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CONVERGENCE RATES AND FINITE-DIMENSIONAL APPROXIMATION FOR A CLASS OF ILL-POSED VARIATIONAL INEQUALITIES

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1. Introduction. Let X be a real reflexive Banach space and let X^* be its dual space. For the sake of simplicity, norms in X and X^* are denoted by the same symbol $\|\cdot\|$. Let $A: X \to X^*$ be a continuous operator satisfying the condition

$$\langle A(x)-A(y), x-y\rangle \geq m_A \|x-y\|^s \quad \forall x, y \in X, \quad m_A > 0, \tag{1}$$

where $s \in \mathbb{R}$, $1 < s < +\infty$, and the symbol $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ denote the value of the linear and continuous functional $x^* \in X^*$ at the point $x \in X$.

Consider the variational inequality: find an element x_0 such that

$$\langle A(x_0), x-x_0\rangle \geq 0 \quad \forall x \in S_0, \quad x_0 \in S_0, \tag{2}$$

where S_0 is a convex and closed subset of the space X. Condition (1) guarantees the existence and uniqueness of a solution x_0 of (2) for an arbitrary convex and closed subset S_0 [1, 2]. In the case where S_0 is the set of solutions of the operator equation

$$F(x) = f_0, \quad f_0 \in X^*, \tag{3}$$

with a monotone operator F from X into X^* (see definition in [3]), problem (2) was studied in [2, 4–7]. It is well known [8] that problem (3), without additional conditions on the structure of F such as strong or uniform monotonicity, is ill-posed. By this we mean that solutions of (3) do not depend continuously on the data (F, f_0). Therefore, problem (2) in this case is also ill-posed. In numerical computation, we often know approximations (F_h, f_δ) of (F, f_0) such that

$$\| F(x)-F_h(x)\| \leq hg(\|x\|) \quad \forall x \in X, \quad \|f_\delta - f_0\| \leq \delta, \quad h, \delta \to 0, \tag{4}$$

where $g(t)$ is some real continuous and positive function. Assume that

$$g(t) \leq \tilde{a} + \tilde{b}\,|t|^\mu, \quad 0 \leq \mu \leq s-1, \quad \tilde{a} \geq 0, \quad \tilde{b} \geq 0, \quad \tilde{a} + \tilde{b} > 0,$$

and F_h are continuous, but they are usually non-monotone. Then, the sets of solutions of the equations

$$F_h(x) = f_\delta,$$

$$F_h(x) + \alpha A(x) = f_\delta, \quad \alpha > 0$$

(the last equation is used for construction of regularized solutions in the cases where $F_h \equiv F$ or F_h are monotone [8, 9]) may be empty. Even in the case where these sets are not empty, they may not be a good approximations for the set S_0. What should be done for finding an approximate solution for (2) and (3) in this situation?

The main aim of this paper is to answer this question. In Section 2, applying the Liskovets approach [10], we study a method for regularization and establish the convergence rate for the regularized solutions. In Section 3, the convergence rate is presented in combination with finite-dimensional approximations of the space X. An example is given in Section 4 for illustration.

Below, the symbols $\rightarrow$ and $\rightharpoonup$ denote the strong and weak convergence for any sequence, respectively.

2. Convergence Rates for Regularized Solutions.

Definition [10]. *An element* x_ω, $\omega = \omega(h, \alpha, \delta, \varepsilon)$ *in* X *is called a regularized solution of* (2) *and* (3) *if*

$$\langle F_h(x_\omega) - \alpha A(x_\omega) - f_\delta, x - x_\omega \rangle + \varepsilon g(\| x_\omega \|) \| x - x_\omega \| \geq 0 \quad \forall x \in X, \tag{5}$$

for every fixed $\varepsilon \geq h > 0$, $\alpha > 0$, *and* $\delta \geq 0$.

We have the following result:

Theorem 1. For *each* $h, \alpha > 0, \delta \geq 0$, *and* $\varepsilon \geq h$, *the set* S_Δ *of solutions of the variational inequality* (5) *is not empty, and the set* $\{ x_\omega \}$, *where the element* $x_\omega \in S_\Delta$ *is arbitrarily chosen, has only one strongly limit point* x_0 *if* δ / α *and* $\varepsilon / \alpha \rightarrow \rightarrow 0$ *as* $\alpha \rightarrow 0$.

Proof. First, we prove that $S_\Delta \neq \varnothing$. Indeed, since the equation $F(x) + \alpha A(x) = f_\delta$ has the solution $x_{\alpha\delta}$ for every $\alpha > 0$ [2], we have

$$0 = \langle F(x_{\alpha\delta}) + \alpha A(x_{\alpha\delta}) - f_\delta, x - x_{\alpha\delta} \rangle \leq$$

$$\leq \langle F_h(x_{\alpha\delta}) + \alpha A(x_{\alpha\delta}) - f_\delta, x - x_{\alpha\delta} \rangle + \varepsilon g(\| x_{\alpha\delta} \|) \| x - x_{\alpha\delta} \| \geq 0 \quad \forall x \in X,$$

i.e., $x_{\alpha\delta} \in S_\Delta$. Let x_ω, $\omega = \omega(h, \alpha, \delta, \varepsilon)$, be an arbitrary element of S_Δ for every fixed $\varepsilon \geq h > 0$, $\alpha > 0$, and $\delta \geq 0$. We prove that the set $\{x_\omega\}$ is bounded. It follows from (1), (3)–(5) and the monotone property of F that

$$m_A \| x - x_\omega \|^s \leq \langle A(x) - A(x_\omega), x - x_\omega \rangle \leq$$

$$\leq \langle A(x), x - x_\omega \rangle + \frac{\varepsilon}{\alpha} g(\| x_\omega \|) \| x - x_\omega \| + \frac{1}{\alpha} \langle F_h(x_\omega) - f_\delta, x - x_\omega \rangle \leq$$

$$\leq \langle A(x), x - x_\omega \rangle + \frac{\delta + (h + \varepsilon) g(\| x_\omega \|)}{\alpha} \| x - x_\omega \| \quad \forall x \in S_0. \tag{6}$$

Since $\mu + 1 \leq s$, the last inequality, the properties of $g(t)$, and the conditions of this theorem guarantee the boundedness of $\{x_\omega\}$. Since X is reflexive, there exists a subsequence of $\{x_\omega\}$ converging weakly to an element of X. For simplicity, we write $x_\omega \rightharpoonup \tilde{x}$. We prove that $\tilde{x}$ is a solution of the stated problem. It easily follows from (5) that

$$\langle F(x_\omega) + \alpha A(x_\omega) - f_\delta, x - x_\omega \rangle + (h + \varepsilon) g(\| x_\omega \|) \| x - x_\omega \| \geq 0$$

$$\forall x \in X, \quad \varepsilon \geq h > 0, \ \alpha > 0, \quad \text{and} \quad \delta \geq 0.$$

By virtue of the monotone property of F and A, we get

$$\langle F(x)-f_0, x-x_\omega\rangle + \alpha\langle A(x), x-x_\omega\rangle +$$
$$+ (\delta+(h+\varepsilon)g(\|x_\omega\|))\|x-x_\omega\| \geq 0 \quad \forall x \in X.$$

By passing to the limit in this inequality as δ, ε, and $\alpha \to 0$, we obtain

$$\langle F(x)-f_0, x-\tilde{x}\rangle \geq 0 \quad \forall x \in X.$$

By virtue of the Minty lemma [3], $\tilde{x} \in S_0$. It follows from (6) that

$$\langle A(x), x-\tilde{x}\rangle \geq 0 \quad \forall x \in S_0.$$

In this inequality, we replace x by $tx+(1-t)\tilde{x}$, $t \in (0, 1)$, and divide then the obtained inequality by t. Then, by passing to the limit as $t \to 1$, we also get (2), i.e., $\tilde{x}$ is a solution of the stated problem. Since this problem has only one solution (because A satisfies condition (1), i.e., A is uniformly monotone), the entire set $\{x_\omega\}$ has the strongly limit point $\tilde{x}$ and $\tilde{x} = x_0$. Replacing x by x_0 in (6), we can conclude that the set $\{x_\omega\}$ has one strongly limit point x_0 as δ/α, ε/α, and $\alpha \to 0$.

If A is a dual mapping of X, i.e., $\langle A(x), x\rangle = \|A(x)\|\,\|x\| = \|x\|^s$, $s \geq 2$, $x \in X$ [8], then A also satisfies condition (1) for most cases of X [11] and our problem has the following form: find a norm-minimal element $x_0 \in S_0$. The Tikhonov regularization in the infinite-dimensional space X for this particular case is considered in [8–10], but, the question about its convergence rates was not mentioned. The answer is contained in the following result:

Theorem 2. *Suppose that the following conditions are satisfied:*

(i) F_h *are Fréchet differentiable in some neighborhood* $\mathcal{U}_0$ *of* x_0, $s-1$ *times if* $s=[s]$, $[s]$ *is the integer part of* s, *and* $[s]$ *times if* $s \neq [s]$,

(ii) there exists a constant $\tilde{L} > 0$ *such that*

$$\left\| F_h^{(k)}(x_0) - F_h^{(k)}(y) \right\| \leq \tilde{L}\|x_0-y\| \quad \forall y \in \mathcal{U}_0,$$

$k=s-1$ *if* $s=[s]$, $k=[s]$ *if* $s \neq [s]$, *and if* $[s] \geq 3$, *then*

$$F_h^{(2)}(x_0) = \ldots = F_h^{(k)}(x_0) = 0,$$

(iii) there exist elements z_h *such that* $F_h'^{*}(x_0)z_h = A(x_0)$ *and if* $s=[s]$, *then* $\tilde{L}\|z_h\| < m_A s!$.

Then, if α *is chosen so that* $\alpha \sim (\delta+\varepsilon)^p$, $0<p<1$, *we have*

$$\|x_\omega - x_0\| = O((\delta+\varepsilon)^\theta), \quad \theta = \min\{(1-p)/(s-1), \rho/s\}.$$

Proof. By virtue of (6) and condition (iii) of this theorem, we can write

$$\|x_\omega - x_0\|^s \leq \left(\frac{\delta+(h+\varepsilon)g(\|x_\omega\|)}{\alpha}\|x_\omega - x_0\| + \langle z_h, F_h'(x_0)(x_0-x_\omega)\rangle\right)\Big/ m_A.$$

In the case $s=[s]$, since F_h are $s-1$ times Fréchet differentiable at x_0 and

$$F_h^{(2)}(x_0) = \ldots = F_h^{(s-1)}(x_0) = 0,$$

we get

$$F_h'(x_0)(x_0-x_\omega) = F_h(x_0) - F_h(x_\omega) + r_\omega$$

with

$$\| r_\omega \| \leq \tilde{L} \| x_\omega - x_0 \|^s / s!.$$

Thus,

$$\| x_\omega - x_0 \|^s \leq \left(\frac{\delta + (h + \varepsilon) g(\| x_\omega \|)}{\alpha} \| x_\omega - x_0 \| + \right.$$

$$\left. + \langle z_h, F_h(x_0) - F_h(x_\omega) \rangle + \tilde{L} \| z_h \| \frac{\| x_\omega - x_0 \|^s}{s!} \right) \Big/ m_A .$$

Therefore,

$$\left(1 - \frac{\tilde{L}}{m_A \, s!} \| z_h \| \right) \| x_\omega - x_0 \|^s \leq \left(\frac{\delta + (h + \varepsilon) g(\| x_\omega \|)}{\alpha} \| x_\omega - x_0 \| + \right.$$

$$\left. + \| z_h \| (\delta + hg(\| x_0 \|)) + \langle z_h, f_\delta - F_h(x_\omega) \rangle \right) \Big/ m_A .$$

Replacing x by $x_\omega + z_h$ in (5), we obtain

$$\langle z_h, f_\delta - F_h(x_\omega) \rangle \leq \varepsilon g(\| x_\omega \|) \| z_h \| + \alpha \langle A(x_\omega), z_h \rangle.$$

Finally, we get

$$\left(1 - \frac{\tilde{L}}{m_A \, s!} \| z_h \| \right) \| x_\omega - x_0 \|^s \leq \left(\frac{\delta + (h + \varepsilon) g(\| x_\omega \|)}{\alpha} \| x_\omega - x_0 \| + \right.$$

$$\left. + \| z_h \| (\delta + hg(\| x_0 \|) + \varepsilon g(\| x_\omega \|) + \alpha \| A(x_\omega) \|) \right) \Big/ m_A . \qquad (7)$$

Using the relation [12]

$$a^p \leq ba^q + c \Rightarrow a^p = O(b^p / (p - q) + c)$$

for $a, b, c > 0$ and $p > q > 0$, and the local boundedness of any hemicontinuous monotone operator, we obtain

$$\| x_\omega - x_0 \| = O((\delta + \varepsilon)^\theta).$$

If $s \neq [s]$, then

$$\| r_\omega \| \leq \frac{\tilde{L}}{([s] + 1)!} \| x_\omega - x_0 \|^{[s]+1}.$$

In this case, the left-hand side of (7) has the form

$$\left(1 - \frac{\tilde{L}}{m_A ([s] + 1)!} \| z_h \| \| x_\omega - x_0 \|^{[s]+1-s} \right) \| x_\omega - x_0 \|^s.$$

Since $\| x_\omega - x_0 \| \to 0$ and $[s] + 1 - s > 0$, we obtain

$$1 - \frac{\tilde{L}}{m_A ([s] + 1)!} \| z_h \| \| x_\omega - x_0 \|^{[s]+1-s} \geq 1/2$$

for sufficiently small α. This remark completes the proof of the theorem.

We mention that all conditions on F_h in this theorem can be imposed on F, i.e., this theorem remains valid if we omit the index h in F_h and z_h.

Let us consider the problem of finite-dimensional approximations for (5).

3. Finite-Dimensional Approximations. The variational inequality (5) can be approximated by the sequence of finite-dimensional inequalities

$$\left\langle F_h^n(x_\omega^n) + \alpha A^n(x_\omega^n) - f_\delta^h,\ x^n - x_\omega^n \right\rangle + \varepsilon g\left(\left\| x_\omega^n \right\|\right)\left\| x^n - x_\omega^n \right\| \ge 0 \tag{8}$$
$$\forall x_n \in X_n,\quad \varepsilon \ge h,\ \alpha > 0,$$

where $F_h^n = P_n^* F_h P_h$, $A_n = P_n^* A P_h$, $f_\delta^n = P_n^* f_\delta$, P_n denotes the linear projection from X onto its subspace X_n satisfying the condition

$$X_n \subset X_{n+1},\quad P_n x \to X,\quad n \to +\infty \quad \forall x \in X,$$

P_n^* is the adjoint of P_n, $\| P_n \| \le C$, and C is a positive constant. By Theorem 1, there exists x_ω^n satisfying (8) for every $\alpha > 0$.

Now we establish whether

$$\lim_{\substack{\alpha, h, \delta \to 0 \\ n \to +\infty}} x_\omega^n = x_0,$$

and its convergence rates.

Obviously, the answer to this question depends on the relation between $h, \alpha, \varepsilon, \delta$, and n. In this section, applying the idea of W. Engl and C. Groetsch in [13], we give an answer to this question.

Theorem 3. *Assuming that the following conditions are satisfied:*

(i) if F is Fréchet differentiable in some neighborhood O_0 of S_0, $s-1$ times if $s = [s]$ ($[s]$ is the integer part of s), $[s]$ times if $s \ne [s]$,

(ii) there exists a constant $\tilde{L} > 0$ such that

$$\| F^{(k)}(x) - F^{(k)}(y) \| \le \tilde{L} \| x - y \| \quad \forall x \in S_0, \quad y \in O_0,$$

$k = s-1$ if $s = [s]$, $k = [s]$ if $s \ne [s]$, and if $[s] \ge 3$, then

$$F^{(2)}(x) = \ldots = F^{(k)}(x) = 0,$$

(iii) $\alpha = \alpha(n) \to 0$ so that

$$(\gamma_n(x) + \| (I - P_n)x \|^{[s]})\alpha^{-1} \to 0 \quad \forall x \in S_0,$$

as $n \to +\infty$, where $\gamma_n(x)$ is defined by

$$\gamma_n(x) = \| F'(x)(I - P_n)x \|.$$

Then x_0 is a strongly limit point of $\left\{x_\omega^n\right\}$.

Proof. It follows from (8) that, for each $x \in S_0$,

$$\alpha m_A \| x_\omega^n - x^n \|^s \le$$
$$\le \varepsilon g(\| x_\omega^n \|) \| x_\omega^n - x^n \| + \alpha \langle A^n(x^n), x^n - x_\omega^n \rangle + \left\langle F_h^n(x_\omega^n) - f_\delta^h,\ x^n - x_\omega^n \right\rangle.$$

Thus,

$$\alpha m_A \| x_\omega^n - x^n \|^s \le \varepsilon g(\| x_\omega^n \|) \| x_\omega^n - x^n \| + \alpha \langle A(x^n), x^n - x_\omega^n \rangle +$$
$$+ \left\langle F_h(x_\omega^n) - F(x_\omega^n) + F(x_\omega^n) - F(x^n) + F(x^n) - F(x) + f_0 - f_\delta,\ x^n - x_\omega^n \right\rangle. \tag{9}$$

Since

$$\left\| F_h(x_\omega^n) - F(x_\omega^n) \right\| \le h g(\| x_\omega^n \|),$$
$$\left\langle F(x_\omega^n) - F(x^n), x^n - x_\omega^n \right\rangle \le 0,$$

it follows from (9) that

$$\alpha m_A \|x_\omega^n - x^n\|^s \leq$$
$$\leq ((h+\varepsilon) g(\|x_\omega^n\|) + \delta + \|F(x^n) - F(x)\|) \|x_\omega^n - x^n\| + \alpha\langle A(x^n), x^n - $$

If $s = [s]$, we can write

$$F(x^n) = F(x) + F'(x)(x^n - x) + r_n$$

with

$$\|r_n\| \leq \frac{\tilde{L}}{s!} \|(I - P_n)x\|^s.$$

Therefore, it is easy to see from (10) that

$$\|x_\omega^n - x^n\|^s \leq \frac{1}{\alpha m_A}\Big((\varepsilon + h) g(\|x_\omega^n\|) + \delta + \|F'(x)(I - P_n)x\| +$$
$$+ \frac{L}{s!}\|(I - P_n)x\|^s\Big)\|x_\omega^n - x^n\| + \langle A(x^n), x^n - x_\omega^n\rangle / m_A \quad \forall x \in S_0.$$

Obviously, this inequality, the property of $g(t)$, and condition (iii) of this th guarantee the boundedness of $\{x_\omega^n\}$. Therefore, there exists a subsequence of that converges weakly to some element x_1. For the sake of simplicity, we $x_\omega^n \to x_1$ as $n \to +\infty$ and $\delta, \alpha, \varepsilon \to 0$. It follows from (8) that

$$\langle F(x^n) - f_\delta, x^n - x_\omega^n\rangle + \alpha\langle A(x^n), x^n - x_\omega^n\rangle + (h+\varepsilon) g(\|x_\omega^n\|)\|x^n - x_\omega^n\| \geq$$

After passing to the limit as $n \to +\infty$ and $\alpha, \delta, \varepsilon \to 0$ in this inequali continuity of F and the weak convergence of $\{x_\omega^n\}$ give

$$\langle F(x) - f_0, x - x_1\rangle \geq 0 \quad \forall x \in X.$$

By the Minty lemma, $x_1 \in S_0$. Replacing x^n by $x_1^n = P_n x_1$ in (11), we see t set $\{x_\omega^n\}$ strongly converges to x_1. On the other hand, from (11), we also obta inequality

$$\langle A(x), x - x_1\rangle \geq 0 \quad \forall x \in S_0.$$

The last inequality yields (2). Then $x_1 = x_0$ and the entire set $\{x_\omega^n\}$ converges

If $s \neq [s]$, we have

$$\|r_n\| \leq \frac{\tilde{L}}{([s]+1)!}\|(I - P_n)x\|^{[s]+1}.$$

In this case, on the right-hand side of (11), we have $[s]+1$ instead of s. This r completes the proof of this theorem.

Now, we answer the question about the convergence rate for $\{x_\omega^n\}$.

Theorem 4. *Suppose that the following conditions are satisfied:*

(i) conditions (i), (ii) of Theorem 3;

(ii) there exist a positive constant d and an element z such that

$$\|A(x) - A(x_0)\| \leq d\|x - x_0\|^{\tilde{s}}, \quad \tilde{s} > 0 \quad \forall x \in \mathcal{U}_0, \quad F'^*(x_0)z = A(x_0$$

and if $s = [s]$, then $\tilde{L}\|z\| < s!$;

(iii) α *is chosen so that* $\alpha \sim (\varepsilon + \delta + \gamma_n)^{1/2}$, *where*

$$\gamma_n = \max\{\|(I-P_n)x_0\|, \|(I^* - P_n^*)f_0\|, \|(I-P_n)z\|\}.$$

Then

$$\| x_\omega^n - x_0 \| = O((\varepsilon+\delta)^{1/2s} + \gamma_n^{\bar{s}}), \quad \bar{s} = \min\left\{\frac{\tilde{s}}{s-1}, \frac{1}{2s}\right\}.$$

Proof. It follows from condition (ii), that, for sufficiently large n with $x_0^n \in \mathcal{U}_0$,

$$\langle A(x_0^n), x_0^n - x_\omega^n \rangle \le d\gamma_n^{\bar{s}} \| x_0^n - x_\omega^n \| + \langle A(x_0), x_0^n - x_\omega^n \rangle.$$

Thus, in the case $s = [s]$, inequality (11) with $x = x_0$ has the form

$$\| x_\omega^n - x_0^n \|^s \le O(\varepsilon + \delta + \gamma_n + \gamma_n^s + \gamma_n^{\bar{s}}) \frac{\| x_\omega^n - x_0^n \|}{\alpha} + \frac{\langle A(x_0), x_0^n - x_\omega^n \rangle}{m_A}. \quad (12)$$

It is easy to see now that, in this case,

$$\langle A(x_0), x_0^n - x_\omega^n \rangle = \langle z, F'(x_0)(x_0^n - x_0) \rangle + \langle z, F'(x_0)(x_0 - x_\omega^n) \rangle \le$$

$$\le O(\gamma_n) + \langle z, F(x_0) - F(x_\omega^n) \rangle + \frac{L}{s!} \| z \| \| x_\omega^n - x_0 \|^s,$$

$$\| x_\omega^n - x_0 \|^s = \| x_\omega^n - x_0^n \|^s + O(\gamma_n)$$

and

$$\langle z, F(x_0) - F(x_\omega^n) \rangle =$$

$$= \langle z, f_0 - f_\delta \rangle + \langle z, f_\delta - F_h(x_\omega) \rangle + \langle z, F_h(x_\omega) - F_h(x_\omega^n) \rangle + \langle z, F_h(x_\omega^n) - F(x_\omega^n) \rangle \le$$

$$\le O(\delta + h) + \alpha \langle z, A(x_\omega) \rangle + \varepsilon g(\| x_\omega \|) \| z \| + \langle z, F_h(x_\omega) - F_h(x_\omega^n) \rangle.$$

We estimate

$$\langle z, F_h(x_\omega) - F_h(x_\omega^n) \rangle = \langle z, F_h(x_\omega) - F_h^n(x_\omega^n) \rangle + \langle z, F_h^n(x_\omega^n) - F_h(x_\omega^n) \rangle,$$

where

$$\langle z, F_h^n(x_\omega^n) - F_h(x_\omega^n) \rangle = \langle (P_n - I) z, F_h(x_\omega^n) \rangle \le$$

$$\le \gamma_n \| F_h(x_\omega^n) \| \le \gamma_n \| F_h(x_\omega^n) - F(x_\omega^n) \| + \| F(x_\omega^n) \| = O(\gamma_n)$$

and

$$\langle z, F_h(x_\omega) - F_h^n(x_\omega^n) \rangle = \langle z, F_h(x_\omega) \rangle + \langle z, -F_h^n(x_\omega^n) \rangle.$$

Replacing x in (5) by $x_\omega - z$, we get

$$\langle z, F_h(x_\omega) \rangle = \langle z, f_\delta - \alpha A(x_\omega) \rangle + \varepsilon g(\| x_\omega \|) \| z \|.$$

On the other hand,

$$\langle z, -F_h^n(x_\omega^n) \rangle = \langle z^n, -F_h^n(x_\omega^n) \rangle.$$

Then replacing x^n by $z^n + x_\omega^n$ in (8), we obtain

$$\langle z^n, -F_h^n(x_\omega^n) \rangle \le \langle z^n, \alpha A^n(x_\omega^n) - f_\delta^n \rangle + \varepsilon g(\| x_\omega^n \|) \| z^n \| =$$

$$= \langle z, \alpha A^n(x_\omega^n) - f_\delta^n \rangle + \varepsilon g(\| x_\omega^n \|) \| z^n \|.$$

Therefore,

$$\langle z, F_h(x_\omega) - F_h^n(x_\omega^n) \rangle \le \langle z, f_\delta - \alpha A(x_\omega) + \alpha A^n(x_\omega^n) - f_\delta^n \rangle + O(\varepsilon) \le$$

$$\le O(\varepsilon + \delta + \alpha) + \langle z, f_0 - f_0^n \rangle = O(\varepsilon + \delta + \alpha\gamma_n).$$

Hence, (12) can be written in the form

$$\left(1 - \frac{L\|z\|}{m_A s!}\right) \| x_\omega^n - x_0^n \|^s \le$$

$$\le O(\delta + \varepsilon + \gamma_n + \gamma_n^s + \alpha\gamma_n^{\bar{s}}) \frac{\| x_\omega^n - x_0^n \|}{\alpha} + O((\varepsilon + \delta + \gamma_n)^{1/2}).$$

Then

$$\| x_\omega^n - x_0^n \|^s \le$$

$$\le O((\varepsilon + \delta + \gamma_n)^{1/2} + \gamma_n^{s/2} + \gamma_n^{\bar{s}}) \| x_\omega^n - x_0^n \| + O((\varepsilon + \delta + \gamma_n)^{1/2}).$$

Applying again the relation in [12] to the last inequality, we obtain

$$\| x_\omega^n - x_0^n \| = O((\varepsilon + \delta)^{1/2s} + \gamma_n^{\bar{s}}).$$

Therefore,

$$\| x_\omega^n - x_0 \| = O((\varepsilon + \delta)^{1/2} + \gamma_n^{\bar{s}}).$$

The case $s \neq [s]$ can be proved by analogy with the proof of Theorem 1.

4. Example. Consider the eigenvalue problem: find an eigenfunction $\varphi_0 \neq 0$ for an eigenvalue λ_0 of the problem $K_1 \varphi = \lambda_0 K_2 \varphi$, where K_i, $i = 1, 2$, are the operators defined by measurable kernel functions $k_i(t, s)$

$$(K_i \varphi)(t) \equiv \int_\Omega k_i(t, s) \varphi(s)\, ds;$$

$$\iint_{\Omega\times\Omega} | k_i(t, s) |^q \, dt\, ds \le +\infty, \quad 1 < q < +\infty,$$

and Ω is a bounded and closed subset of $\mathbb{R}^n$, the operator $F = K_1 - \lambda_0 K_2$ has the domain of definition $X = L_p(\Omega)$, $p^{-1} + q^{-1} = 1$, and the range in $X^* = L_q(\Omega)$. Suppose that F is nonnegative, i.e., $\langle F\varphi, \varphi \rangle \ge 0$ and φ_0 is chosen so that it minimizes the functional $\langle B\varphi, \varphi \rangle / 2 + \| \varphi - \varphi^* \|_{L_p(\Omega)}^s / s$, $2 < s$, where φ^* is not an eigenfunction for the eigenvalue λ_0, and B is a linear selfadjoint bounded nonnegative operator.

In this case, $A + B + U^s$, where U^s is the dual mapping of $L_p(\Omega)$ satisfying the condition

$$\langle U^s(\varphi), \varphi \rangle = \| \varphi \|_{L_p(\Omega)}^s, \quad \| U^s(\varphi) \|_{L_q(\Omega)} = \| \varphi \|_{L_p(\Omega)}^{s-1}.$$

Since

$$\| U^s(\varphi_1) - U^s(\varphi_2) \|_{L_q(\Omega)} \le C_1(R) \| \varphi_1 - \varphi_2 \|_{L_p(\Omega)},$$

where $C_1(R)$ is a positive increasing function of $R = \max\{\|\varphi_1\|_{L_p(\Omega)}, \|\varphi_2\|_{L_p(\Omega)}\}$, we have $d = \|B\| + C_1(R)$. If $p = 2$, then $s = 2$, $m_A = 1$, $C_1(R) \equiv 1$, and, furthermore [11],

$$1 < p < 2: \ s = 2, \ m_A = p - 1, \ \ C_1(\rho) = p 2^{2p-1} e^p L^{p-1},$$

$$e = \max\{2^p, 2\rho\}, \quad 1 < L < 3.18, \quad \tilde{s} = p - 1,$$

$$2 < p: \ s = p, \ m_A = 2^{2-p}/p,$$

$$C_1(\rho) = 2^p \rho^{p-2}\{p[p - 1 + \max\{p, L\}]\}^{-1}, \quad \tilde{s} = 1.$$

Moreover, F is smooth with infinite order, and $F^k(\varphi) = 0$ for $\varphi \in S_0$. In this case, S_0 is a subspace generated by eigenfunctions for the eigenvalue λ_0.

Remark. The condition $\tilde{L}\|z\| < 1$ in [13] can not be omitted for the variational method of Tikhonov regularization. But, in this case, if $s \neq [s]$, the condition $\tilde{L}\|z\| < m_A s!$ is not necessary. This is also a great advantage in using the operator version of Tikhonov regularization for solution of nonlinear monotone iii-posed problems.

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