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On Sushchansky p-groups

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. We study Sushchansky p-groups introduced in [Sus79]. We recall the original definition and translate it into the language of automata groups. The original actions of Sushchansky groups on p-ary tree are not level-transitive and we describe their orbit trees. This allows us to simplify the definition and prove that these groups admit faithful level-transitive actions on the same tree. Certain branch structures in their self-similar closures are established. We provide the connection with, so-called, G groups [BGŠ03] that shows that all Sushchansky groups have intermediate growth and allows to obtain an upper bound on their period growth functions.

Introduction

Sushchansky p-groups were introduced in [Sus79] as one of the pioneering examples of finitely generated infinite torsion groups, providing counter-examples to the General Burnside problem. Initially, this problem was solved by E.S. Golod in [Gol64] using the Golod-Shafarevich theorem. Simpler and easier to handle counter-examples were constructed by S. V. Aleshin in [Ale72] by means of automata. The use of automata groups to resolve Burnside's problem was earlier suggested by V. M. Glushkov in [Glu61]. But only after the results of R. I. Grigorchuk from [Gri80, Gri83] automata groups became the subject of deeper investigation. It

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happened that this class contains groups with many extraordinary properties, like infinite torsion groups, groups of intermediate growth, groups of finite width, just-infinite groups, etc.

V.I. Sushchansky used a different language, namely the language of tableaux, introduced by L. Kaluzhnin to study properties of iterated wreath products [Kal48]. For each prime $p > 2$, V.I. Sushchansky constructed a finite family of infinite p-groups generated by two tableaux. Each such a tableau naturally defines an automorphism of a rooted tree and, as was already noticed in [GNS00], can be represented by a finite initial automaton. We describe these automata and study Sushchansky groups and their actions on rooted trees by means of this well-developed language.

The structure of the paper is as follows. In Section 1 we recall the original definition of Sushchansky groups. In Section 2 we describe the corresponding automata. The associated action on a rooted tree is not level-transitive and in Section 3 we describe its orbit tree and show that there exists a faithful level-transitive action given by finite initial automata. The self-similar closure is studied in Section 4. The main results are presented in Section 5. It was pointed out in [Gri85a] that all Sushchansky p-groups have intermediate growth, but only the main idea of the proof was given. Here we provide a complete proof of this fact together with new estimates on the growth function, thus contributing to the Milnor question [Mil68], which was solved in [Gri83] by R.I. Grigorchuk. Also we give an upper bound on the period growth function. The main idea is to use G groups of intermediate growth introduced in [BŠ01] (see also [BGŠ03]). For each Sushchansky p-group we construct a G group of intermediate growth and prove that their growth functions are equivalent.

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1. Original definition via tableaux

Let $X = \{0, 1, \ldots, p-1\}$ be a finite alphabet for some prime p. We identify X with the finite field $\mathbb{F}_p$.

The set X^* of all finite words over X has a natural structure of a rooted p-ary tree. Every automorphisms $g \in \operatorname{Aut} X^*$ of this tree induces an automorphism $g|_v$ of the subtree vX^* by the rule $g|_v(w) = u$ if and only if $g(vw) = g(v)u$. This automorphism is called *the restriction of g on word v* (in some papers the word *section* or *state* is used).

The Sylow p-subgroup P_∞ of the profinite group $\operatorname{Aut} X^*$ is equal to the infinite wreath product of cyclic groups of order p, i.e. $P_\infty = \wr_{i\geq 1} C_p^{(i)}$. Using this description one can construct special “tableau” representation of P_∞. The “tableau” representation was initially introduced by L. Kaluzhnin for Sylow p-subgroups of symmetric groups of order p^m in [Kal48].

The group P_∞ is isomorphic to the group of triangular tableaux of the form:

$$u = [a_1, a_2(x_1), a_3(x_1, x_2), \ldots],$$

where $a_1 \in \mathbb{F}_p$, $a_{i+1}(x_1, \ldots, x_i) \in \mathbb{F}_p[x_1, \ldots, x_i]/\langle x_1^p - x_1, \ldots, x_i^p - x_i\rangle$. The multiplication of tableaux is given by the formula:

$$[a_1, a_2(x_1), a_3(x_1, x_2), \ldots] \cdot [b_1, b_2(x_1), b_3(x_1, x_2), \ldots] =$$

$$= [a_1 + b_1, a_2(x_1) + b_2(x_1 + a_1), a_3(x_1, x_2) + b_3(x_1 + a_1, x_2 + a_2(x_1)), \ldots].$$

The action of the tableau u on the tree X^* is given by:

$$u(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n, \tag{1}$$

where $y_1 = x_1 + a_1$, $y_2 = x_2 + a_2(x_1), \ldots, y_n = x_n + a_n(x_1, \ldots, x_{n-1})$, where all calculations are made by identifying X with the field $\mathbb{F}_p$.

For the duration of the rest of the paper we fix a prime $p > 2$.

Fix some order $\lambda = \{(\alpha_i, \beta_i), i = 1, \ldots, p^2\}$ on the set of pairs $\{(\alpha, \beta) | \alpha, \beta \in \mathbb{F}_p\}$. For $j > p^2$ we define $(\alpha_j, \beta_j) = (\alpha_i, \beta_i)$ where $i \equiv j \mod p^2$. Define two tableaux

$$A = [1, x_1, 0, 0, \ldots], \quad B_\lambda = [0, 0, b_3(x_1, x_2), b_4(x_1, x_2, x_3), \ldots],$$

where the coordinates of B_λ are defined by its values in the following way:

a) $b_3(2, 1) = 1$;

b) $b_i(0, 0, \ldots, 0, 1) = 1$ if $\beta_i \neq 0$;

c) $b_i(1, 0, \ldots, 0, 1) = -\frac{\alpha_i}{\beta_i}$ if $\beta_i \neq 0$ and $b_i(1, 0, \ldots, 0, 1) = 1$ if $\beta_i = 0$;

d) all the other values are zeroes.

The group $G_\lambda = \langle A, B_\lambda\rangle$ is called *the Sushchansky group of type λ*. The following theorem is proven in [Sus79].

Theorem 1. G_λ is infinite periodic p-group for any type λ.

2. Automata approach

Another language dealing with groups acting on rooted trees is the language of automata groups. For a definitions we refer to the survey paper [GNS00]. Many groups related to Burnside and Milnor Problems happen to be in the class of groups generated by finite automata. The Sushchansky groups are not an exception and we describe the structure of the corresponding automata in this section.

The action of every automorphism g of the rooted tree X^* can be encoded by an initial automaton whose states are the restrictions of g on the finite words over X. In the case when this set is finite we call g a *finite-state* automorphism. The action of such an automorphism is encoded by a finite automaton.

It is known that (see [GNS00]) that $\mathrm{Aut}\, X^* \cong \mathrm{Aut}\, X^* \wr \mathrm{Sym}(X)$, which gives a convenient way to represent every automorphism in the following form:

$$g = (g|_0, g|_1, \ldots, g|_{p-1})\pi_g,$$

where $g|_0, g|_1, \ldots, g|_{p-1}$ are the restrictions of g on the letters of X and π_g is the permutation of X induced by g.

The multiplication of automorphisms written in this way is performed as follows. If $h = (h|_0, h|_1, \ldots, h|_{p-1})\pi_h$ then

$$gh = (g|_0 h|_{\pi_g(0)}, \ldots, g|_{p-1} h|_{\pi_g(p-1)})\pi_g\pi_h.$$

Now we proceed with an explicit construction of automata associated to Sushchansky groups. Let $\sigma = (0, 1, \ldots, p-1)$ be a cyclic permutation of X. With a slight abuse of notation, depending on the context, σ will also denote the automorphism of X^* of the form $(1, 1, \ldots, 1)\sigma$.

Given the order $\lambda = \{(\alpha_i, \beta_i)\}$ define words $u, v \in X^{p^2}$ in the following way:

$$u_i = \begin{cases} 0, & \text{if } \beta_i = 0; \\ 1, & \text{if } \beta_i \neq 0. \end{cases} \qquad v_i = \begin{cases} 1, & \text{if } \beta_i = 0; \\ -\frac{\alpha_i}{\beta_i}, & \text{if } \beta_i \neq 0. \end{cases}$$

The words u and v encode the actions of B_λ on the words $00\ldots01*$ and $10\ldots01*$, respectively. Using the words u and v we can construct automorphisms $q_1, \ldots, q_{p^2}, r_1, \ldots, r_{p^2}$ of the tree X^* by the following recurrent formulas:

$$q_i = (q_{i+1}, \sigma^{u_i}, 1, \ldots, 1), \qquad r_i = (r_{i+1}, \sigma^{v_i}, 1, \ldots, 1), \tag{2}$$

for $i = 1, \ldots, p^2$, where the indices are considered modulo p^2, i.e. $i = i + np^2$ for any n.

Formula (1) implies that q_i and r_i are precisely the restrictions of B_λ on the words $00(0)^{i-1+np^2}$ and $10(0)^{i-1+np^2}$, respectively, for any $n \geq 0$.

The action of the tableau A is given by:

$$A = (1, \sigma, \sigma^2, \ldots, \sigma^{p-1})\sigma;$$

while B_λ acts trivially on the second level and the action on the rest is given by the restrictions:

$$B_\lambda|_{00} = q_1, \quad B_\lambda|_{10} = r_1, \quad B_\lambda|_{21} = \sigma$$

and all the other restrictions are trivial. In particular, the automorphisms A and B_λ are finite-state and Sushchansky group G_λ is generated by two finite initial automata. Denote the union of these two automata by $\mathcal{A}_{u,v}$. Its structure is shown in Figure 1. The particular automaton for $p = 3$ and the lexicographic order on $\{(\alpha, \beta) | \alpha, \beta \in \mathbb{F}_p\}$ is given in Figure 2 (all the arrows not shown in the figures go to the trivial state 1).

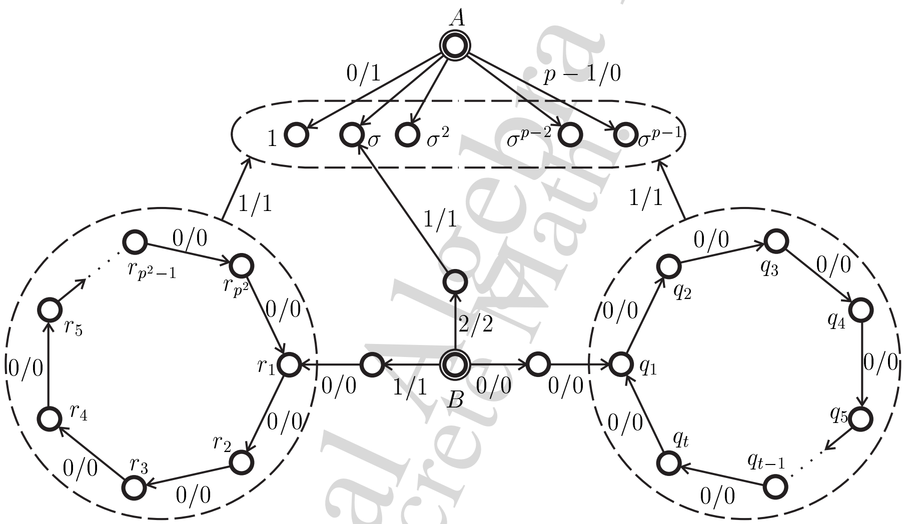

Figure 1: The Structure of Sushchansky automata

Notice that the word v cannot be periodic since it contains exactly $p-1$ zeros and $p-1 \nmid p^2$. On the contrary u may be periodic with period p. In this case we have $q_i = q_{i+p}$ and the minimization of $\mathcal{A}_{u,v}$ contains $p^2 + 2p + 5$ states. If u is not periodic then $\mathcal{A}_{u,v}$ contains $2p^2 + p + 5$ states. Let t be the length of the minimal period in u (thus either $t = p$ or $t = p^2$).

Lemma 2. The group $\langle q_1, \ldots, q_t, r_1, \ldots, r_{p^2} \rangle$ is elementary abelian p-group.

Proof. All q_i, r_j have order p because

$$q_i^p = (q_{i+1}^p, 1, 1, \ldots, 1), \qquad r_i^p = (r_{i+1}^p, 1, 1, \ldots, 1),$$

and therefore q_i^p and r_i^p act trivially on the tree.

All q_i, r_j commute with each other, because

$$\begin{aligned} q_i q_j &= (q_{i+1}q_{j+1}, \sigma^{u_i+u_j}, 1, \ldots, 1), & q_j q_i &= (q_{j+1}q_{i+1}, \sigma^{u_i+u_j}, 1, \ldots, 1); \\ r_i r_j &= (r_{i+1}r_{j+1}, \sigma^{v_i+v_j}, 1, \ldots, 1), & r_j r_i &= (r_{j+1}r_{i+1}, \sigma^{v_i+v_j}, 1, \ldots, 1); \\ q_i r_j &= (q_{i+1}r_{j+1}, \sigma^{u_i+v_j}, 1, \ldots, 1), & r_j q_i &= (r_{j+1}q_{i+1}, \sigma^{u_i+v_j}, 1, \ldots, 1), \end{aligned}$$

so the corresponding pairs act equally on the tree. □

The last lemma implies that the order of B_λ is p. Since

$$A^p = (\sigma^{\frac{p(p-1)}{2}}, \sigma^{\frac{p(p-1)}{2}}, \ldots, \sigma^{\frac{p(p-1)}{2}})$$

and p is odd, the order of A is also p.

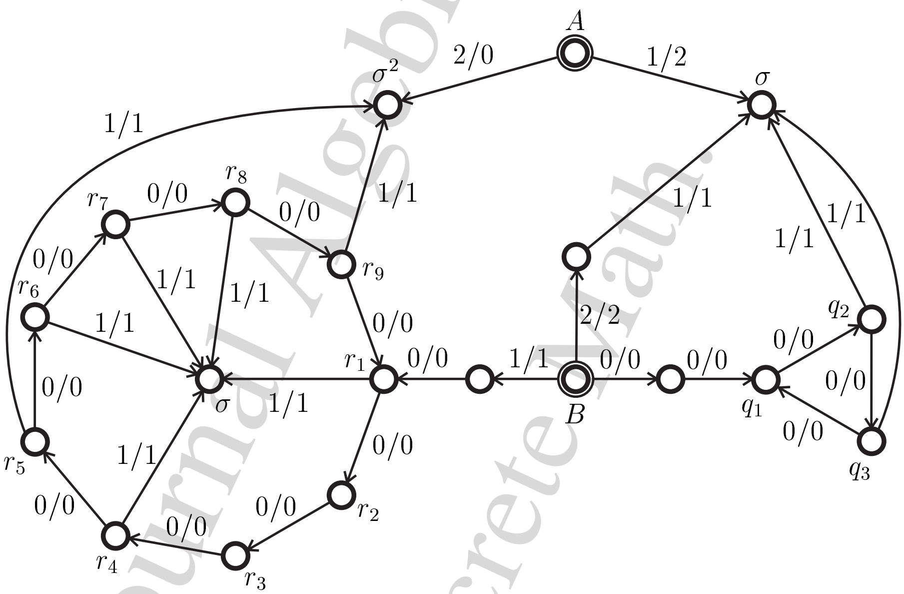

Figure 2: Sushchansky automaton for $p = 3$ corresponding to the lexicographic order

3. Actions on rooted trees

Here we describe the structure of the action of G_λ on a p-ary tree by means of the orbit tree. This notion is defined in [Ser03] and used in [GNS01] to establish a criterion determining when two automorphisms of a rooted tree are conjugate. Here we use it to simplify the definition of Sushchansky groups and show that they admit a faithful level-transitive action on a regular rooted tree.

Definition 1. Let G be a group acting on a regular p-ary tree X^*. The *orbit tree* of G is a graph whose vertices are the orbits of G on the levels of X^* and two orbits are adjacent if and only if they contain vertices that are adjacent in X^*.

Proposition 3. The structure of the orbit tree of G_λ does not depend on the type λ and is shown in Figure 3.

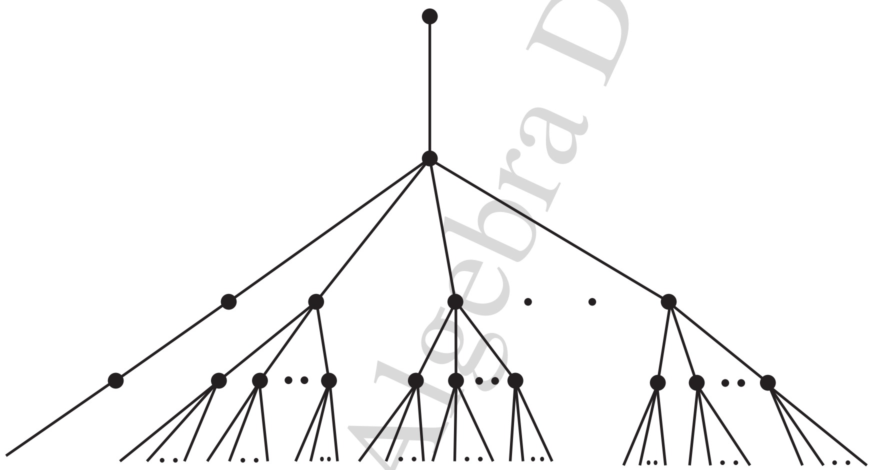

Figure 3: The Orbit Tree of Sushchansky group

Proof. Let T_O be the orbit tree of G_λ. Denote by $\mathrm{Orb}(w)$ the orbit of the word $w \in X^*$ under the action of G_λ. Define the set

$$V = \{xyw \in X^* | xy \in \mathrm{Orb}(00) \text{ and } w \in X^*\} \cup \{\emptyset\}, \tag{3}$$

where $\emptyset$ is the root of the tree.

The generator B_λ stabilizes the second level of the tree and hence the orbit $\mathrm{Orb}(00)$ coincides with the orbit of 00 under the action of the group generated by A. The set V and its compliment $W = X^* \setminus V$ are invariant under the action of G_λ.

Notice that $\{00, 10, 21\} \subset \mathrm{Orb}(00)$ and the generator B_λ acts trivially on all words that lie in the set W. Since the restrictions of A on all words of length ≥ 2 are trivial, every element $g \in G_\lambda$ that acts trivially on the second level of the tree must stabilize all the vertices of the set W. Hence, the orbits of G_λ on W coincide with the ones of A. Automorphism A acts transitively on the first level and has order p. Therefore the orbit of any word $w \in W$ consists of p vertices, namely the images of w under the action of the cyclic group of order p generated by A. Therefore the first two levels of T_O are exactly as shown in Figure 3 and $p-1$ vertices on the second level of T_O are the roots of regular p-ary trees.

Let us prove that G_λ acts transitively on the levels of the set V, i.e. for every $n \geq 1$ the group G_λ acts transitively on the set

$$V_n = \{xyw \in X^{n+1} | xy \in \mathrm{Orb}(00) \text{ and } w \in X^{n-1}\}.$$

We use induction on n. For $n = 1$ there is nothing to prove. Assume G_λ acts transitively on V_n and consider the $(n+1)$-th level. Since by construction either $u_{n-1} = 1$ or $v_{n-1} = 1$, the restriction of B_λ on either $00\ldots01$ or $10\ldots01$ is equal to σ. Denote this word as s (here $s \in V_n$) and note that B stabilizes s. To prove the induction step it suffices for an arbitrary $s'z' \in V_{n+1}$, where $s' \in V_n$ and $z' \in X$, to construct an element $g \in G_\lambda$ such that $g(s0) = s'z'$. By the inductive assumption there is an element $h \in G_\lambda$ such that $h(s) = s'$. Suppose $h^{-1}(s'z') = sz$ for some letter $z \in X$. Then for $g = (B_\lambda)^z h$ (here we consider z as an integer) we have

$$\begin{aligned} g(s0) &= h((B_\lambda)^z(s0)) = h(s(B_\lambda)^z|_s(0)) = h(s(B_\lambda|_s)^z(0)) = \\ &= h(s\sigma^z(0)) = h(sz) = s'z' \end{aligned}$$

as required. □

The set V has a natural structure of a rooted p-ary tree T, where the root $\emptyset$ is connected by an edge with every vertex in $\mathrm{Orb}(00)$ and there is an edge between w and wx for all $w \in V$ and $x \in X$. In other words, there is a natural 1-to-1 correspondence between V and vertices of T given by $xyw \mapsto xw$ for $xy \in \mathrm{Orb}(00)$ and $w \in X^*$. Since the set V is invariant under the action of G_λ, the group G_λ acts by automorphisms on the tree T. This action has simpler structure and the following proposition holds.

Proposition 4. The action of Sushchansky group G_λ on the tree T is faithful, level transitive and has the following form

$$\begin{aligned} A &= \sigma, \\ B_\lambda &= (q_1, r_1, \sigma, 1, \ldots, 1), \\ q_i &= (q_{i+1}, \sigma^{u_i}, 1, \ldots, 1), \\ r_i &= (r_{i+1}, \sigma^{v_i}, 1, \ldots, 1). \end{aligned} \tag{4}$$

Proof. The expressions (4) follow directly from the definition of Sushchansky groups.

Let us prove that this action is faithful. Take an arbitrary nontrivial element $g \in G_\lambda$. If g acts non-trivially on the second level of X^*, then the exponent of A in g is not divisible by p. But then g acts non-trivially on the first level of T as well because it is fixed under B_λ and A acts there as σ. If g acts trivially on the second level of X^* then it acts trivially on the complement of V in X^* according to Proposition 3. Therefore to be nontrivial it must act nontrivially on T.

We proved in Proposition 3 that G_λ acts transitively on every set V_n, which is precisely the nth level of the tree T. $\square$

4. Self-similar closure

The Sushchansky group G_λ is not generated by all the states of $\mathcal{A}_{u,v}$ and is not self-similar (see definition below). However, we can embed it into a larger self-similar group where we can use some known techniques to derive some important results about G_λ itself. In particular that G_λ is amenable (Corollary 8) and that the word problem is solvable in polynomial time (Corollary 9). For the definitions not given here and more information on self-similar groups we refer to [Nek05] and [BGŠ03].

Definition 2. A group $G < \operatorname{Aut} X^*$ is called *self-similar* if $g|_u \in G$ for any $g \in G$ and word $u \in X^*$. The *self-similar* closure of $G < \operatorname{Aut} X^*$ is the group generated by all the restrictions of all the elements of G on words in X^*.

Let $\tilde{G}_\lambda$ be the self-similar closure of G_λ, i.e. $\tilde{G}_\lambda$ is generated by all the states of the automaton $\mathcal{A}_{u,v}$. Consider also the self-similar subgroup $K = \langle q_1, \ldots, q_t, r_1, \ldots, r_{p^2}, \sigma\rangle$ of $\tilde{G}_\lambda$.

Lemma 5. The group K is not periodic.

Proof. First, consider the case $t = p$. Then all u_i's are equal to 1 except one equal to 0. In particular, $\sum_{i=1}^{p} u_i = p - 1$. Then the element $g = q_1 q_2 \cdots q_t \sigma^{p-1}$ has representation

$$g = (q_1 q_2 \cdots q_t, \sigma^{p-1}, 1, \ldots, 1)\sigma^{p-1}.$$

Therefore

$$g^p = (q_1 q_2 \cdots q_t \sigma^{p-1}, *, \ldots, *) = (g, *, \ldots, *).$$

Since g is nontrivial it must have infinite order.

In case $t = p^2$, exactly p of u_i's are zeros. We mark the vertices of the cycle of q_i's in the automaton by the corresponding u_i's. There are at most $\binom{p}{2}$ different distances between the zeros in the cycle. But the length of the cycle is p^2 so there are

$$\frac{p^2-1}{2} > \frac{p^2-p}{2} = \binom{p}{2}$$

possible distances in the cycle, so let d be a distance that is not attained as a distance between two zeros.

Now consider the element $g = q_1 q_{d+1} \sigma^{u_{p^2}+u_d}$. It can be written as

$$g = (q_2 q_{d+2}, \sigma^{u_1+u_{d+1}}, 1, \ldots, 1)\sigma^{u_{p^2}+u_d}.$$

Since the distance between states q_{p^2} and q_d in the cycle is exactly d at least one of u_{p^2} and u_d is nonzero so $\sigma^{u_{p^2}+u_d}$ is a cycle of length p. Hence

$$g^p = (q_2 q_{d+2} \sigma^{u_1+u_{d+1}}, *, \ldots, *).$$

Therefore if the order $|g|$ of g is finite, then it is not smaller than $p \cdot |q_2 q_{d+2} \sigma^{u_1+u_{d+1}}|$.

Now we repeat this procedure p^2 times and on the i-th iteration we get

$$q_i q_{d+i} \sigma^{u_{i-1}+u_{d+i-1}} = (q_{i+1} q_{d+i+1}, \sigma^{u_i+u_{d+i}}, 1, \ldots, 1)\sigma^{u_{i-1}+u_{d+i-1}}.$$

Again, the distance between q_{i-1} and q_{d+i-1} is exactly d so $\sigma^{u_{i-1}+u_{d+i-1}}$ is a cycle of length p and

$$(q_i q_{d+i} \sigma^{u_{i-1}+u_{d+i-1}})^p = (q_{i+1} q_{d+i+1} \sigma^{u_i+u_{d+i}}, *, \ldots, *).$$

Therefore

$$|q_i q_{d+i} \sigma^{u_{i-1}+u_{d+i-1}}| \geq p \cdot |q_{i+1} q_{d+i+1} \sigma^{u_i+u_{d+i}}|.$$

But after p^2 steps we will meet g again. So its order cannot be finite. □

Definition 3. A group G acting on the tree X^* is called *weakly regular branch* over its subgroup P, if

1. G acts transitively on each level X^n, $n \geq 0$;

2. $P \succ P \times P \times \cdots \times P$ as geometric embedding induced by the restriction on some level X^k.

In case if P is a subgroup of finite index in G, the group G is said to be *regular branch* over P.

Proposition 6. $\tilde{G}_\lambda$ is a weakly regular branch group over K^p.

Proof. First of all note that Lemma 5 guarantees that K^p is nontrivial. At least one (in fact more) of the u_i's is non zero, say u_1. Then the relations (2) and

$$\sigma q_1 \sigma^{p-1} = (\sigma^{u_1}, 1, \ldots, 1, q_2)$$

show that the set of restrictions of the elements of K, that stabilize the first level X of the tree, on letter 0 includes the generators of K and hence the whole group K (therefore conjugating by $\sigma \in K$ yields that K is self-replicating, i.e. for any $x \in X$ the projection of $\mathrm{St}_x(K)$ onto the vertex x coincides with K). Thus for any $v \in K$ there is $w \in K$ of the form

$$w = (v, \sigma^i, 1, \ldots, 1, q_2^j)$$

for some i and j. But then by Lemma 2

$$w^p = (v^p, \sigma^{ip}, 1, \ldots, 1, q_2^{jp}) = (v^p, 1, \ldots, 1).$$

Therefore $K^p \succ K^p \times 1 \times \cdots \times 1$. Since σ acts transitively on the first level and belongs to the normalizer of K^p in K (because $\sigma^{-1} v^p \sigma = (\sigma^{-1} v \sigma)^p$) by conjugation we get

$$K^p \succ K^p \times K^p \times \cdots \times K^p,$$

as geometric embedding.

The transitivity of $\tilde{G}_\lambda$ on levels follows from the fact that its subgroup K acts nontrivially on the first level and is self-replicating, and hence, level transitive. Another explanation comes from the known fact that a self-similar subgroup of $\wr_{i \geq 1} C_p^{(i)}$ acts level-transitively if and only it is infinite (see [BGK+06]). The proof of the last fact is similar to the proof of transitivity in Proposition 3. $\square$

We summarize some general properties of $\tilde{G}_\lambda$ in the following proposition:

Proposition 7. The self-similar closure of G_λ is neither torsion, nor torsion free, level-transitive group of tree automorphisms. Moreover, it is generated by a bounded automaton, hence it is contracting and amenable.

Proof. The first three assertions are already proved above. The automaton $\mathcal{A}_{u,v}$ is bounded by Corollary 14 in [Sid00] (see the definition there as well). As a corollary $\tilde{G}_\lambda$ is contracting (see [BN03]) and amenable (see [BKNV06]). $\square$

Corollary 8. G_λ is amenable.

Note also that the last corollary follows from Theorem 16.

Corollary 9. The word problem in G_λ is solvable in polynomial time.

Proof. See Proposition 2.13.10 in [Nek05]. □

5. Intermediate growth

Let G be a group finitely generated by a set S. *The growth function of* G is defined by

$$\gamma_G(n) = \left|\{g \in G | g = s_1 s_2 \dots s_k \text{ for some } s_i \in S \cup S^{-1}, k \leq n\}\right|.$$

Two functions γ_1 and γ_2 are called *equivalent* if there exists a constant $C > 0$ such that $\gamma_1(\frac{1}{C}n) \leq \gamma_2(n) \leq \gamma_1(Cn)$ for all n. The growth function γ_G depends both on G and on S, but the equivalence class of γ_G does not depend on S.

In 1968 John Milnor asked about the existence of finitely generated groups with growth that is intermediate between polynomial and exponential. The first examples of such groups were provided by R.I. Grigorchuk in [Gri83], where he constructed uncountable family of such groups. In particular, it was shown, that there are groups of intermediate growth generated by automata with 5 states, namely, G_ω for $\omega = (012)^\infty$ (not to be confused with Sushchansky groups G_λ). These examples were generalized to the notion of G groups [BGŠ03]. Under some finiteness restriction all G groups have intermediate growth.

Recently it was proved [BP06] that there is a 4-state automaton over a 2-letter alphabet generating a group of intermediate growth. This group itself is isomorphic to the iterated monodromy group of the map $f(z) = z^2 + i$. But it is still an open question whether there is a group of intermediate growth generated by a 3-state automaton over a 2-letter alphabet.

In view of the examples above it is not very surprising that the two of the pioneering examples of infinite finitely generated periodic groups introduced by S.V. Aleshin in [Ale72] and V.I. Sushchansky in [Sus79] also have intermediate growth. For Aleshin group it follows from the intermediate growth of Grigorchuk group and the result of Y.I. Merzlyakov [Mer83], who proved that Aleshin group contains a subgroup of finite index isomorphic to the subdirect product of four copies of Grigorchuk group. Also the relation between these two groups was studied in [Gri85b].

As was mentioned above in [Gri85a] R.I. Grigorchuk pointed out that all Sushchansky groups have intermediate growth, but only the idea of proof was given. In this paper we give a complete proof of this fact based on the results from [BŠ01].

At the present moment the main method of obtaining the upper bounds for growth functions of groups was originated by R.I. Grigorchuk in [Gri84]. Different modifications of this method in [Bar98, MP01, BŠ01] allowed to improve existing estimates and to prove the estimates for new groups.

As for the lower bounds for growth functions, there are several techniques. In [Gri84] R.I. Grigorchuk uses self-similarity to obtain the lower bound of the form $e^{\sqrt{n}}$ for most of his groups. Moreover, he shows that any group G that is abstractly commensurable with its own power G^k for some $k \geq 2$ has a growth function not smaller that e^{n^α} for some $0 < \alpha \leq 1$.

In [Gri89] R.I. Grigorchuk used bounds on the coefficients of Hilbert-Poincaré series of graded algebras associated with groups to bound their growth functions. Namely, it was obtained that any residually p-group whose growth function is not bounded above by polynomial, must grow at least as $e^{\sqrt{n}}$.

Y.G. Leonov [Leo01], L. Bartholdi and Z. Šunić [Bar98, BŠ01] used more advanced techniques (common in spirit to the ones used in [Gri84]) also based on certain self-similarity of the groups acting on trees. In obtaining the lower bounds for the growth functions of these groups the important role was played by the property, which is in some sense opposite to contraction. The main idea is that the restrictions of elements can not be much shorter than the elements themselves.

A. Erschler used random walks and Poisson boundary to approach to this question. In particular, in [Ers04] it was shown that the growth function of Grigorchuk group G_ω for $\omega = (01)^\infty$, which is generated by 5-state automaton, grows faster than e^{n^α} for any $\alpha < 1$. The upper estimate of the same sort was obtained for this group in spirit of [Gri84], which shows that groups G_ω for $\omega = (012)^\infty$ and $\omega = (01)^\infty$ have essentially different growth functions.

Recall the definition of a G group.

Definition 4. Let R be a subgroup of $Sym(X)$, D be any group with a sequence of homomorphisms $w_i : D \to \mathrm{Sym}(X)$, $i \geq 1$. Then R acts on the first level of X^* and D acts on X^* in the following way. Each $d \in D$ defines the automorphism $\hat{d}$ that acts trivially on the first level and is given by its restrictions

$$\hat{d}\big|_{0^i1} = w_i(d), i \geq 1$$

and all the other restrictions act trivially on X. Denote $\hat{D} = \{\hat{d} \mid d \in D\}$.

The group $G = \langle R, \hat{D} \rangle$ is called a G *group* if the following conditions are satisfied:

(i) The groups R and $w_i(D), i \geq 1$, act transitively on X.

(ii) For each $d \in D$ the permutation $w_i(d)$ is trivial for infinitely many indices.

(iii) For each nontrivial $d \in D$ the permutation $w_i(d)$ is nontrivial for infinitely many indices.

The groups R and D are called *the root part* and *the directed part* of G correspondingly.

Note that in [BGŠ03] the definition of a G group is given in slightly more general settings. The results in [BŠ01] and [BGŠ03] imply the following theorem.

Theorem 10. All G groups with finite directed part have intermediate growth.

There is a lower bound for the growth of such groups given in [BGŠ03]:

$$\gamma_G(n) \succeq e^{n^\alpha}, \tag{5}$$

where $\alpha = \frac{\log(|X|)}{\log(|X|)+\log(2)}$.

The sequence of homomorphisms w_i in the definition of a G group is called r-homogeneous, if for every finite subsequence of r consecutive homomorphisms $w_{i+1}, w_{i+2}, \ldots, w_{i+r}$ every element of D is sent to the identity by at least one of the homomorphisms from this finite subsequence. In particular, if the sequence of homomorphisms $\{w_i, i \geq 1\}$ defining a G group is periodic with period r, it is also r-homogeneous.

It is proved in [BŠ01] that in case of r-homogeneous sequence of defining homomorphisms there is an estimate of the upper bound on the growth function. Moreover, in this case if the directed part has finite exponent there is an upper bound on the torsion growth function $\pi(n)$ (the maximal order of an element of length at most n).

Theorem 11 (η-estimate)**.** Let G be a G group defined by an r-homogeneous sequence of homomorphisms. Then the growth function of the group G satisfies

$$\gamma_G(n) \preceq e^{n^\beta}, \tag{6}$$

where $\beta = \frac{\log(|X|)}{\log(|X|)-\log(\eta_r)} < 1$ and η_r is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$.

If the directed part D of G has finite exponent q, then the group G is torsion and there exists a constant $C > 0$, such that the torsion growth function satisfies

$$\pi(n) \leq Cn^{\log_{1/\eta_r}(q)}. \tag{7}$$

Sushchansky groups G_λ are not G groups, because the automorphism B_λ cannot be expressed as $\hat{d}$ for some homomorphisms w_i. On the other hand, the automorphisms q_i and r_i can, and the following proposition shows that the self-similar closure of G_λ contains a subgroup which is a G group. Since the simplified definition of G_λ from Proposition 4 does not simplify considerably the proofs in this section, we use the original definition in order to make this section independent.

Proposition 12. The group $H = \langle q_1, r_1, \sigma \rangle$ is a G group with finite directed part defined by a periodic sequence of homomorphisms with period p^2.

Proof. We prove that the subgroups $\langle q_1, r_1 \rangle$ and $\langle \sigma \rangle$ are the directed and the root parts of H.

First observe that $\langle q_1, r_1 \rangle \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$. Indeed, the group $\langle q_1, r_1 \rangle$ is elementary abelian p-group by Lemma 2. Suppose that $r_1 \in \langle q_1 \rangle$, $r_1 = q_1^k$. Comparing restrictions on words $0\ldots01$ we get $v_i = ku_i$. Contradiction, since $u_i = 0$ and $v_i = 1$ for i with $\beta_i = 0$.

Consider the periodic sequence of homomorphisms $w_i : \langle q_1, r_1 \rangle \to \mathrm{Sym}(X)$ with period p^2 given by $w_i(q_1) = \sigma^{u_i}$ and $w_i(r_1) = \sigma^{v_i}$. Then for any $d \in \langle q_1, r_1 \rangle$ the associated $\hat{d}$ from the definition of a G group coincides with the automorphism d. To complete the proof we need to check the conditions (i)–(iii) from the definition of a G group.

(i) The root part generated by σ acts transitively on X. Furthermore, for any $i \geq 1$

$$\begin{aligned} w_i(q_1) &= \sigma, \quad \text{if } \beta_i \neq 0; \\ w_i(r_1) &= \sigma, \quad \text{if } \beta_i = 0. \end{aligned}$$

In any case $w_i(\langle q_1, r_1 \rangle)$ contains σ and thus acts transitively on X.

(ii),(iii) Let $d = q_1^k r_1^l$, $k, l \in \mathbb{Z}_p$, be an arbitrary nontrivial element of $\langle q_1, r_1 \rangle$. Since the sequence w_i is periodic it suffices to show at least one occurrence of trivial and one occurrence of nontrivial $w_i(d)$.

Find i such that

$$\begin{aligned} (\alpha_i, \beta_i) &= (1, 0), \quad \text{if } l = 0; \\ (\alpha_i, \beta_i) &= (k, l), \quad \text{if } l \neq 0. \end{aligned}$$

Then

$$w_i(d) = \begin{cases} w_i(q_1^k) = \sigma^{ku_i} = 1, & \text{if } l = 0; \\ w_i(q_1^k r_1^l) = \sigma^{ku_i + lv_i} = \sigma^{k+l(-k/l)} = 1, & \text{if } l \neq 0. \end{cases}$$

For a nontrivial occurrence find i such that

$$(\alpha_i, \beta_i) = (0,1), \quad \text{if } l = 0;$$
$$(\alpha_i, \beta_i) = (1,0), \quad \text{if } l \neq 0.$$

Then

$$w_i(d) = \begin{cases} w_i(q_1^k) = \sigma^{ku_i} = \sigma^k, & \text{if } l = 0; \\ w_i(q_1^k r_1^l) = \sigma^{ku_i + lv_i} = \sigma^l, & \text{if } l \neq 0. \end{cases}$$

□

The last proposition shows that the growth function of H satisfies inequalities (5) and (6), for $r = p^2$. Also note that it is proved in [BGŠ03] that a G group is torsion if and only if its directed part D is torsion. Therefore, the group H is torsion. The next proposition exhibits another branch structure inside $\tilde{G}_\lambda$.

Proposition 13. The group $H = \langle q_1, r_1, \sigma \rangle$ is regular branch over its commutator subgroup H'.

Proof. Let $H_k = \langle q_k, r_k, \sigma \rangle$, $k = 1, \dots, p^2$ be the subgroups of $\tilde{G}_\lambda$. First we show that

$$H'_k \succeq H'_{k+1} \times H'_{k+1} \times \cdots \times H'_{k+1} \tag{8}$$

for all k. Indeed, at least one of u_k and v_k is nonzero. Suppose $u_k \neq 0$. Then relations $q_k = (q_{k+1}, \sigma^{u_k}, 1, \dots, 1)$ and $r_k = (r_{k+1}, \sigma^{v_k}, 1, \dots, 1)$ imply

$$\begin{aligned} [q_k, r_k] &= ([q_{k+1}, r_{k+1}], 1, \dots, 1), \\ [q_k, (q_k^{\sigma^{-1}})^{1/u_k}] &= ([q_{k+1}, \sigma], 1, \dots, 1), \\ [r_k, (q_k^{\sigma^{-1}})^{1/u_k}] &= ([r_{k+1}, \sigma], 1, \dots, 1). \end{aligned}$$

Since the projection of the stabilizer of the first level in H_k on the leftmost vertex coincides with H_{k+1} we get $H'_k \succeq H'_{k+1} \times 1 \times \cdots \times 1$. Conjugation by $\sigma \in H_k$ implies inclusion (8). Since $H_1 = H_{p^2+1} = H$, we obtain $H' \succeq H' \times H' \times \cdots \times H'$ as geometric embedding induced by the restriction on X^{p^2}.

The transitivity of H on the levels is proved by the method used in Proposition 3.

Now H is a torsion p-group, hence, so is H/H', which is abelian. But each torsion finitely generated abelian group is finite. Thus, H' is a subgroup of finite index in H. □

When we deal with a group G of automorphisms of X^*, it is sometimes difficult to say something about the whole group, but we know something about the group P generated by all the restrictions of the elements in G on some level k of the tree. In case G is self-similar, P is a subgroup of G and if G is self-replicating, P coincides with G. Some properties of P are inherited by G itself. In particular, if P is finite or torsion then so is G (the converse is not true). But what we are interested in here is that the growth of G can be estimated in terms of the growth of P.

Let S be a finite generating set of G. Then P is generated by the set $\tilde{S}$ of the restrictions of all elements of S on all vertices of k-th level X^k of the tree. The following lemma holds.

Lemma 14. The growth function $\gamma_G(n)$ of the group G with respect to S is bounded from above by

$$\gamma_G(n) \preceq \left(\gamma_P(n)\right)^{|X|^k}, \tag{9}$$

where $\gamma_P(n)$ is the growth function of the group P with respect to $\tilde{S}$. In particular, the growth type of G (finite, polynomial, intermediate or exponential) cannot exceed the one of P.

Proof. Let $g \in G$ be an element of length n with respect to the generating set S. This element induces a permutation π_k of the k-th level of the tree and $|X|^k$ restrictions $g|_v, v \in X^k$, on words of length k. Moreover, different automorphisms correspond to different tuples $(\pi_k, \{g|_v, v \in X^k\})$ of restrictions and permutations. Each such a restriction is a word of length not greater than n with respect to the generating set $\tilde{S}$ of P. So for each vertex $v \in X^k$ the number of possible restrictions on v is bounded from above by $\gamma_P(n)$. □

The following corollary shows an easy way to construct new examples of groups with intermediate (finite, polynomial, exponential) growth.

Corollary 15. Let F be a finite set of automorphisms from $\operatorname{Aut} X^*$, whose restrictions on some level k belong to G (in particular, F could be a set of finitary automorphisms). Then

$$\gamma_G(n) \precsim \gamma_{\langle G,F\rangle}(n) \precsim \left(\gamma_G(n)\right)^{|X|^k}.$$

where $\gamma_{\langle G,F\rangle}(n)$ is the growth function of the group $\langle G, F\rangle$ with respect to $S \cup F$.

In particular the previous corollary shows that if a group G is generated by a finite automaton, then the growth type of this group depends only on the nucleus (see definition in [Nek05]) of this automaton.

An interesting question is whether it is true that if G grows faster than polynomially then $\gamma_G(n) \sim \gamma_{\langle G,F\rangle}(n)$.

We are ready to prove the main results.

Theorem 16. All Sushchansky p-groups have intermediate growth. The growth function of each Sushchansky p-group G_λ satisfies

$$e^{n^\alpha} \preceq \gamma_{G_\lambda}(n) \preceq e^{n^\beta},$$

where $\alpha = \frac{\log(p)}{\log(p)+\log(2)}$, $\beta = \frac{\log(p)}{\log(p)-\log(\eta_r)}$ and η_r is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$, where $r = p^2$.

Proof. The group generated by all the restrictions of elements of G_λ on the second level is $H = \langle q_1, r_1, \sigma\rangle$, which is a G group of intermediate growth by Proposition 12 and Theorems 10 and 11, whose growth function satisfies inequalities (5) and (6). Therefore by Lemma 14 the Sushchansky group G_λ has subexponential growth function, which satisfies inequality

$$\gamma_G(n) \precsim (\gamma_H(n))^{p^2} \precsim \gamma_H(n). \tag{10}$$

The last part of this inequality follows from Proposition 13, where it is proved that H is regular branch over H'.

Now consider the subgroup $L = \langle B_\lambda, AB_\lambda A^{p-1}, A^2B_\lambda A^{p-2}\rangle$ of G_λ. This subgroup stabilizes the second level of the tree and the restrictions of the generators on the second level look like:

$$\begin{array}{lcl} B_\lambda & = & (q_1, *, \ldots, *), \\ AB_\lambda A^{p-1} & = & (r_1, *, \ldots, *), \\ A^2B_\lambda A^{p-2} & = & (\sigma, *, \ldots, *). \end{array}$$

Each word of length n in L will be projected on the corresponding word of length n in H. Therefore $\gamma_L(n) \geq \gamma_H(n)$ for all $n \geq 1$. But L is a finitely generated subgroup of G_λ. Thus

$$\gamma_H(n) \precsim \gamma_L(n) \precsim \gamma_G(n). \tag{11}$$

Inequalities (10) and (11) imply

$$\gamma_G(n) \sim \gamma_H(n). \tag{12}$$

□

Finally, it was mentioned above that the group H is torsion as a G group with torsion directed part. But periodicity of H implies that G_λ is periodic as well. This gives a different proof of Theorem 1 proved by V.I. Sushchansky. The theory of G groups allows to sharpen this result.

Theorem 17. There is a constant $C > 0$, such that the torsion growth function of each Sushchansky p-group G_λ satisfies inequality

$$\pi_{G_\lambda}(n) \leq C n^{\log_{1/\eta_r}(p)},$$

where η_r is the same as in the previous theorem.

Proof. By Proposition 12 the group H is a G group defined by a p^2-homogenous sequence of homomorphisms, whose directed part $\langle q_1, r_1 \rangle$ is an elementary abelian p-group (see Lemma 2). Therefore by Theorem 11 the torsion growth function $\pi_H(n)$ satisfies inequality

$$\pi_H(n) \leq C_1 n^{\log_{1/\eta_r}(p)}$$

for some constant C_1.

For any element g of length n in G_λ, g^p stabilizes the second level of the tree and the restrictions of g^p at the vertices of the second level are the elements of H, whose length is not bigger than pn. Hence, the order of g^p cannot be bigger than the least common multiple of the orders of $g|_v$, $v \in X^2$. Since the orders of these restrictions are the powers of p, the least common multiple coincides with the maximal order among the restrictions. This implies

$$\mathrm{Order}(g) = p \cdot \mathrm{Order}(g^p) \leq p\pi_H(pn) \leq pC_1(pn)^{\log_{1/\eta_r}(p)} \leq C n^{\log_{1/\eta_r}(p)}$$

for $C = C_1 p^{\log_{1/\eta_r}(p)+1}$.

□

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