



which gives the subbigraph  of Δ.

Thus, excluding (G3), (G5), (G9), we obtain only two such problems, the proof in these cases being combinatorial and simple. □

Corollary 1. *Let the bimodule problem $\mathcal{A} \in Q_r$ contain no critical subproblem* (G1)–(G15) *and let the Tits form $\chi_{\widehat{\mathcal{A}}}(X)$ of the universal covering $\widehat{\mathcal{A}}$ be weakly positive. Then the bimodule problem $\widehat{\mathcal{A}}$ is absolutely simply connected.*

4. Conclusive remarks

The authors are positive that such a geometrical technique may be effectively used in some classes of bimodule problems endowed with a multiplicative basis.

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