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Free products of finite groups acting on regular rooted trees

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. Let finite number of finite groups be given. Let n be the largest order of their composition factors. We prove explicitly that the group of finite state automorphisms of rooted n-tree contains subgroups isomorphic to the free product of given groups.

1. Introduction

In last time many authors pay attention to automorphism groups or rooted tree. There are two main reasons for it. First, these groups help to obtain many deep purely algebraic results. Second, in studying automorphism groups of rooted trees many interesting connections between algebra, automata theory, dynamical systems, functional analysis etc. arise. For detailed overview see [GNS] and references therein.

We continue in the presented paper to study the subgroup structure of the automorphism group of a rooted tree. We consider free products of finite groups acting on regular rooted trees by the so-called finite state automorphisms. This is a natural question, since in the automorphism group of a regular rooted tree "almost all" subgroups are free [3, 4] and free products are rich in free subgroups. For previous results on free products acting on rooted trees see [5, 6, 2, 7, 8].

The main result of our paper is the following sufficient condition of existence of a faithful action on the regular rooted tree T_n by finite state automorphisms.

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Theorem 5.2. *Let $H_1, \dots, H_k$ be finite groups and suppose that the orders of all composition factors of each H_i are bounded above by n. Then the free product $H_1 * \dots * H_k$ acts faithfully on the regular rooted tree T_n by finite state automorphisms.*

The work is organized as follows. In Section 2 all necessary definitions and notations are given. In Section 3 we give some necessary conditions of existence of a faithful action by finite state automorphisms on T_n of a given free product of finite groups. The section 4 is the main part of the paper. It contains an explicit construction used in the proof of the main result and of the corollaries in Section 5. Section 6 contains an example of a faithful action, constructed by the methods of Section 4. And the last Section 7 is devoted to some open question concerning free products and groups of finite state automorphisms of regular rooted trees.

2. Preliminaries

For more details on the contents of this section see, for example, [GNS]. Let X be a finite alphabet, $|X| = n \geq 2$. Denote by X^* the free monoid generated by X. In other words, X^* is the set of all finite words over the alphabet X, including the empty word Λ, with the operation of concatenation. We identify X^* with the disjoint union of the Cartesian powers of X

$$\bigcup_{i \geq 0} X^i,$$

where $X^0 = \{\Lambda\}$. The length $|v|$ of a word $v \in X^*$ is $i \geq 0$ such that $v \in X^i$. Also consider the set $X^{\mathbb{N}}$ of all sequences (or the so called ω-words) of the form

$$x_1 x_2 x_3 \dots,$$

where $x_i \in X, i \geq 1$.

Define a partial order "$<$" on X^* by the rule

$$\text{for } u, v \in X^* \ u < v \quad \text{iff} \quad u = vv_1 \text{for some } v_1 \in X^*, v_1 \neq \Lambda.$$

Then the diagram of the poset $(X^*, <)$ is a regular rooted tree denoted T_n. The set of vertices of T_n is X^*, Λ is the root and two vertices u, v are connected if and only if for some $x \in X$ we have $u = vx$ or $v = ux$. All vertices of this tree are naturally partitioned into levels, where the ith level is X^i, $i \geq 0$. Each vertex from the ith level ($i \geq 0$) is adjacent with n vertices from the $(i + 1)$st level and with one vertex from the $(i - 1)$st level, except for $i = 0$.

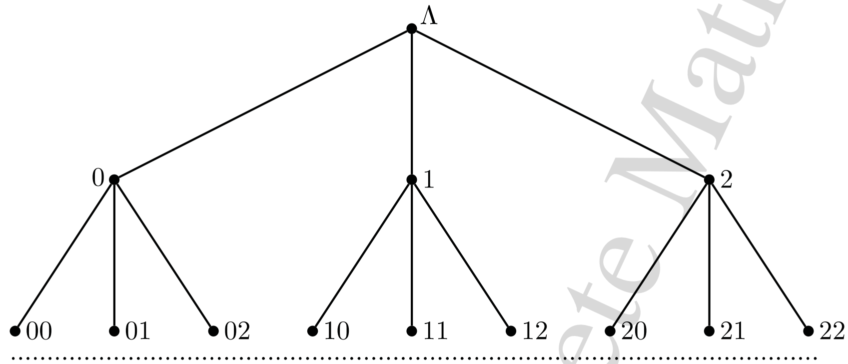

Figure 1: Regular rooted tree T_3, constructed for the alphabet $X = \{0, 1, 2\}$

Denote the automorphism group of the rooted tree T_n by GA_n. Every automorphism of T_n fixes the root Λ. This means, that for given automorphism $f \in GA_n$, word $u \in X^*$ and $x \in X$ we have

$$(ux)^f = u^f x^{\pi(u)},$$

where $\pi(u)$ is a permutation from the symmetric group S_n, depending only on the word u. This property leads us to a definition of the action of GA_n on $X^{\mathbb{N}}$. Namely, for $f \in GA_n$ and $w = x_1 x_2 x_3 \dots x_n \dots \in X^{\mathbb{N}}$, then

$$w^f = x_1^{\pi_\Lambda} x_2^{\pi_{x_1}} x_3^{\pi_{x_1 x_2}} \dots x_n^{\pi_{x_1 x_2 \dots x_{n-1}}}.$$

We obtain in this way the permutation group $(GA_n, X^{\mathbb{N}})$. As a permutation group this group is isomorphic to the infinitely iterated wreath product of symmetric groups S_n:

$$(GA_n, X^{\mathbb{N}}) \simeq \left(\wr_{i=1}^{\infty} S_n, X^{\mathbb{N}} \right).$$

In particular, this means that for arbitrary $k \in \mathbb{N}$ each automorphism $f \in GA_n$ can be written as a pair

$$f = (f_k, f^k),$$

where $f_k \in \wr_{i=1}^{k} S_n$ and $f^k : X^k \to \wr_{i=k+1}^{\infty} S_n$, i.e. for every $u \in X^k$ we have $f^k(u) \in GA_n$. We call the automorphism $f^k(u)$ the state of the automorphism f in u. In particular, $f(\Lambda) = f$ so that f is a state of itself. An automorphism $f \in GA_n$ is called finite state if it has only finitely many different states. All finite state automorphisms form a subgroup of

GA_n that is called the finite state automorphism group and is denoted by FGA_n. This group contains a proper subgroup $FinGA_n$ of finitary automorphisms. An automorphism $f \in GA_n$ is called finitary if there exist $k \geq 0$ such, that for all $u \in X^k$ the state $f^k(u)$ is the identity. In other words, f does not change the mth letter in every (finite or ω-) word over X for all $m > k$. This group is locally finite since it can be decomposed as the direct limit of wreath products $\wr_{i=1}^{k} S_n$, $k \geq 1$ with the natural embeddings.

3. Some necessary conditions

Proposition 3.1. Let $H_1, \ldots, H_k$ $(k \geq 2)$ be finite subgroups of the finite state automorphism group FGA_n. All composition factors of H_i $(1 \leq i \leq k)$ are subgroups of S_n.

Proof. We will prove that for some $m \geq 1$ the group H_1 is embeddable into $\wr_{i=1}^{m} S_n$. As a subgroup of FGA_n the group H_1 acts on the levels of T_n. Suppose that all of these actions are non-faithful. Note that the kernel of the action on the lth level contains the kernel of the action on $(l+1)$-st level, $l \geq 1$. Since H_1 is finite, the series of kernels have to stabilize on some nontrivial subgroup of H_1. This means that the action of H_1 on T_n is not faithful which contradicts the selection of H_1. Hence for some $m \geq 1$ the group H_1 acts faithfully on the set X^m. This precisely means embeddability of H_1 into the wreath product $\wr_{i=1}^{m} S_n$. As it follows from [9], this implies the statement of the proposition. $\square$

Remark 3.2. Note, that this condition for finite group H is also sufficient for being a subgroup of FGA_n.

Proposition 3.3. Suppose that the group $\mathfrak{G}$ splits into a free product:

$$\mathfrak{G} = H_1 * \cdots * H_k$$

of finite groups. Then at most one of the groups $H_1, \ldots, H_k$ is a subgroup of $FinGA_n$.

Proof. Assume that two groups (for example H_1, H_2) are contained in $FinGA_n$. It follows immediately from the local finiteness of $FinGA_n$ that the group generated by them is finite. This is a contradiction. $\square$

4. Sufficient condition

We start with two useful lemmata about wreath products.

Let (G, M) be a transitive permutation group and let $\wr_{i=1}^{m}(G, M)$ be its (permutational) wreath power. For $m \geq 2$, the elements of $\wr_{i=1}^{m}(G, M)$ will be written as pairs $(g, f(x))$, where $g \in \wr_{i=1}^{m-1}(G, M)$ and $f(\cdot) : M^{m-1} \to G$. We write h instead of $f(x)$ in case $f(x) \equiv h$ for some $h \in G$. Let (H, N) be a subgroup of $\wr_{i=1}^{m}(G, M)$. The canonical embedding

$$\tau : (H, N) \hookrightarrow \wr_{i=1}^{m+1} (G, M)$$

is given by

$$\tau(h) = (h, e), \quad h \in H.$$

Here e denotes the identity element of H. Also define the j-diagonal (for $j \geq 0$) embedding

$$\psi_{m,j} : (H, N) \hookrightarrow \wr_{i=1}^{j} (G, M).$$

Here ψ_0 is the identity mapping and for $j > 0$

$$\psi_j(h) = (\psi_{j-1}(h), h), \quad h \in H.$$

Lemma 4.1. Let (H, N) be a subgroup of the wreath power $\wr_{i=1}^{m}(G, M)$. Then for every $j \geq 1$ the wreath power $\wr_{i=1}^{m+j}(G, M)$ contains a subgroup (H_1, N_1) isomorphic to (H, N) as a permutation group.

Proof. Fix an arbitrary $m \in M$. Define the subset

$$N_1 = N \times \underbrace{\{m\} \times \cdots \times \{m\}}_{j \text{ times}} \subset M^{m+j}.$$

Denote by H_1 the image of H under τ^j, the jth iteration of the canonical embedding. Then (H_1, N_1) is the necessary subgroup. $\square$

Lemma 4.2. Let (H, N) be a subgroup of the wreath power $\wr_{i=1}^{m}(G, M)$. Then for every $w \in M^m$ there exists a subgroup (H_1, N_1) of $\wr_{i=1}^{m}(G, M)$ isomorphic to (H, N) as a permutation group, such that $w \in N_1$.

Proof. Consider the case $w \notin N$. Chose an arbitrary $u \in N$. Since the group (G, M) is transitive, so is the group $\wr_{i=1}^{m}(G, M)$. Hence it contains a permutation π such that $\pi(u) = w$. Then $(\pi^{-1} H \pi, \pi(N))$ is the required group. $\square$

Consider now finite groups $H_1, \ldots, H_k$ such that all the composition factors of H_i $(1 \leq i \leq k)$ are subgroups of S_n. This means that H_i is a subgroup of $\left(\wr_{i=1}^{m_i} S_n, X^{m_i}\right)$ for some $m_i \in \mathbb{N}$ and $1 \leq i \leq k$. Suppose additionally that H_i acts regularly on some subset $A_i \subseteq X^{m_i}$, $1 \leq i \leq k$. Using Lemma 4.1 we may assume that $m_1 = \ldots = m_k = m$. By Lemma 4.2 we have that the intersection $\bigcap_{i=1}^{m} A_i$ is not empty. Denote some common element by w.

Define now for every i, $1 \leq i \leq k$, the embedding

$$\varphi_i : H_i \hookrightarrow \wr_{i=1}^{mk} S_n$$

as the composition $\varphi_i = \psi_i \cdot \tau^{m-i}$. This means that at first we take the i-diagonal embedding and then use the canonical embedding $m-i$ times.

Consider the following k subsets of the set X^{mk}:

$$\begin{array}{rcl} M_1 & = & \{uww\ldots w : u \in A_1, u \neq w\} \\ M_2 & = & \{uuw\ldots w : u \in A_2, u \neq w\} \\ \multicolumn{3}{c}{\dotfill} \\ M_k & = & \{uuu\ldots u : u \in A_k, u \neq w\}. \end{array}$$

Lemma 4.3. The sets $M_1, \ldots, M_k$ are nonempty and pairwise disjoint.

Proof. Follows from nontriviality of the given groups and regularity of their actions on the sets $A_1, \ldots, A_k$. $\square$

It is easy to see that

$$M_i = \{(\underbrace{w\ldots w}_{k \text{ times}})^{\varphi_i(h)} : h \in H_i, h \neq e\}, \quad 1 \leq i \leq k.$$

Now define the following subsets of the set X^{mk}:

$$D_i = \bigcup_{\substack{1 \leq j \leq k \\ j \neq i}} \{v^{\varphi_i(h)} : v \in M_j, h \in H_i\}, \quad 1 \leq i \leq k.$$

Lemma 4.4. The set D_i, $1 \leq i \leq k$ is a union of the orbits of the action of $\varphi_i(H_i)$ on X^{mk}.

Proof. Let $u \in D_i$. Then for some $h_1 \in H_i$, $h_2 \in H_j$, $h_2 \neq e$, $j \neq i$, we have

$$u = ((w^k)^{\varphi_i(h_2)})^{\varphi_i(h_1)}.$$

This implies the required assertion. $\square$

Note that neither M_i, nor D_i $(1 \leq i \leq k)$ contains the word w_k.

We will use a presentation of GA_n as a wreath product

$$GA_n \simeq \left(\wr_{i=1}^{mk} (S_n, X) \right) \wr \left(\wr_{i=1}^{\infty} (S_n, X) \right) \simeq \left(\wr_{i=1}^{mk} (S_n, X) \right) \wr GA_n.$$

Define finally for each i $(1 \leq i \leq k)$ two maps $f_{i1}, f_{i2} : H_i \longrightarrow GA_n$. Namely, for arbitrary element $h \in H_i$ denote $f_{i1}(h)$ by h' and $f_{i2}(h)$ by h''. Then as elements of the wreath product above they have the form

$$h' = (e, \bar{h}), \quad h'' = (\varphi_i(h), \bar{h})$$

and the map $\bar{h} : X^{mk} \longrightarrow GA_n$ acts by the rule

$$\bar{h}(v) = \begin{cases} h'', & \text{if } v \in D_i \\ h', & \text{otherwise} \end{cases}.$$

Then for every $v \in X^{\mathbb{N}}$ presented in the form $v = v_1 v_2 v_3 \ldots$, where $v_i \in X^{mk}$, $i \geq 1$, we get:

$$v^{h'} = v_1^e (v_2 v_3 \ldots)^{\bar{h}(v_1)},$$

$$v^{h''} = v_1^{\varphi_i(h)} (v_2 v_3 \ldots)^{\bar{h}(v_1)}.$$

Proceeding this way we have

$$v^{h'} = v_1^e v_2^{\pi_1} v_3^{\pi_2} \ldots,$$

$$v^{h''} = v_1^{\varphi_i(h)} v_2^{\pi_1} v_3^{\pi_2} \ldots,$$

where

$$\pi_j = \begin{cases} \varphi_i(h), & \text{if } v_j \in D_i \\ e, & \text{otherwise} \end{cases}, \quad j \geq 2.$$

Lemma 4.5. The maps f_{i1}, f_{i2} are faithful representations of H_i in FGA_n.

Proof. Let $h \in H_i$. The sets of states of the automorphisms h' and h'' are equal to the set of states $\{h'(v), h''(v) : |v| \leq km\}$. This implies that f_{i1}, f_{i2} are maps into the group FGA_n of finite state automorphisms.

Let $h_1, h_2 \in H_i$ be arbitrary elements. For $v = v_1 v_2 v_3 \ldots \in X^{\mathbb{N}}$, where $v_i \in X^{mk}$, $i \geq 1$, we have

$$(v^{h'_1})^{h'_2} = (v_1^e (v_2 v_3 \ldots)^{\bar{h_1}(v_1)})^{h'_2} = v_1^e ((v_2 v_3 \ldots)^{\bar{h_1}(v_1)})^{\bar{h_2}(v_1)}$$

and

$$(v^{h_1''})^{h_2''} = (v_1^{\varphi_i(h_1)}(v_2v_3\ldots)^{\bar{h_1}(v_1)})^{h_2''} = v_1^{\varphi_i(h_1h_2)}((v_2v_3\ldots)^{\bar{h_1}(v_1)})^{\bar{h_2}(v_1^{\varphi_i(h_1)})}.$$

By Lemma 4.4 $\bar{h_1}(v_1)$ and $\bar{h_2}(v_1^{\varphi_i(h_1)})$ are equal to h_1' and h_2' or to h_1'' and h_2'', respectively. Since φ_i is an embedding, we inductively obtain that both f_{i1} and f_{i2} are faithful representations. □

Denote by $\mathfrak{G}_1$ and $\mathfrak{G}_2$ the subgroups of FGA_n generated by the images $f_{11}(H_1), \ldots, f_{k1}(H_k)$ and $f_{12}(H_1), \ldots, f_{k2}(H_k)$, respectively.

Theorem 4.6. The groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are isomorphic to the free product $H_1 * \cdots * H_k$.

We need for the proof the following generalization of the well-known "ping-pong" lemma.

Lemma 4.7. [6] Let a permutation group G on a set Ω be generated by its proper subgroups $G_1, \ldots, G_m$ $(m \geq 2)$ and at least one of them has order greater then 2. If there exist pairwise disjoint nonempty subsets $\Omega_1, \ldots, \Omega_m$ of Ω such that for $i \in \{1, \ldots, m\}$ the next condition holds:

$$\omega^g \in \Omega_i \text{ for } \omega \in \Omega_j, j \neq i, \text{ and } g \in G_i, g \neq 1,$$

then the group G splits into the free product

$$G = G_1 * \cdots * G_m.$$

Proof of theorem 4.6. Let us prove our statement for the group $\mathfrak{G}_2$.

The group $\mathfrak{G}_2$ as a subgroup of FGA_n is a permutation group acting on the set $X^{\mathbb{N}}$. Define for the subgroups $f_{12}(H_1), \ldots, f_{k2}(H_k)$ the following subsets $\Omega_1, \ldots, \Omega_k$ of $X^{\mathbb{N}}$:

$$\Omega_i = (X^{mk})^* M_i w^{\mathbb{N}} = \\ = \{u_1 \ldots u_l vww \ldots : l \geq 0, u_1, \ldots, u_l \in X^{mk}, v \in M_i\}, 1 \leq i \leq k.$$

Since $w \notin M_i$, for $1 \leq i \leq k$, the subsets are well defined.

By Lemma 4.3 the subsets $\Omega_1, \ldots, \Omega_k$ are pairwise disjoint.

Consider arbitrary indices i, j $(1 \leq i, j \leq k,\ i \neq j)$. Let $u = u_1 \ldots u_l vww \ldots \in \Omega_j, l \geq 0, u_1, \ldots, u_l \in X^{mk}, v \in M_j$ and $h \in H_i$. We will show that $u^{f_{i2}(h)} \in \Omega_i$. Since $M_j \subset D_i$ and $w_k \notin D_i$ using the rule of action of $f_{i2}(h)$ on $X^{\mathbb{N}}$ we can write the word $u^{f_{i2}(h)}$ in the form:

$$u_1' \ldots u_l' v' (w^k)^{\varphi_i(h)} ww \ldots.$$

Here $u'_1, \dots, u'_l, v'$ are some elements of X^{mk}. This presentation implies that $u^{f_{i2}(h)} \in \Omega_i$ by the definition of the set M_i.

Applying Lemma 4.7 we immediately obtain a factorization

$$\mathfrak{G}_2 = f_{12}(H_1) * \dots * f_{k2}(H_k).$$

By Lemma 4.5 it means that

$$\mathfrak{G}_2 \simeq H_1 * \dots * H_k$$

which completes the proof. □

5. Embedding theorems

In this section we present some interesting results following from the construction of the previous section and in particular from theorem 4.6.

We start with the following corollary of the Kaloujnine-Krasner wreath product embedding theorem [10].

Lemma 5.1. *Let H be a finite group, let m be the length of its composition series and suppose that the orders of the composition factors of H are less than or equal to n. Then for an alphabet X ($|X| = n$) the wreath product $\left(\wr_{i=1}^{m} S_n, X^m\right)$ contains a regular subgroup (Γ, A) with $\Gamma \simeq H$.*

Proof. Let

$$H = H_1 \triangleright H_2 \triangleright \dots \triangleright H_m \triangleright \{1\}$$

be some composition series of H. Due to the Kaloujnine–Krasner Theorem H acts regularly on the set

$$(H_1/H_2) \times (H_2/H_3) \times \dots \times (H_{m-1}/H_m) \times H_m$$

as a subgroup of the standard wreath product

$$(H_1/H_2) \wr (H_2/H_3) \wr \dots \wr (H_{m-1}/H_m) \wr H_m.$$

Since by assumption all factors of the wreath product have size $\le n$ it naturally embeds into $\left(\wr_{i=1}^{m} S_n, X^m\right)$. This completes the proof. □

Theorem 5.2. *Let $H_1, \dots, H_k$ be finite groups and suppose that the orders of all their composition factors are bounded above by n. Then the free product $H_1 * \dots * H_k$ acts faithfully on the regular rooted tree T_n by finite state automorphisms.*

Proof. By Lemma 5.1 all given groups satisfy the conditions of the previous section. Use then the construction described there and Theorem 4.6 gives the required conclusion. □

Corollary 5.3. Let $n = 2, 3$ or 4. Then finite groups $H_1, \dots, H_k$ are embeddable into the group of finite state automorphisms FGA_n if and only if their free product $H_1 * \dots * H_k$ is embeddable.

Proof. Sufficiency is obvious.

We have to prove necessity. By Proposition 3.1 all groups $H_1, \dots, H_k$ have subnormal series with factors from the symmetric group S_n. But in our case (S_2, S_3 or S_4) it follows that the composition factors of all given groups have size $\leq n$. Now apply Theorem 5.2. □

Corollary 5.4. Finite soluble groups $H_1, \dots, H_k$ are embeddable into the group of finite state automorphisms FGA_n if and only if their free product $H_1 * \dots * H_k$ is embeddable.

Proof. It is sufficient to prove necessity. By Proposition 3.1 all the groups $H_1, \dots, H_k$ have subnormal series with factors from the symmetric group S_n. This means that orders of these factors divide $n!$. Consider the composition series which are refinements of the subnormal ones. Since groups $H_1, \dots, H_k$ are soluble their composition factors are cyclic of prime order. It follows from above that their orders divide $n!$. Hence these orders are bounded by n from above. The rest is to apply Theorem 5.2. □

In case $n = p$ is prime we have the natural Sylow p-subgroup of the group GA_p, namely the wreath product $\wr_{i=1}^{\infty} C_p$ of infinity many copies of the cyclic group of order p. Denote it by $Syl(GA_p)$. We can also consider the finite state part of this group, i.e., the intersection $FSyl(GA_p) = Syl(GA_p) \cap FGA_p$. It is easy to see that both groups $Syl(GA_p)$ and $FSyl(GA_p)$ contain every finite p-group. As in the previous section, for any finite p-groups $H_1, \dots, H_k$ we can use the completely analogous construction and to prove the following

Theorem 5.5. Let $H_1, \dots, H_k$ be finite p-groups. Then the group $FSyl(GA_p)$ contains subgroups isomorphic to the free product $H_1 * \dots * H_k$.

6. An example

Let $H_1 = \langle a | a^2 = 1 \rangle$ and $H_2 = \langle b | b^3 = 1 \rangle$ be cyclic groups of orders 2 and 3, respectively. Then H_1, H_2 act regularly on the subsets $A_1 = \{1, 2\}$

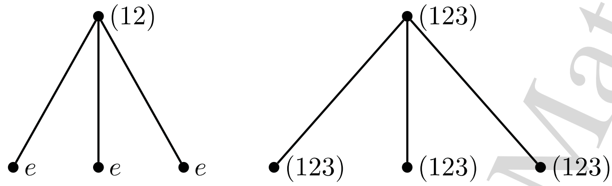

Figure 2:

and $A_2 = \{1,2,3\}$ of $X = \{1,2,3\}$ as the subgroups $\langle(12)\rangle$ and $\langle(123)\rangle$ of the symmetric group S_3. Using the notation of Section 4 we have $n = 3$, $k = 2$ and $m = 1$. Consider $\varphi_1(a) = ((12); e, e, e)$ and $\varphi_2(b) = ((123); (123), (123), (123))$ (Figure 2).

Choose $w = 1$. Then we get $M_1 = \{21\}$, $M_2 = \{22, 33\}$ and $D_1 = \{22, 33, 12\}$, $D_2 = \{21, 32, 13\}$. Denote $a_i = f_{1i}(a)$ and $b_i = f_{1i}(b)$, $i = 1, 2$. As elements of FGA_3 these automorphisms have the following recurrent form (Figure 3):

$$\begin{aligned}
a_1 &= (e; a_1, a_2, a_1, a_1, a_2, a_1, a_1, a_1, a_2)\\
a_2 &= (\varphi_1(a); a_1, a_2, a_1, a_1, a_2, a_1, a_1, a_1, a_2)\\
b_1 &= (e; b_1, b_1, b_2, b_2, b_1, b_1, b_1, b_2, b_1)\\
b_2 &= (\varphi_2(b); b_1, b_1, b_2, b_2, b_1, b_1, b_1, b_2, b_1)
\end{aligned}$$

By Theorem 4.6 we have $\langle a_i, b_i\rangle = \langle a_i\rangle * \langle b_i\rangle \simeq C_2 * C_3$, $i = 1, 2$.

Due to [11] the subgroups $\langle b_i^2 a_i b_i^2, a_i b_i^2 a_i b_i^2 a_i\rangle$ are free of rank 2.

7. Some open questions

Let us formulate some questions arising in connection with the obtained results. Firstly, the question about possibility of omitting any additional conditions in Theorem 5.2.

Question 7.1. Let finite groups $H_1, \dots, H_k$ be embeddable into FGA_n. Is their free product $H_1 * \cdots * H_k$ embeddable into FGA_n?

In particular,

Question 7.2. Is the free product $A_n * A_n$ embeddable into FGA_n for $n \geq 5$?

Here by A_n we denote the alternating group of degree n.

If the answer to Question 7.1 is negative then

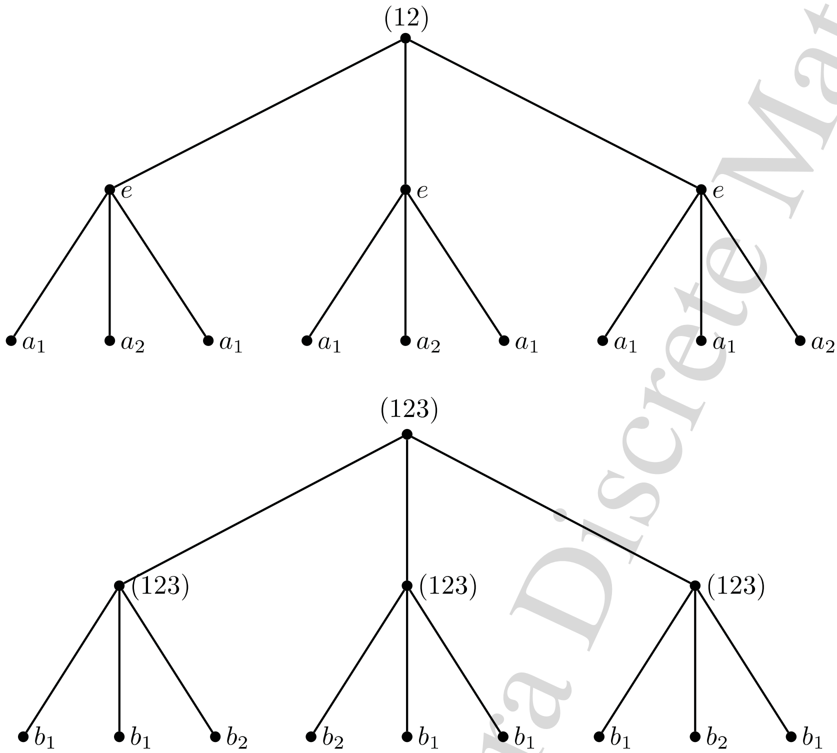

Figure 3: Automorphisms a_2 and b_2

Question 7.3. For given finite groups $H_1, \ldots, H_k$ compute the smallest number n such that the free product $H_1 * \cdots * H_k$ is embeddable into FGA_n.

By Theorem 5.2 this number does not exceed the order of the largest composition factor of $H_1, \ldots, H_k$.

And finally a question about closeness under operation of free product in the class of finite state automorphism groups.

Question 7.4. Let $H_1 \leq FGA_n$ and $H_2 \leq FGA_m$ $(n, m \geq 2)$. Is it true that $H_1 * H_2 < FGA_k$ for some $k \geq 2$?

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