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FINITELY REPRESENTED DYADIC SETS AND THEIR MULTIELEMENTARY REPRESENTATIONS*

СКІНЧЕННО ЗОБРАЖУВАЛЬНІ ДІАДИЧНІ МНОЖИНИ ТА ЇХ МУЛЬТІЕЛЕМЕНТАРНІ ЗОБРАЖЕННЯ

We obtaine the direct reduction of representations of a dyadic set S such that $|\operatorname{Ind} C(S)|<\infty$ to the bipartite case.

Отримано пряме зведення зображень діадичної множини S, що задовольняє $|\operatorname{Ind} C(S)|<\infty$, до бікомпонентного випадку.

Introduction. The dyadic sets (= biinvolutive posets) were introduced in [1] (see also [2]), and for any such S the poset $C(S)$ and the natural map $\Theta : \operatorname{Ob}\operatorname{Rep} C(S) \to$ $\to \operatorname{Ob}\operatorname{Rep} S$ were constructed (see § 1). The representations in $\operatorname{Im}\Theta$ are called *multielementary* [3] (= normal [4]). In [1, 5] the main statement ([2], 5.8) was proved

$$|\operatorname{Ind} S| < \infty \quad \text{iff} \quad |\operatorname{Ind} C(S)| < \infty.$$

For a special case it has been proved in [3]; here is only one (up to equivalence and duality) non-multielementary indecomposable Ex. In the general case non-multielementary representations of finitely-represented (= matrix-finite [4]) S are in some sense the modifications of Ex, they were classified in [6, 7].

Here we propose the direct reduction of the general case to bipartite one. For any string $P \subset S$ (see § 2) we define the bundle Π_P and the natural map Θ_P: $\operatorname{Ob}\operatorname{Rep}\Pi_P \to \operatorname{Ob}\operatorname{Rep} S$. Put $K_P = \operatorname{Im}\Theta_P$, $K = \oplus_{P\subset S} K_P$. The Theorem of § 6 states that *if* $|\operatorname{Ind} C(S)| < \infty$, *the set* K *is dense in* $\operatorname{Rep} S$. $\operatorname{Ind}\Pi_P$ coincides with $\operatorname{Ind} S_\Pi$ for some bipartite S_Π (such that $C(S_\Pi) \simeq C(S) < C(S)$, see § 2, 3), therefore, by results of [8], $|\operatorname{Ind} C(S)| < \infty$ implies $|\operatorname{Ind} S| < \infty$ and we get some description of $\operatorname{Ind} S$.

Non-equivalent representations of $C(S)$ can become equivalent under the Θ. In the appendix we show (without complete proof) how to exclude such ambiguity.

§ 1. A triple $S = (S, \le, \approx)$, where S is a finite set, $\le$ an order on S and $\approx$ an equivalence on $\le \subset S \times S$ is called a *dyadic set* if

1) each equivalence class in $S \times S$ contains at most two elements;

2) for any $s, t, p, s', p' \in S$ such that $s \le t \le p$, $s' \le p'$ and $(s, p) \approx (s', p')$ there exists unique $t' \in S$ that satisfied $s' \le t' \le p'$, $(s, t) \approx (s', t')$ and $(t, p) \approx (t', p')$;

3) $(s, t) \approx (s, t')$ implies $t = t'$, and $(s, t) \approx (s', t)$ implies $s = s'$.

If $(s, s) \approx (s', s')$, we set $s \approx s'$, and set $\{s, s^*\} = S^{\approx}(s)$ for $s \in S$ such that $|S^{\approx}(s)| = 2$ $(S^{\approx}(s) = \{x \in S \mid x \approx s\})$. We will denote $s \lhd t$ if $|\le^{\approx}(s, t)| = 1$, and $s \Rightarrow t$ if $s < t$ and $|\le^{\approx}(s, t)| = 2$. In the latter case $\le^{\approx}(s, t) = \{(s, t), (s^*, t^*)\}$, the pair $\varphi = (s, t)$ is called *edge* and $\varphi^* = (s^*, t^*)$ is the edge, *dual* to φ. Put $\overset{\circ}{S} =$ $= \{s \in S \mid |S^{\approx}(s)| = 2\}$.

Note that for any dyadic set S it is possible to construct a vectroid (= subcategory of $\operatorname{mod} k$ that is a spectroid in terms [2]) $\mathcal{V} = \operatorname{Vect} S$ [3], attaching to any class $\{a_1, \dots, a_n\} \in S/\approx (n \le 2)$ a vectorspace A with basis $\{a_1, \dots, a_n\}$ and to any class

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$\{(a_1, b_1), \dots, (a_n, b_n)\} \in \leq/\approx$ a basic morphism $X \in \mathcal{V}(A, B)$, $a_1 X = b_1, \dots, a_n X = b_n$. For this $\mathcal{V}$, $S(\mathcal{V}) \simeq S$.

We define quasiorder $\leq$ on the $S \times \mathbb{N}$ by setting $(s, i) \leq (t, j)$ if $s \leq t$, and equivalence on $\leq$: $((s, i), (t, j)) \approx ((s', i'), (t', j'))$ iff $(s, t) \approx (s', t')$ and $i = i'$, $j = j'$. Set $(s, i) \lhd (t, j)$, if $|\leq^{=}((s, i), (t, j))| = 1$ (equivalent to $s \lhd t$), and $(s, i) \approx (t, j)$ if $((s, i), (s, i)) \approx ((t, j), (t, j))$ (equivalent to $s \approx t$ and $i = j$).

Suppose φ to be a function $S \to \mathbb{N} \cup \{0\}$ such that $\varphi(s) = \varphi(t)$ if $s \approx t$, then we define $S_\varphi = \{(s, i) \in S \times \mathbb{N} \mid i \leq \varphi(s)\}$. *Representation* of S (over field k) of dimension φ is a matrix T (whose sets of rows row T or columns col T may be empty) and bijection $t\colon \operatorname{col} T \to S_\varphi$. Note that if T' is a matrix obtained from T by permutations α of rows and β of columns, than $(T', t') \in \operatorname{Rep} S$, $t' = t\beta$.

If (R, r) is another representation of S, where $r\colon \operatorname{col} R \to S_{\varphi'}$, then the pair of matrices (A, B) is called a *morphism* from (T, t) to (R, r) if $AR = TB$ and for entries b_{ij} of matrix B hold 1) $b_{ij} = 0$ if $t(i) \not\leq r(j)$; 2) $b_{ij} = b_{i'j'}$ if $(t(i), r(j)) \approx (t(i'), r(j'))$. Representations of S with defined above morphisms form the category $\operatorname{Rep} S$; note that $\operatorname{Rep} S \simeq \operatorname{Rep} \mathcal{V}$, $\mathcal{V} = \operatorname{Vect} S$, where the objects of $\operatorname{Rep} \mathcal{V}$ is triples (V, f, X), $V \in \operatorname{mod} k$, $X \in \oplus \mathcal{V}$ and $f \in \operatorname{mod} k(V, X)$, see [3].

Given representation (T, t) of S, bijection t translate relations from $S \times \mathbb{N}$ to $\operatorname{col} T$. Put $\bar{t}(x) = a$, where $t(x) = (a, i)$, $x \in \operatorname{col} T$; $\mathring{\operatorname{col}}\, T = \bar{t}^{-1}(\mathring{S})$. For $x \in \mathring{\operatorname{col}}\, T$, x^* is such column of T that $x^* \approx x$. We will write $x \asymp y$ iff x and y are incomparable elements of $\operatorname{col} T$ or of any other (quasi)poset; for the elements of $\operatorname{col} T$ or any other set with equivalence $\approx$ we will write $x \sim y$ iff $x \approx y$ and $x \neq y$.

Given morphism $(A, B) \in \operatorname{Rep} S((T, t), (R, r))$, $B = (b_{xy})$ and pair $a, b \in S$, in the natural way the matrix B^{ab} is defined, $B^{ab}_{ij} = b_{t^{-1}(a,i) r^{-1}(b,j)}$.

For $(T, t) \in \operatorname{Rep} S$ a subset $X \subset \operatorname{col} T$ is called *a block of* (T, t) if for any $x \in X$ $T_{px} \neq 0$ implies $T_{py} = 0$ for any $y \notin X$. A *block composition* $\mathcal{T}$ of (T, t) is the set $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$, where $\mathcal{T}_i$ are the blocks of (T, t) and $\operatorname{col} T = \coprod_{i=1}^{n} \mathcal{T}_i$; the triple $(T, t, \mathcal{T})$ is called *a block representation.* Uniquely correspond to any block $\mathcal{T}_i$ is the set $\mathcal{T}_i^* \subset \operatorname{row} T$ ($p \in \mathcal{T}_i^*$ if $T_{px} \neq 0$ for some $x \in \mathcal{T}_i$) and (up to permutations of rows and columns) a matrix T_i, $\operatorname{col} T_i = \mathcal{T}_i$, $\operatorname{row} T_i = \mathcal{T}_i^*$.

Block $\mathcal{T}_i$ is *small* if it is either $|\mathcal{T}_i| = 1$, $|\mathcal{T}_i^*| = 0$ or $|\mathcal{T}_i| = |\mathcal{T}_i^*| = 1$, $T_i = (1)$ or $\mathcal{T}_i = \{x_1, x_2\}$, $x_1 \asymp x_2$, $x_1 \not\sim x_2$, $T_i = (1, 1)$. Column y of block representation $(T, t, \mathcal{T})$ is said to be *linked* with column x, if there exists a sequence of columns $x = x_1, \dots, x_{2\alpha} = y$, $\alpha \geq 1$, such that $x_{2i-1} \sim x_{2i}$ $(i = \overline{1, \alpha})$, and x_{2i}, x_{2i+1} are contained in the same block $(i = \overline{1, \alpha - 1})$.

The set $\mathcal{M}$ of block representations consits of such $(T, t, \mathcal{T})$ that all blocks T_i, $i > 1$, are small; $\mathcal{T}_i \cap \mathring{\operatorname{col}}\, T \neq \varnothing$ for $i > 1$; for any pair of dual columns $x, x^* \in \mathring{\operatorname{col}}\, T$ precisely one is linked with a column from $\mathcal{T}_1$.

For $(T, t, \mathcal{T}) \in \mathcal{M}$ the block $\mathcal{T}_1$ will be called *main,* the blocks $\mathcal{T}_i$, $i > 1$, *supplementary.* The representation (T, t) is *multielementary* if there exists $(T, t, \mathcal{T}) \in \mathcal{M}$. The set of all multielementary representations is denoted by M. A block representation $(T, t, \mathcal{T}) \in \mathcal{M}$ and $(T, t) \in M$ are called *elementary* if $\mathcal{T}_1$ is a small block.

Let $\hat{S} = S \cup \{0, 1\}$, $0 \lhd s \lhd 1$ for any $s \in \hat{S}$, and $\hat{S}^{\approx}(0) = \{0\}$, $\hat{S}^{\approx}(1) = \{1\}$. Let us introduce a poset $C(S)$. Elements of $C(S)$ are *zigzags* in S, i. e. the sequences $(s_0, \dots, s_m)$, $m \in \mathbb{N} \cup \{0\}$ of elements $\hat{S}$ such that $s_{i-1}^* \asymp s_i$, $i = \overline{1, m-1}$, and either $s_{m-1}^* \asymp s_m$ and $s_m \in S \setminus \mathring{S}$, or $s_m \in \{0, 1\}$ and $m > 0$. For $\mathcal{S}, \mathcal{J} \in C(S)$, $\mathcal{S} = (s_0, \dots, s_m)$, $\mathcal{J} = (t_0, \dots, t_n)$, we set $\mathcal{S} \leq \mathcal{J}$ iff there exists $j \in \mathbb{N} \cup \{0\}$, $j \leq \min(m, n)$ such that $s_i \Rightarrow t_i$ or $s_i = t_i \in \mathring{S}$ if $i < j$, and either $s_j \lhd t_{j-1}^*$, or $s_{j-1}^* \lhd t_j$, or $s_j \lhd t_j$. Denote by $\gamma\colon C(S) \to S$ the map $\gamma(s_0, s_1, \dots, s_m) = s_0$. Following [4] we will further suppose that $|C(S)| \leq \infty$.

***Remark* 1.** In the similar way the (stronger) order $\bar{\lesssim}$ on $C(S)$ can be introduced: for $\mathcal{S} = (s_0, \dots, s_m)$, $\mathcal{T} = (t_0, \dots, t_n) \in C(S)$ we set $\mathcal{S} \bar{\lesssim} \mathcal{T}$ iff $j \in \mathbb{N} \cup \{0\}$ exists such that $j \leq \min(m, n)$, $s_i \leq t_i$ for $i < j$ and at least one of the following conditions hold: 1) $s_0 \lhd t_0$, 2) $s_1 < t_0^*$, 3) $s_0^* < t_1$, 4) $s_j < t_j$, $j > 0$, 5) $j = m = n$, $s_n = t_n$.

In the natural way a map $\Theta\colon \operatorname{Ob}\operatorname{Rep} C(S) \to \operatorname{Ob}\operatorname{Rep} S$ can be constructed (see, for instance, [4]). In fact Θ may be considered as the map: $\operatorname{Ob}\operatorname{Rep} C(S) \to \mathcal{M}$. However, there is no a natural functor $\operatorname{Rep} C(S) \to \operatorname{Rep} S$. Nevertheless, the results of [3] imply

Proposition 1. *The category* $\operatorname{Rep}' C(S)$ *and the functor* $I\colon \operatorname{Rep}' C(S) \to \operatorname{Rep} S$ *exist such that* $\operatorname{Ob}\operatorname{Rep}' C(S) = \operatorname{Ob}\operatorname{Rep} C(S)$, $\operatorname{Rep} C(S)$ *is epivalent to* $\operatorname{Rep}' C(S)$, $M = \operatorname{Im} I$ *(up to permutation of rows and columns).*

Proof. We may set $\operatorname{Rep}' C(S) = \operatorname{Rep}(\mathcal{E}l, N)$, where a module N over spectroid $\mathcal{E}l$ of elementary representations of S and a functor $I\colon \operatorname{Rep}(\mathcal{E}l, N) \to \operatorname{Rep} S$ were constructed in [3], § 5, 6.

It is easy to see that $I(T) = \Theta(T)$ *for* $T \in \operatorname{Ob}\operatorname{Rep}' C(S)$.

§ 2. Let $P \subset S$ be a *string*, i. e. the subset $\{p_1, \dots, p_k\} \subset \mathring{S}$, such that

1) $p_1 \Rightarrow p_2 \Rightarrow \dots \Rightarrow p_k$;

2) if $p \in P$, $x \in S$ and $x \Rightarrow p$ or $p \Rightarrow x$, then $x \in P$.

In this case the dual set $P^* = \{p_1^*, \dots, p_k^*\}$ is also the string. Since $|C(S)| < \infty$, any $s \in \mathring{S}$ is contained in some uniquely determined string [1]; for pair of dual strings P, P^* for all $p \in P$ either $p < p^*$, or $p^* < p$ (if $x \asymp x^*$ then $(x, \dots, x, \dots, x, 1) \in C(S)$, and $|C(S)| = \infty$).

We say that $s \in \mathring{S}$ is *seminormal* if $\{\mathcal{B} \in C(S) \mid \gamma(\mathcal{B}) \asymp s\}$ is a chain[1]. Point s is *co-seminormal* if s^* is seminormal. The definition immediately imply

Proposition 2. *If* $x \asymp y \in \mathring{S}$ *and* x *is seminormal then* y *is co-seminormal.*

Proposition 3. *If* $|\operatorname{Ind} C(S)| < \infty$, *then either* x *or* x^* *is seminormal.*

Proof. If x, x^* both are not seminormal, then there exist $\mathcal{A}$, $\mathcal{B}$, $C = (c_1, \dots, c_n)$, $\mathcal{D} = (d_1, \dots, d_m) \in C(S)$, $\mathcal{A} \asymp \mathcal{B}$, $x \asymp \{\gamma(\mathcal{A}), \gamma(\mathcal{B})\}$, $C \asymp \mathcal{D}$, $x^* \asymp \{\gamma(C), \gamma(\mathcal{D})\}$. So $\mathcal{A}$, $\mathcal{B}$, $(x, c_1, \dots, c_n)$, $(x, d_1, \dots, d_m)$ are pairwise incomparable in $C(S)$, $|\operatorname{Ind} C(S)| = \infty$.

The string $P \subset S$ is called *normal* if any $p \in P$ is seminormal, the dual string P^* will be called *conormal.* Representation $(T, t, \mathcal{T}) \in \mathcal{M}$ is *conormal* if for any $x \in \mathcal{T}_1$ element $\bar{t}(x)$ is co-seminormal or $\bar{t}(x) \in S \setminus \mathring{S}$.

[1] In [5] the stronger notion of *normal point* was introduced and the proposition stronger then prop. 3 was proved.

Proposition 4. *If* $|\operatorname{Ind} C(S)| < \infty$ *and* (T, t) *is elementary, then conormal* $(T, t, \mathcal{T}) \in \mathcal{M}$ *exists.*

Proof. See [9], prop. 2.3.

A *bundle (of posets)* $\Pi = \Pi_1 \sqcup \Pi_2$ is the pair of posets $(\Pi_1, \leq_1)$, $(\Pi_2, \leq_2)$ with given equivalence $\approx$ on $\leq = \leq_1 \sqcup \leq_2$, such that

1) $(s, t) \sim (s', t')$ implies $s, t \in \Pi_i$, $s', t' \in \Pi_j$, $i \neq j$;

2) for any $s, t, p, s', p' \in \Pi$ such that $s \leq t \leq p$, $s' \leq p'$ and $(s, p) \approx (s', p')$ there exists unique $t' \in \Pi$ that satisfied $s' \leq t' \leq p'$, $(s, t) \approx (s', t')$ and $(t, p) \approx (t', p')$;

3) $(s, t) \approx (s, t')$ implies $t = t'$, and $(s, t) \approx (s', t)$ implies $s = s'$;

As for dyadic sets, given bundle Π, we introduce the relations $\approx, \lhd, \Rightarrow$ and $*$ on Π and $\Pi \times \mathbb{N}$ (for instance, $s \approx s'$ iff $(s, s) \approx (s', s')$).

Suppose $\varphi = \varphi_1 \sqcup \varphi_2 : \Pi \to \mathbb{N} \cup \{0\}$ to be a function such that $\varphi(s) = \varphi(s^*)$, we define $\Pi_\varphi = \{(s, i) \in \Pi \times \mathbb{N} \mid i \leq \varphi(s)\}$, $(\Pi_1)_\varphi = \{(s, i) \in \Pi_\varphi \mid s \in \Pi_1\}$, $(\Pi_2)_\varphi = \{(s, i) \in \Pi_\varphi \mid s \in \Pi_2\}$. *A representation of* Π of dimension φ is the quadruple (T_1, t_1, T_2, t_2), where T_1, T_2 are the matrices and $t_i : \operatorname{col} T_i \to (\Pi_i)_\varphi$ are the bijections. Given another representation (R_1, r_1, R_2, r_2) of Π, the quadruple of matrices (A_1, A_2, B_1, B_2) is said to be a *morphism* from (T_1, t_1, T_2, t_2) to (R_1, r_1, R_2, r_2) if $A_1 T_1 = R_1 B_1$, $A_2 T_2 = R_2 B_2$ and for entries b_{ij}^α of matrices B_α hold 1. $b_{ij}^\alpha = 0$ if $t_\alpha(i) \not\leq r_\alpha(j)$; 2. $b_{ij}^\alpha = b_{i'j'}^\beta$ if $(t_\alpha(i), r_\alpha(j)) \approx (t_\beta(i'), r_\beta(j'))$. So, we have defined the category $\operatorname{Rep} \Pi$.

Let C be an aggregate, and $L = (L_1, L_2)$ be a pair of C-modules. *A representation* of pair $(C, L) = (C, L_1, L_2)$ is a collection (V_1, f_1, V_2, f_2, X), where $V_1, V_2 \in \operatorname{mod} k$, $X \in C$ and $f_1 : V_1 \to L_1(X)$, $f_2 : V_2 \to L_2(X)$ are linear maps. A morphism from (V_1, f_1, V_2, f_2, X) to another representation (W_1, g_1, W_2, g_2, Y) is a triple $(\varphi_1, \varphi_2, \psi)$, where $\varphi_i \in \operatorname{mod} k(V_i, W_i)$, $\psi \in C(X, Y)$ and $f_i \circ L_i \psi = \varphi_i \circ g_i$, $i = 1, 2$. So, we have defined the category $\operatorname{Rep}(C, L)$ of the pair's representations.

The bundle $\Pi = \Pi_1 \sqcup \Pi_2$ allows us to construct an aggregate C, the object set of it spectroid is $\Pi / \approx$, for $s_1, s_2 \in \Pi$ $C(\Pi^{\approx}(s_1), \Pi^{\approx}(s_2))$ = linear hull of the set $\{(x, y) \mid x \in \Pi^{\approx}(s_1), y \in \Pi^{\approx}(s_2), x \leq y\}$ and the pair of modules L_1, L_2, $L_i(\Pi^{\approx}(s))$ = linear hull of $\Pi_i^{\approx}(s)$, the action of C on L_i is obvious. It is evident that $\operatorname{Rep}(C, L) \simeq \operatorname{Rep} \Pi$.

Let K be a module over aggregate C, $i : K' \to K$ an inclusion of submodule and $\pi : K \to K'' = K / K'$ a projection on factor. K' is said to be a *component* in K if for any $X, Y \in C$ and a map $\alpha \in \operatorname{mod} k(K''X, K'Y)$ there exists $\xi \in C(X, Y)$ such that $K\xi = \pi X \circ \alpha \circ iY$. For arbitrary submodule $K' \subset K$ (not supposed to be a component), the functor

$$F_{K'} : \operatorname{Rep}(C, K) \to \operatorname{Rep}(C, K', K'');$$

$$F_{K'}(V, f, X) = (\operatorname{Ker}(f \circ \pi X), f', \operatorname{Im}(f \circ \pi X), \operatorname{can}, X),$$

is defined, where f' is unique linear map for which $f' \circ iX = f | \operatorname{Ker}(f \circ \pi X)$, and $\operatorname{can} : \operatorname{Im}(f \circ \pi X) \to K''(X)$ is a subspace inclusion.

Lemma 1. *Given a component* $K' \subset K$, *functor* F *induced an injection* $\operatorname{Ind}(C, K)$ *into* $\operatorname{Ind}(C, K', K'')$. *The only isoclass does not contained in the image of* F *is the class of* $(0, 0, k, 0, 0)$.

Proof is easy (see [8], § 7).

Corollary 1. *Given the bundle* Π, *the dyadic set* S_Π *and the functor* $\operatorname{Rep} S_\Pi \to \operatorname{Rep} \Pi$ *exist that the image of induced injection* $\operatorname{Ind} S_\Pi \to \operatorname{Ind} \Pi$ *does not contain the only class of* (T_1, t_1, T_2, t_2), *where* $\operatorname{col} T_1 = \operatorname{col} T_2 = \operatorname{row} T_1 = \varnothing$, $|\operatorname{row} T_2| = 1$.

Proof. S_Π is built as follows: $S = \Pi$, $\leq_S = \leq_\Pi \cup \preceq$, where $s \preceq t$ iff $s \in \Pi_1$, $t \in \Pi_2$, $\approx_S = \approx_\Pi$.

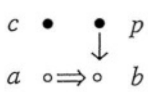
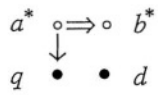
Stated Corollary allows us to transfer definitions for dyadic sets to bundle case, for instance, we set $C(\Pi) = C(S_\Pi)$. The image of $M(S_\Pi)$ in $\operatorname{Rep} \Pi$ will be called a set of *multielementary representations of bundle* Π.

A bundle Π is *bipartite* if it contains exactly 2 strings P and P^*. It is clear that Π is bipartite iff S_Π is bipartite in sense of [8].

We say that exceptional representation Ex of bundle Π is such (T_i, t_i, T_j, t_j) (where $\{i, j\} = \{1, 2\}$) that

$$T_i = \begin{array}{c} \begin{matrix} a & b & c & p \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad T_j = \begin{array}{c} \begin{matrix} a^* & b^* & d & q \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

(we write here over $x \in \operatorname{col} T_i$ the element $\bar{t}_i(x) \in \Pi_i$), where Π_i, Π_j contain full subposets

 $\subset \Pi_i$,  $\subset \Pi_j$,

and $(a, b) \sim (a^*, b^*)$.

Proposition 5. *If a bundle* Π *is a bipartite and* $|\operatorname{Ind} C(\Pi)| < \infty$., *then*

a. *Any* $(T_1, t_1, T_2, t_2) \in \operatorname{Ind} \Pi$ *is either multielementary or exceptional.*

b. *If* $\operatorname{Rep} \Pi$ *contains a faithful multielementary* $(T_1, t_1, T_2, t_2) \in \operatorname{Ind} \Pi$, *then either* P *or* P^* *is normal.*

Proof. See [8], Th. 2.

Remark 2. Moreover, [8] implies $\Theta_\Pi \colon \operatorname{Ob} \operatorname{Rep} C(\Pi) \to \operatorname{Ob} \operatorname{Rep} \Pi$, $C(\Pi) = C(S_\Pi) = C_1 < C_2$ ($C_i = \gamma^{-1}(\Pi_i)$, $\operatorname{Ind} C(\Pi) = \operatorname{Ind} C_1 \sqcup \operatorname{Ind} C_2$). If the string P is normal, then Θ_Π induces the bijection between $\operatorname{Ind} C_2 \setminus \{0\}$ and $\operatorname{Ind} \Pi \setminus \operatorname{Ind} (\Pi_1 \setminus \mathring{\Pi}_1)$.

§ 3. Let $P \subset S$ be a string, $p < p^*$ for all $p \in P$. We define the set $K_P = K_{P^*}$ as consisting of block representations $(T, t, \{\mathcal{T}_1, \ldots, \mathcal{T}_k\})$, $k \geq 2$ such that

– for $i > 2$ the blocks T_i are small;

– for $i > 2$, $\mathcal{T}_i \cap \mathring{\operatorname{col}}\, T \neq \varnothing$;

– if $x \in \mathring{\operatorname{col}}\, T$ then x or x^* is linked with $y \in \mathcal{T}_1 \cup \mathcal{T}_2$;

– for $x < x^*$ such conditions are equivalent:

a. both x, x^* are linked with some elements of $\mathcal{T}_1 \cup \mathcal{T}_2$;

b. $x \in \mathcal{T}_1 \cap \bar{t}^{-1}(P)$;

c. $x^* \in \mathcal{T}_2 \cap \bar{t}^{-1}(P^*)$;

For $(T, t, \mathcal{T}) \in K_P$ the blocks $\mathcal{T}_1$, $\mathcal{T}_2$ will be called *main*, the blocks $\mathcal{T}_i$, $i > 2$, *complementary.* Let K be a set of representations $(T_1, t_1) \oplus \ldots \oplus (T_n, t_n)$, where for some string $Q_i \subset S$ and a block composition $(T_i, t_i, \mathcal{T}_i) \in K_{Q_i}$.

Define the bundle $\Pi_P = \Pi_1 \sqcup \Pi_2$, $\Pi_1 = (C(S)\setminus\gamma^{-1}(P) \cup P$, $\Pi_2 = (C(S)\setminus\gamma^{-1}(P^*)) \cup P^*$. If $x \in \Pi_1 \setminus P$, $y \in P$, then $x < y$ $(x > y)$ iff $\gamma(x) < y$ $(\gamma(x) > y)$, and the same for Π_2; order on P, P^*, $\Pi_1 \setminus P$ and $\Pi_2 \setminus P^*$ is induced by the orders on S and $C(S)$; $(p_1, p_2) \sim (q_1, q_2)$ if $q_1, q_2 \in P^*$ and $(p_1, p_2) \sim (q_1, q_2)$ in S. It is easy to see that $C(\Pi_P) = C_1 < C_2$, where $C_1 \simeq C_2 \simeq C(S)$. As in multielementary case, the map Θ_P: $\mathrm{ObRep}\,\Pi_P \to \mathrm{Ob\,Rep}\,S$ can be constructed, $\mathrm{Im}\,\Theta_P = K'_P$, where K'_P consists of such (T, t) that allow a block composition $(T, t, \mathcal{T}) \in K_P$: For $i = 1, 2$ the matrix T_i attached to block $\mathcal{T}_i \in \mathcal{T}$ coincides with the matrix T_i of $(T_1, t_1, T_2, t_2) \in \mathrm{Rep}\,\Pi_P$; each complementary block of $\mathcal{T}$ is attached in the natural way to some element b_j, $j > 0$ in zigzag $\mathcal{B} = (b_0, \ldots, b_n) \in \mathrm{Supp}\,\overline{T_1} \cup \mathrm{Supp}\,\overline{T_2}$, where $\overline{T_i} \in \mathrm{Rep}\,\Pi_i$ corresponds to T_i.

Proposition 6. *The category* $\mathrm{Rep}'\,\Pi_P$ *and the functor* J_P: $\mathrm{Rep}\,\Pi_P \to \mathrm{Rep}\,S$ *exist such that* $\mathrm{Ob\,Rep}'\,\Pi_P \to \mathrm{Ob\,Rep}\,\Pi_P$, $\mathrm{Rep}\,\Pi_P$ *is epivalent to* $\mathrm{Rep}\Pi_P$, $\mathrm{Im}\,J_P = K'_P$ *up to permutation of rows and columns* (J_P *and* Θ_P *coincide as objects map).*

Proof. See [9], prop. 3.1.

***Corollary* 2.** *Given* $X, Y \in \mathrm{Rep}\,\Pi_P$, *then*

1) $\Theta_P(X \oplus Y) \simeq \Theta_P(X) \oplus \Theta_P(Y)$;

2) *if* $X \simeq Y$, *then* $\Theta_P(X) \simeq \Theta_P(Y)$.

Representation $T_{\mathrm{ex}} = (T, t) \in \mathrm{Rep}\,S$ is *exceptional* if there exist a string P in S and exceptional representation $\mathrm{Ex} \in \mathrm{Rep}\,\Pi_P$ such that $\Theta_P(\mathrm{Ex}) = T_{\mathrm{ex}}$[2].

It is easy to see that (since $|C(S)| < \infty$) *only a finite number of exceptional representations of dyadic* S *exists.*

Proposition 7. *Any* $(T, t) \in K$ *is equivalent to a direct sum of exceptional and multielementary representations.*

Proof. Bounding to direct summand we can suppose that for some block composition $\mathcal{T}$ and string P we have $(T, t, \mathcal{T}) \in K_P$. Then $(T_1, t_1, T_2, t_2) \in \mathrm{Rep}\,\Pi_P$ exist such that $\Theta_P(T', t') = (T, t)$. But $(T_1, t_1, T_2, t_2) \simeq \mathrm{Ex}_1 \oplus \ldots \oplus \mathrm{Ex}_n \oplus (R_1, r_1, R_2, r_2)$, where (R_1, r_1, R_2, r_2) is multielementary representation of bundle. Since Θ_P maps multielementary to multielementary and exceptional to exceptional, we finish the proof, using the corollary 2.

***Remark* 3.** If the dyadic set S is *weakly completed poset,* i. e. S does not contain edges, it is known that the sets M, and, consequently, K are dense in $\mathrm{Rep}\,S$, as it was proved in [10] (see also [1], p. 4).

The natural transformations P: $\mathcal{M} \to K_P$, P^*: $\mathcal{M} \to K_P$ are defined: if $(T, t, \mathcal{T}) \in \mathcal{M}$, then $P((T, t, \mathcal{T})) = (T, t, \mathcal{T}')$, $P^*((T, t, \mathcal{T})) = (T, t, \mathcal{T}'')$, where $\mathcal{T}_1 = \mathcal{T}'_1 = \mathcal{T}''_2$,

$$\mathcal{T}'_2 = \{x \in \textstyle\bigcup_{i \ge 2} \mathcal{T}_i \mid x \sim y \text{ for some } y \in \mathcal{T}_1,\ \bar{t}(x) \in P^*\},$$

$$\mathcal{T}''_1 = \{x \in \textstyle\bigcup_{i \ge 2} \mathcal{T}_i \mid x \sim y \text{ for some } y \in \mathcal{T}_1,\ \bar{t}(x) \in P\}.$$

[2] Every exceptional representation is indecomposable; this fact is not used.

After the proper permutation the other columns form complementary blocks $\mathcal{T}_i'$, $\mathcal{T}_i''$, $i > 2$.

A subset $X \subset C(S)$ is called *locally linear,* if there are no $\mathcal{B}_1, \mathcal{B}_2 \in X$, such that $\mathcal{B}_1 \asymp \mathcal{B}_2$ in $C(S)$ and $\gamma(\mathcal{B}_1) = \gamma(\mathcal{B}_2)$. Representation $T \in \operatorname{Rep} C(S)$ is *locally linear,* if $\operatorname{Supp} T \subset C(S)$ is a locally linear subset. Denote by $\overline{\operatorname{Rep}}\, C(S)$, $\overline{\operatorname{Ind}}\, C(S)$ the subsets of locally linear representations contained in resp. sets.

For $\mathcal{B} \in C(S)$, $\mathcal{B} = (z_0, \dots, z_m)$, set $h(\mathcal{B}) = m$ if $z_m \in S$, $h(\mathcal{B}) = m-1$ if $z_m \in \in \{0, 1\}$. For $T \in \operatorname{Rep} C(S)$, *height* of T is the integer $h(T) = \sup\{h(\mathcal{B}) \mid \mathcal{B} \in \in \operatorname{Supp} T\}$. A representation $T \in \overline{\operatorname{Ind}}\, C(S)$ is said to be *of minimal height* if $h(T) = = \inf\{h(T') \mid T' \in \overline{\operatorname{Ind}}\, C(S),\ \Theta(T') \simeq \Theta(T)\}$.

Proposition 8. *If* $|\operatorname{Ind} C(S)| < \infty$, *the set* $\Theta(\overline{\operatorname{Ind}}\, C(S))$ *is dense in* $\operatorname{Ind} M$.

Proof. We remind that if $\mathcal{A} = (a, a_1, \dots, a_n) \in C(S)$, then $\partial^2_{a^*}\mathcal{A} = (a_1, \dots, a_n)$ [3]. For $T \in \operatorname{Ind} C(S)$ we denote $B(T) = \{\mathcal{A} \in \operatorname{Supp} T \mid \mathcal{A} \asymp \mathcal{A}' \in \operatorname{Supp} T,\ \gamma(\mathcal{A}) = = \gamma(\mathcal{A}')\}$, $\overline{h}(T) = \sum_{\mathcal{A} \in B(T)} h(\mathcal{A})$. We will prove our statement by induction by $\overline{h}(T)$. $\overline{h}(T) \neq 0$. Taking some proper $\approx$ – closed subset of S, we can suppose that $\Theta(T)$ is a faithful representation. There exist $\mathcal{B}_1, \mathcal{B}_2 \in \operatorname{Supp} T$ such that $\mathcal{B}_1 \asymp \mathcal{B}_2$ and $\gamma(\mathcal{B}_1) = \gamma(\mathcal{B}_2) = q \in \mathring{S}$. Let Q be a string in S, $q \in Q$ and Q^* the dual string, and, for defineteness, $q < q^*$. Representation $\Theta_{\Pi_Q}(T)$ is indecomposable (remark 2). It may be not faithful for Π_Q, but it is faithful for some bipartite bundle $\Pi_Q' \subset \Pi_Q$ $(Q, Q^* \subset \Pi_Q')$. The definition of the order on $C(S)$ immediately implies that $\partial^2_{q^*}\mathcal{B}_1 \asymp \partial^2_{q^*}\mathcal{B}_2$, and since $\gamma(\partial^2_{q^*}\mathcal{B}_i) \asymp q^*$, $i = 1, 2$, the string Q^* in Π_Q' is not normal, and, therefore, the string Q in Π_Q' is normal (see prop. 5). By remark 2 there exists $\overline{T} \in \operatorname{Ind} C_2 \subset \operatorname{Ind} C(\Pi)$, $C_2 \simeq C(S)$ such that $Q^*(\overline{T}) = Q(T)$ and $\Theta(T) \simeq \Theta(\overline{T})$. Our constructions imply $\operatorname{Supp} \overline{T} \setminus \gamma^{-1}(Q^*) = \{\partial^2_{a^*}\mathcal{A} \mid \mathcal{A} \in \operatorname{Supp} T,$ $a \in Q\}$. It is easy to see that $\gamma^{-1}(Q^*) \cap B(\overline{T}) = \varnothing$ (since Q is normal in Π_Q'); $\partial^2_{a^*}\mathcal{A} \in B(\overline{T})$ implies $\mathcal{A} \in B(T)$. Thus, $\overline{h}(\overline{T}) < \overline{h}(T)$.

***Remark* 4.** Define a subset $U(S) \subset C(S)$, we have $\mathcal{S} = (s_0, \dots, s_m) \in U(S)$ iff for any $i, j = \overline{0, m-1}$, $s_i^* \neq s_j$. The proof also shows that *for any* $T \in \operatorname{Ind} U(S)$ *representation* $\overline{T} \in \operatorname{Ind} U(S) \cap \overline{\operatorname{Ind}}\, C(S)$ *exists such that* $\Theta(T) \simeq \Theta(\overline{T})$ (see [9], remark 3.2).

Proposition 9. *Let* $|\operatorname{Ind} C(S)| < \infty$, P *be a string,* $\operatorname{Supp} T \subset P$, *and* $h(T) > > 1$. *Then* T *is not of minimal height.*

Proof. Construct $\overline{T}$ as in the proof of prop. 8 (clearly, P is normal in Π_P'). If $\mathcal{A} \in \operatorname{Supp} \overline{T} \cap \gamma^{-1}(P)$, then $h(\mathcal{A}) = 1$, if $\mathcal{B} \in \operatorname{Supp} \overline{T} \cap \gamma^{-1}(P)$, then $B = \partial^2_{x^*}\mathcal{D}$, $\mathcal{D} \in \operatorname{Supp} T$, $h(\mathcal{B}) = h(\mathcal{D}) - 1$. $\overline{T} \in \overline{\operatorname{Ind}}\, C(S)$ since $T \in \overline{\operatorname{Ind}}\, C(S)$.

§ 4. Let C be a poset, $(\mathcal{K}, M)$ a corresponding module over aggregate, $\mathcal{K} = = \oplus \operatorname{Vect} C$, $M = \oplus \operatorname{Vect} C$, considered as module over himself, $x \in C$.

Module M contains submodule N: for $y \in C$

$$N(y) = \begin{cases} 0, & y \leq x; \\ M(y), & y > x \text{ or } y \asymp x, \end{cases}$$

and a factormodule $L = M / N$ is defined.

We will call a representation $T = (V, f, X) \in \operatorname{Rep}(\mathcal{K}, M) \simeq \operatorname{Rep} C$ *x-projective*, if

1) $\operatorname{Supp} T \cap C^{\geq}(x) = \varnothing$;

2) the linear map $V \xrightarrow{f} M(X) \xrightarrow{\text{can}} L(X)$ is injective.

Condition 2 is equivalent to linear independence of rows of submatrix of representation matrix T, which consists of columns that are less x. In the dual manner, the definition of *x-injectivity* can be done. In order to do this, we need to consider submodule $N_1 \subset M$,

$$N_1(y) = \begin{cases} 0, & y < x \text{ or } y \asymp x; \\ M(y), & y \geq x, \end{cases}$$

and demand the fulfilment of two conditions:

1) $\operatorname{Supp} T \cap C^{\leq}(x) = \varnothing$;

2) the linear map $V \xrightarrow{f} M(X) \xrightarrow{\text{can}} M/N_1(X)$ is surjective.

Proposition 10. *If $T \in \operatorname{Rep}(\mathcal{K}, M)$, and $\operatorname{Supp} T \cap C^{\geq}(x) = \varnothing$, then T is x-projective iff any representation of the form $T' = (V, (f\varphi), X \oplus x)$ is isomorphic to $T \oplus (0, 0, x)$.*

The proof is trivial.

In analogous way the dual proposition 1* for x-injective representations is formulated (with substitution $(k, 1, x)$ for $(0, 0, x)$).

Proposition 11. *If $x, y \in C$, $x \leq y$, $T \in \operatorname{Ind} C$, $\operatorname{Supp} T \cap C^{\leq}(x) = \varnothing$, $\operatorname{Supp} T \cap C^{\geq}(y) = \varnothing$, and T is neither x-injective nor y-projective, then there exists an indecomposable $T' \in \operatorname{Ind} C$ such that $\operatorname{Supp} T' = \operatorname{Supp} T \cup \{x, y\}$.*

In order to prove this statement, an indecomposable with the support $\operatorname{Supp} T \cup \{x\}$ should be considered. The condition 2 from the definition of y-projectivity does not hold for this representation because it does not for T.

We remind that the edge $\varphi = a \Rightarrow b$ is 1) *maximal* iff there is no x such that $b < x$, $a \Rightarrow x$ and no y such that $y < a$, $y \Rightarrow b$; 2) *equipped* by u iff $u \asymp \{a, b\}$. Set $\operatorname{eq}(\varphi) = |\{\mathcal{B} \in C(S) \mid \gamma(\mathcal{B}) \asymp \{a, b\}\}|$. If $\operatorname{eq}(\varphi)$ is at least 2 we say that φ is twice equipped. If the dual edge $\varphi = a^* \Rightarrow b^*$ is equipped, the edge φ is called *coequipped.*

Fix a maximal edge $\varphi = a \Rightarrow b$ in dyadic S, and suppose $|\operatorname{Ind} C(S)| < \infty$. For definiteness, we suppose that $a < a^*$, and, therefore, $b < b^*$, let P be a string containing φ.

Lemma 2. *Let $\mathcal{A}, \mathcal{B} \in C(S)$, $\gamma(\mathcal{A}) \leq a$, $\gamma(\mathcal{B}) \geq b^*$. Then there is no $T \in \operatorname{Ind} C(S)$ such that $\mathcal{J} = \operatorname{Supp} T \supset \{\mathcal{A}, \mathcal{B}\}$.*

Proof. Suppose that such T exists. Clearly, $\mathcal{A} < \mathcal{B}$. Define $\overline{\mathcal{J}} = \{\mathcal{X} \in \mathcal{J} \mid \mathcal{X} \asymp \{\mathcal{A}, \mathcal{B}\}\}$. The Kleiner's list of faithful finitely represented posets shows that $\overline{\mathcal{J}} \neq \varnothing$. Let $\mathcal{X} \in \overline{\mathcal{J}}$, $x \in \gamma(\mathcal{X})$.

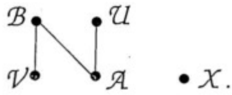
1. $x \notin P \cup P^*$. If $x \in P$, and $x \leq b$, then $x \lhd b^*$ and $\mathcal{X} < \mathcal{B}$; if $x > b$, then owing to maximality of $a \Rightarrow b$ we have $a \lhd x$, and $\mathcal{A} < \mathcal{X}$.

2. $x \asymp \{a, b, a^*, b^*\}$. Otherwise $a < x$ or $x < b^*$. If, for definiteness, $a < x$, then 1 implies that $a \lhd x$, so, $\mathcal{A} < \mathcal{X}$.

3. $|\overline{\mathcal{J}}| = 1$ and $x \notin \mathring{S}$. Otherwise 2 implies that the edges $a \Rightarrow b$ and $a^* \Rightarrow b^*$ are twice equipped (that contradicts to $|\operatorname{Ind} C(S)| < \infty$, [1]).

4. $b \lhd a^*$, in the opposite case $b \asymp a^*$, and $\mathcal{J}$ contains subset $(1, 3, 3)$ — $\{\mathcal{X}; (b, 0), (b, 1), (b, x); (a^*, 0), (a^*, 1), (a^*, x)\}$, $|\operatorname{Ind} C(S)| = \infty$.

5. $\mathcal{X}$ contains subset $\{\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}, \mathcal{X}\}$, where



It follows from 3 and the Kleiner's list of faithful finitely represented posets (see [2], 5.4).

6. $\gamma(\mathcal{U}) \geq a^*$. Indeed, if $\gamma(\mathcal{U}) \notin P^*$, then $\gamma(\mathcal{U}) \not\asymp \{b^*, a^*\}$, and two incomparable points x and $\gamma(\mathcal{U})$ equip the edge $a^* \Rightarrow b^*$ ($|\operatorname{Ind} C(S)| < \infty$). If $\gamma(\mathcal{U}) \in P^*$, then $\gamma(\mathcal{U}) < a^*$, and $\gamma(\mathcal{U}) \lhd b^*$ owing to maximality of $a \Rightarrow b$; in this case $\mathcal{U} < \mathcal{B}$.

7. In analogous way, it is checked that $\gamma(\mathcal{V}) \leq b$.

8. Now, $\gamma(\mathcal{V}) \leq b \lhd a^* \leq \gamma(\mathcal{U})$, and we obtain the contradiction $\mathcal{V} < \mathcal{U}$.

Proposition 12. *If $T \in \operatorname{Ind} C(S)$ is not $(b^*, 0)$-projective, then it is $(a, 1)$-injective.*

Proof. By the lemma 2 either (1) $\operatorname{Supp} T \cap C(S)^{\geq}(b^*, 0) = \varnothing$, or (2) $\operatorname{Supp} T \cap$ $\cap\, C(S)^{\leq}(a, 1) = \varnothing$. Suppose that the (1) holds (for (2) the proof is similar). If T is not $(b^*, 0)$-projective, then by proposition 10 an indecomposable with support $\operatorname{Supp} T \cup \{(b^*, 0)\}$ exists, and again by the lemma 2, $\operatorname{Supp} T \cap C(S)^{\leq}(a, 1) = \varnothing$. Either T is $(a, 1)$-injective, or we can construct an indecomposable representation of $C(S)$ with support $\operatorname{Supp} T \cup \{(b^*, 0), (a, 1)\}$, so we have once more contradiction to the lemma 2. The statement is proved.

§ 5. Given a maximal edge $\varphi = a \Rightarrow b$, $a < a^*$ in S, the dyadic set $S_\varphi^\lhd = S^\lhd$ can be obtained from S by „excluding" the edge $a \Rightarrow b$, i. e. $S^\lhd = S$ as sets, $\leq_{S^\lhd} = \leq_S$, $\approx_{S^\lhd} = \approx_S$, and $(x, y) \sim_{S^\lhd} (x', y')$ iff $(x, y) \sim_S (x', y')$, $(x, y) \notin \{\varphi, \varphi^*\}$. We have $\operatorname{Ob} \operatorname{Rep} S = \operatorname{Ob} \operatorname{Rep} S^\lhd$. Moreover, $C(S) = C(S^\lhd)$ as sets, the order on $C(S^\lhd)$ is strengthening of the order on $C(S)$. Hence if $C(S)$ is finitely represented, then $C(S^\lhd)$ is finitely represented too.

Let $(T, t) \in \operatorname{Rep} S$, $x, x^* \in \operatorname{col} T$, $x < x^*$, $x \in \bar{t}^{-1}(\{a, b\})$. If $t(x) = (a, i)$ (resp. $t(x) = (b, i)$) then the pair x, x^* is *not essential* if the morphism $(A, B) \in \operatorname{Rep} S^\lhd(X, (T^-, t^-))$ (resp. $(A, B) \in \operatorname{Rep} S^\lhd((T^+, t^+), X)$ exists such that $B_{i1}^{ab} \neq B_{i1}^{a^*b^*}$, $B_{j1}^{ab} =$ $= B_{j1}^{a^*b^*}$, $j \neq i$, where $\operatorname{row} T^- = \varnothing$, $\operatorname{col} T^- = \{y < y^*\}$, $\bar{t}^-(y) = b$ (resp. $|\operatorname{row} T^+| =$ $= 2$, $\operatorname{col} T^+ = \{z < z^*\}$, $\bar{t}^+(z) = a$, and T^+ is the unit matrix). B^{ab} is the matrix (with one row here) attached to the pair a, b in § 1.

The representation (T, t) is φ-*injective* (resp. φ-*projective*) if any pair x, x^*, where $\bar{t}(x) = a$ (resp. $\bar{t}(x) = b$) is not essential. For arbitrary string $Q \subset S^\lhd$ block representation $(T, t, \mathcal{T}) \in K_Q(S^\lhd)$ is φ-*quasibijective* if any essential pair $x < x^*$ ($\bar{t}(x) \in \{a, b\}$) is such that $x \in \mathcal{T}_1$, $x^* \in \mathcal{T}_2$ (so, if $(T, t, \mathcal{T})$ is φ-quasibijective, then essential pair can exist only if $Q \cap \{a, b\} \neq \varnothing$).

Lemma 3. *If $(T, t) \in \operatorname{Rep} S^\lhd$ is φ-projective or φ-injective, $(R, r) \simeq (T, t)$ in $\operatorname{Rep} S^\lhd$, then $(R, r) \simeq (T, t)$ in $\operatorname{Rep} S$.*

φ-projectivity or φ-injectivity of (T, t) implies the existence of morphism $(A, B) \in \operatorname{Rad} \operatorname{Rep} S^\lhd((T, t), (R, r))$ with arbitrary B^{ab}, that implies the lemma.

Lemma 4. *Given* $(T, t) \in K^{\triangleleft}$, $(T, t) = \oplus\ (T_\alpha, t_\alpha)$, *such that for some string* $\mathcal{Q}_\alpha$ *and block composition* $\mathcal{T}^{\alpha}$ *representation* $(T_\alpha, t_\alpha, \mathcal{T}^\alpha) \in K_Q(S^{\triangleleft})$ *is quasi-bijective, then the isomorphism* $(R, r) \simeq (T, t)$ *in* $\operatorname{Rep} S^{\triangleleft}$ *implies the existence of* $(T', t') \in K$, $(R, r) \simeq (T', t')$ *in* $\operatorname{Rep} S$.

As in lemma 3 we can prove that isomorphism $(A, B)\colon (T, t) \to (R, r)$ exists such that $B_{ij}^{ab} \neq B_{ij}^{a^*b^*}$ only if $t^{-1}(a, i) \in \mathcal{T}_1^{\alpha}$, $t^{-1}(b, j) \in \mathcal{T}_1^{\beta}$. Then the isomorphism $(\overline{A}, \overline{B}) \in \operatorname{Rep} S\,((T', t'), (R, r))$ exists such that $B_{xy} \neq \overline{B}_{xy}$ only if $\bar{t}(x) = a^*$, $\bar{t}(y) = b^*$, $x \in \cup_\alpha \mathcal{T}_2^{\alpha}$, $y \in \cup_\beta \mathcal{T}_2^{\beta}$. The definition of K_P (§ 3) implies that $(T', t') \in K$.

Let $\{x < x^*\} \subset \operatorname{col} T$, $\bar{t}(x) \in \{a, b\}$. If $(T, t, \mathcal{T}) \in \mathcal{M}$, set $\mathcal{T}(x, x^*) = |\{x, x^*\} \cap \mathcal{T}_1| \in \{0, 1\}$, and, if $(T, t, \mathcal{T}') \in K_P$, set $\mathcal{T}'(x, x^*) = |\{x, x^*\} \cap (\mathcal{T}_1 \cup \mathcal{T}_2)| \in \in \{0, 1, 2\}$.

Given $(T, t, \mathcal{T}) \in \mathcal{M}$, $P((T, t, \mathcal{T})) = (T, t, \mathcal{T}') \in K_P$, $P^*((T, t, \mathcal{T}) = (T, t, \mathcal{T}'')) \in K_P$. We say that

$(T, t, \mathcal{T})$ is *M-bijective* if $x, x^* \in \operatorname{col} T$, $\bar{t}(x) \in \{a, b\}$ and $\mathcal{T}(x, x^*) = 0$ imply the pair x, x^* is not essential;

$(T, t, \mathcal{T})$ is *P-bijective* if $x, x^* \in \operatorname{col} T$, $\bar{t}(x) \in \{a, b\}$ and $\mathcal{T}(x, x^*) = \mathcal{T}'(x, x^*) = 1$ imply the pair x, x^* is not essential;

$(T, t, \mathcal{T})$ is *P^*-bijective* if $x, x^* \in \operatorname{col} T$, $\bar{t}(x) \in \{a, b\}$ and $\mathcal{T}(x, x^*) = \mathcal{T}''(x, x^*) = 1$ imply the pair x, x^* is not essential.

***Remark* 5.** It is obvious that $\mathcal{T}(x, x^*) \leq \min\{\mathcal{T}'(x, x^*), \mathcal{T}''(x, x^*)\}$ and, therefore, that if $(T, t, \mathcal{T}) \in \mathcal{M}$ is M-bijective and P- (resp. P^*-) bijective, then $P(T, t, \mathcal{T})$ (resp. $P^*(T, t, \mathcal{T})$) is φ-quasibijective.

Proposition 13. *Given* $Z \in \operatorname{Ind} C(S)$ *that is* $(b^*, 0)$*-projective, then* $\Theta(Z) = (T, t, \mathcal{T})$ *is P-bijective.*

Proof. It is easy to see, ([9], prop. 5.1).

Proposition 13, the dual proposition and proposition 12 imply

***Corollary* 3.** *If* $|\operatorname{Ind} C(S)| < \infty$, *then any* $(T, t, \mathcal{T}) \in \mathcal{M}$ *is either P-bijective or P^*-bijective.*

Let $(T, t, \mathcal{T}) \in \mathcal{M}$. Complementary block $\mathcal{T}_i$ is said to be *bad* if it is either $\mathcal{T}_i = \{u\}$, $\bar{t}(u) \in S^{\approx}(a)$, $|\mathcal{T}_i^*| = 1$, or $\mathcal{T}_i = \{v\}$, $\bar{t}(v) \in S^{\approx}(b)$, $|\mathcal{T}_i^*| = 0$, or $\mathcal{T}_i = \{x, y\}$, $\bar{t}(x) \in S^{\approx}(\{a, b\})$, $|\mathcal{T}_i^*| = 1$, $\bar{t}(y)$ equips that edge from the pair φ, φ^* that contains $\bar{t}(x)$. The following proposition is almost evident.

Proposition 14. $(T, t, \mathcal{T}) \in \mathcal{M}$ *is M-bijective if (and only if) there are no pairs* $x, x^* \in \operatorname{col} T$, $x < x^*$ *such that* $\bar{t}(x) \in \{a, b\}$ *and* x, x^* *belong to bad block.*

§ 6. In § 6 S denotes a dyadic set, $|\operatorname{Ind} C(S)| < \infty$.

The pair of edges $a \Rightarrow b$, $c \Rightarrow d$ we call *crossed,* if $d \asymp \{a, b\}$, $a \asymp \{c, d\}$ (then $c \triangleleft b$). The pair of dual edges $\varphi = a \Rightarrow b$, $\varphi^* = a^* \Rightarrow b^*$ we call *admissible,* if no one of φ, φ^* is crossed with coequipped edge.

***Remark* 6.** If every pair of dual maximal edges is not admissible, then for any edge φ, $\max\{\operatorname{eq}(\varphi), \operatorname{eq}(\varphi^*)\} \geq 3$.

***Remark* 7.** If φ, φ^* is a pair of dual maximal edges and both edges are equipped, then this pair is admissible, because both crossed edges cannot be coequipped, otherwise $|\operatorname{Ind} C(S)| = \infty$ (see [1], № 17).

Proposition 15. *There exists an admissible pair of dual maximal edges.*

Proof. Suppose the contrary. Then for any pair of maximal dual edges $a \Rightarrow b$, $a^* \Rightarrow b^*$ there exists an edge $c \Rightarrow d$ such that the edges $a \Rightarrow b$, $c \Rightarrow d$ are crossed, and the edge $c^* \Rightarrow d^*$ is equipped. Edge $c \Rightarrow d$ is not maximal, because in the opposite case, by remark 7, the pair $c \Rightarrow d$, $c^* \Rightarrow d^*$ is admissible. If $\overline{c} \le c$, $d \le \overline{d}$, and $\overline{c} \Rightarrow \overline{d}$ — maximal edge, then by remark 6 one of the edges $\overline{c} \Rightarrow \overline{d}$, $\overline{c}^* \Rightarrow \overline{d}^*$ is three times equipped, the other contains equipped subedge $c \Rightarrow d$ or $c^* \Rightarrow d^*$ but this contradicts to [1] № 14.

Proposition 16. *If* $\varphi = a \Rightarrow b$, $\varphi^* = a^* \Rightarrow b^*$ *is the admissible pair of dual maximal edges, and* $T_{\text{ex}} = (T, t)$ *is exceptional representation of* $S^{\triangleleft}$, *then there exists string* $Q \subset S^{\triangleleft}$ *and block composition* $\mathcal{T}$ *such that* $(T, t, \mathcal{T})$ *is* φ-*quasibijective,* $(T, t, \mathcal{T}) \in K_Q^{\triangleleft}$.

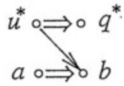
Proof. See [9], prop. 6.2.

Proposition 17. *If* $\varphi = a \Rightarrow b$, $\varphi^* = a^* \Rightarrow b^*$ *is a maximal admissible pair of edges in* S, $(T, t, \mathcal{T}) \in \mathcal{M}$ *is a locally linear indecomposable of minimal height, then* $(T, t, \mathcal{T})$ *is M-bijective.*

Proof. Since proposition 4 we will consider, that if (T, t) is elementary, then $(T, t, \mathcal{T})$ is conormal. If $(T, t, \mathcal{T})$ is not M-bijective, then by proposition 14 there are $x < x^*$ $(\bar{t}(x) \in \{a, b\})$, both in the bad (complementary) blocks. Suppose Supp $T \ni \mathcal{B}$, where $\mathcal{B} = (b_1, \dots, b_m)$ and $1 < n < m$ exists such that $b_{n-1} = q$, $b_n = a$, $b_{n+1} = t$, $q^* \asymp \{a, b\}$ and either $t = 1$ or $t \asymp \{a^*, b^*\}$, another cases may be considered in similar way. If $t \in \overset{\circ}{S}$, then we have [1], № 1^2, therefore, in fact $m = n + 1$. The point q^* is not seminormal, so, $q = b_{n-1}, \dots, b_1$ are not co-seminormal (prop. 2, 3), so, T is not elementary. We will prove that T is not of minimal height.

Since T is not elementary and b_1 is seminormal, there exists $\mathcal{D} \in$ Supp T, $\mathcal{D} \asymp \mathcal{B}$ and $u = \gamma(\mathcal{D})$ such that $u \Rightarrow b_1$ or $b_1 \Rightarrow u$ $(u \ne b_1$ by locally linearity). $\mathcal{B} \asymp \mathcal{D}$ implies $b_2 \asymp u^*$, b_1^*, so, b_2 is not seminormal. But if $n > 2$, then b_2 is also not co-seminormal (see above), that contradicts to prop. 3. Therefore, $n = 2$, $\mathcal{B} = (q, a, t)$, $u^* \asymp a = b_2$.

If $q \Rightarrow u$, then $u^* \asymp b$, $\{q^* \Rightarrow u^*\} \asymp \{a \Rightarrow b\}$, that contradicts to finite representability of $C(S)$ ([1], № 21). We conclude that it is possible only $u^* \Rightarrow q^*$, $u^* < b$, then



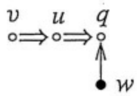
is a crossed pair of edges, and admissibility of φ, φ^* implies that $u \Rightarrow q$ is nonequipped.

Let P be a string in S containing $u \Rightarrow q$. We will show that $\gamma(\text{Supp } T) \subset P$, and then prop. 9 will finish the proof of statement.

If $|\text{Supp } T| = 2$, it is evident. If $|\text{Supp } T| > 2$, then according to the Kleiner's list of faithful finitely represented posets [2], 5.4, there exists $\mathcal{C} \in$ Supp T, $\mathcal{C} \asymp \{\mathcal{B}, \mathcal{D}\}$; if $v = \gamma(\mathcal{C})$, then admissibility of the pair φ, φ^* implies $\text{eq}\,(u \Rightarrow q) = 0$, $v \in P$. Hence (up to rename v and u), we have $a \asymp \{v^* \Rightarrow q^*\}$.

Note that $\text{eq}\,(a^* \Rightarrow b^*) \le 1$, because the edge $a \Rightarrow b$ is equipped by the point $q^* \in \overset{\circ}{S}$ (see [2], № 1); $\text{eq}\,(v^* \Rightarrow q^*) \le 3$ (see [1], № 3). Consequently, $v^* \Rightarrow q^*$ can be equipped (except for a) by the only point $s \notin \overset{\circ}{S}$; if $t \asymp \{a \Rightarrow b\}$ $(t \notin \overset{\circ}{S})$ then t and s can not exist simultaneously.

$\mathcal{B} = (q, a, 1)$ or (q, a, t), by the definition of bad block. Now it is obvious that $C = (v, 1)$. For $\mathcal{D}$ there exist two possibility: $\mathcal{D} = (u, a, 1)$ or (u, s), $s > a$, so, $\mathcal{B}$ lie in the only antichain of cardinality 3 in $\operatorname{Supp} T$ (more longer edge $v \Rightarrow q$ give the case [1], № 22^i). Hence, by [2], 5.4, either $\operatorname{Supp} T = \{\mathcal{B}, \mathcal{C}, \mathcal{D}\}$, or $\operatorname{Supp} T = \{\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$, where $\mathcal{E} \asymp \{\mathcal{C}, \mathcal{D}\}$, $\mathcal{E}$ is comparable with $\mathcal{B}$. We claim that $w = \gamma(\mathcal{E}) \in P$. Indeed, in the opposite case in S there is the following fragment



, $\quad v \Rightarrow q$,

the edge $v \Rightarrow q$ is twice equipped (by a). We obtain the case [1], № $(14^3)^*$.

Theorem 1. *If S is a dyadic set and $|\operatorname{Ind} C(S)| < \infty$, then K is dense in* $\operatorname{Rep} S$.

The theorem is proved by induction by the number of edges in S. The base of induction — the weakly completed case — was studied in [10] (see remark 3).

Choose, owing to prop. 15, a maximal admissible pair φ, φ^*, and build $S^{\lhd}$. Our aim is to show that any $(T, t) \in \operatorname{Ind} S$ is equivalent to $(T', t') \in K$. The induction hypothesis and prop. 7 imply that in $S^{\lhd}$ any $(T, t) \simeq \oplus\,(T_i, t_i)$, each (T_i, t_i) is indecomposable and either exceptional or multielementary, in the latter case, by prop. 8 we can assume $(T_i, t_i) = \Theta(Z_i)$, $Z_i \in \overline{\operatorname{Ind}}\, C(S)$. Due to lemma 4 it is sufficient to prove that, for some string Q in $S^{\lhd}$ and some block composition, $(T_i, t_i, \mathcal{T}_i) \in K_q^{\lhd}$ is φ-quasibijective. For exceptional representations it is checked in prop. 16, for multielementary ones it follows from corollary 3 and prop. 17, using remark 5.

Appendix. In the appendix we denote by S a dyadic set with $|\operatorname{Ind} C(S)| < \infty$.

Given the poset $(C, \leq)$, the *width* of $t \in C$ is $w(t) = w(C^{\asymp}(t)) + 1$.

Lemma 5. *Width of any zigzag in $C(S) \setminus U(S)$ is not greater than* 2.

Proof is a consequence of the properties of the normal points (notion of that points is introduced in [5], 2.2, details of the proof see [9], lemma 7.1).

Proposition 18. *Let $\Pi = \Pi_1 \sqcup \Pi_2$ be a bipartite bundle, T is a faithful indecomposable representation of $C(\Pi)$, $\gamma(T) \subset \Pi_1$, $|\operatorname{Ind} C(\Pi)| < \infty$. Then* $w(\Pi_2 \setminus \overset{\circ}{\Pi}_2) \leq 1$.

See proof of prop. 7.1 in [9], where in Fig. on p. 26 should be $\mathcal{A} < \mathcal{B}$ instead of $\overline{\mathcal{B}} < \overline{\mathcal{A}}$, after this Fig. "can be $\overline{\mathcal{B}} < \overline{\mathcal{A}}$", and 6 lines below in C_4, "$\overline{\mathcal{B}} < \overline{\mathcal{A}}$" instead of "$\overline{\mathcal{A}} < \overline{\mathcal{B}}$".

Owing to $U(S) \subset C(S)$, $U(S)$ carries two order relations: $\leq$ and $\bar{\leq}$ (see remark 1), and order $\bar{\leq}$ is a strengthening of the order $\leq$. So, the natural functor $\operatorname{Rep}(U(S), \leq) \to \operatorname{Rep}(U(S), \bar{\leq})$ is defined, that is bijection on the objects. Composing the inverse bijection map with inclusion $\operatorname{Rep}(U(S), \leq) \to \operatorname{Rep}(C(S), \leq)$ and with Θ, we obtain the map $\Phi: \operatorname{Ind}(U(S), \bar{\leq}) \to \operatorname{Rep}(C(S), \leq) \xrightarrow{\Theta} \operatorname{Rep} S$.

Denote by $\mathcal{L}(S)$ a set of all locally linear representations from $\operatorname{Rep}(U(S), \bar{\leq})$. Representations $T \in \mathcal{L}(S)$ is said to be *binormal*, if

1) $\gamma(\operatorname{Supp} T)$ contains precisely one string P of length greater than 1;
2) $w(\operatorname{Supp} T \setminus \gamma^{-1}(P)) = 1$.

Binormal representation is said to be *positive*, if either $\operatorname{equ}(P) > \operatorname{equ}(P^*)$, or $\operatorname{equ}(P) = \operatorname{equ}(P^*)$ and $p < p^*$ for $p \in P$, where

$$\operatorname{equ}(P) = |\{\mathcal{X} \in \operatorname{Supp} T \setminus \gamma^{-1}(P) \mid \gamma(\mathcal{X}) \asymp P \cap \gamma(\operatorname{Supp} T)\}|,$$

$$\operatorname{equ}(P^*) = |\{\partial_p^2 \mathcal{Y} \in U(S) \mid \mathcal{Y} \in \operatorname{Supp} T, p \in P\}|.$$

Let $\mathcal{L}^+(S)$ be the set of positive binormal indecomposables; $\mathcal{L}^n(S)$ be the set of locally linear non-binormal in $\operatorname{Ind}(U(S), \lessapprox)$. Let $\mathcal{L}_1(S) = \mathcal{L}^+(S) \cup \mathcal{L}^n(S)$.

Theorem 2. *Any multielementary non-elementary*[3] $T \in \operatorname{Ind} S$ *is equivalent to representation in* $\Phi(\mathcal{L}_1(S))$.

Proof. Let $T_1 \in \overline{\operatorname{Ind}}\, C(S)$, $\Theta(T_1) \simeq (T, t)$, and $P \subset S$ be a such string that $\gamma(\operatorname{Supp} T) \cap P \neq \varnothing$. Set $(T, t, \mathcal{T}') = P(T, t, \mathcal{T})$, where $(T, t, \mathcal{T}) \in \mathcal{M}$, and let $Q \in \operatorname{Ind} \Pi_P$ be an indecomposable such that $J_P(Q) = (T, t, \mathcal{T}')$. Define the bundle $\Pi'_P = \operatorname{Supp} Q$.

If T_1 is non-binormal or binormal positive, set $R = T_1$, $P_1 = P^*$. Otherwise, $P \cap \Pi'_P$ is normal, then by remark 2, $R \in \operatorname{Ind} C(S)$ exists such that $P^*(\Theta(R), r, \mathcal{R}) = (T, t, \mathcal{T}')$, where $(\Theta(R), r, \mathcal{R}) \in \mathcal{M}$, and set $P_1 = P$, obtained R is positive. Owing to (T, t) is not elementary and lemma 5, $\operatorname{Supp} R \subset U(S)$; local linearity of R follows from that for T_1 by remark 4.

Normality of P_1 in Π'_P and proposition 18 immediatly imply that $\lessapprox|_{\operatorname{Supp} R} = \leq|_{\operatorname{Supp} R}$. So, indecomposability of R in $\operatorname{Rep} C(S)$ implies it indecomposability in $(C(S), \lessapprox)$. Theorem is proved.

In fact for any T considered above the unique $R \in \mathcal{L}_1(S)$ exists such that $\Phi(R) \simeq T$, but we don't supply the proof in this article.

Proposition 19. *Let* $R \in \mathcal{L}^n(S)$. *Assume for string* $P \subset S$, $|P \cap \gamma(R)| > 1$. *Then* $P \cap \operatorname{Supp} R$ *is not normal in the bundle* $\Pi_P \cap \operatorname{Supp} \Theta_P(R)$.

Proof. See [9], prop. 7.2.

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[3] For elementary representations the non-uniques is completely described in [3].