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RESEARCH ARTICLE



# Groups with many self-normalizing subgroups

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*Dedicated to Leonid A. Kurdachenko on his sixtieth birthday*

Abstract. This paper investigates the structure of groups in which all members of a given relevant set of subgroups are self-normalizing. In particular, soluble groups in which every non-abelian (or every infinite non-abelian) subgroup is self-normalizing are described.

## Introduction

It is well known that a group has only normal subgroups if and only if it is either abelian or the direct product of a quaternion group of order 8 and a periodic abelian group with no elements of order 4. The structure of groups for which the set of non-normal subgroups is small in some sense has been studied by many authors in several different situations; in particular, metahamiltonian groups (i.e. groups whose non-abelian subgroups are normal) were introduced and investigated in a series of papers by G.M. Romalis and N.F. Sesekin (see [8],[9],[10]); further informations on the structure of such groups have been later obtained by N.F. Kuzennyi and N.N. Semko [4].

Dualizing this point of view, we are interested here to (generalized) soluble groups in which all members of a given relevant set of subgroups are self-normalizing. Of course, any group whose non-trivial subgroups are self-normalizing is periodic and simple, and it is elementary to show that locally finite groups with such property either are trivial or have prime order. The consideration of Tarski groups (i.e. infinite simple

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groups whose proper non-trivial subgroups have prime order) shows that there exist infinite groups in which every non-trivial subgroup is self-normalizing; such infinite simple groups can have even a more complicated structure, since it follows from a result of V.N. Obraztsov [5] that any Tarski group can be properly embedded in a simple group in which every proper subgroup either has prime order or is a Tarski group.

As in the case of metahamiltonian groups, we look at the set of all non-abelian subgroups of a group. The structure of (soluble) groups in which every non-abelian subgroup is self-normalizing will be described in Section 2, while the final Section 3 will be devoted to the study of groups whose infinite non-abelian subgroups are self-normalizing.

Let $\mathcal{H}$ be the class of all groups in which every non-abelian subgroup is self-normalizing. Clearly, all groups whose proper subgroups are abelian belong to the class $\mathcal{H}$, and we shall denote by $\mathcal{H}^*$ the class of $\mathcal{H}$-groups containing proper non-abelian subgroups; it will be proved that the structure of soluble non-nilpotent $\mathcal{H}^*$-groups is close to that of minimal non-abelian groups. Moreover, if $\mathcal{H}_\infty$ denotes the class of groups in which all infinite non-abelian subgroups are self-normalizing, it will turn out that any soluble $\mathcal{H}_\infty$-group either has the property $\mathcal{H}$ or is a Černikov group (i.e. if it is a finite extension of an abelian group satisfying the minimal condition on subgroups).

Most of our notation is standard and can be found in [7].

## 1. Self-normalizing non-abelian subgroups

Let $G$ be any $\mathcal{H}$-group. It follows from the definition that $G$ is soluble if and only if its commutator subgroup $G'$ is properly contained in $G$; thus every soluble $\mathcal{H}$-group is actually metabelian. Observe also that any group in the class $\mathcal{H}_\infty$ which properly contains its commutator subgroup either is metabelian or finite-by-abelian.

It is well known that any soluble group whose proper subgroups are abelian is either finite or abelian; the structure of finite minimal non-abelian groups is well-known. We will also need the following result of S.N. Černikov [2] concerning groups with few non-abelian subgroups (see also [6]).

**Lemma 1.1.** Let $G$ be a soluble non-abelian group satisfying the minimal condition on non-abelian subgroups. Then $G$ is a Černikov group.

**Lemma 1.2.** Let $G$ be a non-periodic $\mathcal{H}_\infty$-group. If $G' < G$, then $G$ is abelian.

*Proof.* Without loss of generality, we may suppose that $G$ is finitely generated. If $G$ is nilpotent, all its infinite proper subgroups are abelian, so that in particular $G$ satisfies the minimal condition on non-abelian subgroups and hence it is abelian by Lemma 1.1. Assume, to the contrary, that $G$ is not nilpotent, so that $G/G'$ is a cyclic $p$-group for some prime number $p$, and in particular $G'$ is a finitely generated infinite abelian group. Let $x$ be an element of $G$ such that $G = \langle x, G' \rangle$, and so $G' = [x, G']$. As $\langle x \rangle \cap G' \leq Z(G)$, the factor group $G/\langle x \rangle \cap G'$ is infinite by Schur's theorem and thus it is also a counterexample. Replacing $G$ by $G/\langle x \rangle \cap G'$, we may assume that $\langle x \rangle \cap G' = \{1\}$. Clearly, $G/(G')^p$ is a finite $p$-group, so that the proper subgroup $\langle x, (G')^p \rangle$ is subnormal in $G$. It follows that $\langle x, (G')^p \rangle$ is abelian and hence

$$(G')^p = [x, G']^p = [x, (G')^p] = \{1\}.$$

Therefore $G'$ is finite and this contradiction completes the proof of the statement. □

Recall from [1] (p.52) that a group $G$ is said to be *locally graded* if every finitely generated non-trivial subgroup of $G$ contains a proper subgroup of finite index. The class of locally graded groups is quite large, and contains in particular all locally (soluble-by-finite) groups.

**Lemma 1.3.** If $G$ is a periodic locally graded $\mathcal{H}_\infty$-group, then $G$ is locally finite.

*Proof.* Assume for a contradiction that $G$ contains a finitely generated infinite subgroup $E$. Since $G$ is locally graded, there exists a proper normal subgroup $H$ of $E$ such that $E/H$ is finite. Then $H$ is an infinite proper subgroup of $G$ which is not self-normalizing, so that $H$ is abelian and $E$ is abelian-by-finite. Therefore $E$ is finite, and this contradiction proves that $G$ is locally finite. □

Our next lemma reduces the study of (soluble) $\mathcal{H}$-groups to the case of non-nilpotent groups.

**Lemma 1.4.** If $G$ is a locally nilpotent $\mathcal{H}$-group, then all proper subgroups of $G$ are abelian.

*Proof.* Assume for a contradiction that $G$ contains a proper non-abelian subgroup $X$, and let $g$ be an element of $G \setminus X$. Clearly, there exists a finitely generated non-abelian subgroup $E$ of $X$, and $E$ is a proper subnormal subgroup of $\langle E, g \rangle$, contradicting the property $\mathcal{H}$. □

**Lemma 1.5.** If $G$ is a $\mathcal{H}^*$-group, then the factor group $G/G'$ is cyclic with prime-power order.

*Proof.* Let $x$ and $y$ be elements of $G$ such that $xy \neq yx$. Then $\langle x, y, G' \rangle$ is a non-abelian normal subgroup of $G$, so that $\langle x, y, G' \rangle = G$ and $G/G'$ is finitely generated. Assume that the statement is false, so that $G/G'$ contains two distinct maximal subgroups $M_1/G'$ and $M_2/G'$. Then the normal subgroups $M_1$ and $M_2$ of $G$ must be abelian, and hence $G = M_1M_2$ is nilpotent. It follows from Lemma 1.4 that all proper subgroups of $G$ are abelian, contradicting the hypothesis. □

The following result is elementary and can be found in [3].

**Lemma 1.6.** Let $A$ be a periodic abelian group, and let $\alpha : A \longrightarrow A$ be an automorphism of $A$ of prime order $p$. If $A$ has no elements of order $p$, then

$$A = C_A(\alpha) \times [A, \alpha].$$

We are now in a position to describe all soluble groups in the class $\mathcal{H}$.

**Theorem 1.7.** Let $G$ be a soluble non-nilpotent group. Then $G$ has the property $\mathcal{H}$ if and only if $G = \langle x \rangle \ltimes G'$, where $x$ is an element of prime-power order $p^n$, $x^p$ belongs to $C_G(G')$ and the commutator subgroup $G'$ is a periodic abelian group with no elements of order $p$.

*Proof.* If all proper subgroups of $G$ are abelian, it follows directly from the structure of minimal non-abelian groups that $G$ has the required structure. Suppose now that $G$ is a $\mathcal{H}^*$-group, so that in particular it is periodic by Lemma 1.2. Moreover, Lemma 1.5 yields that there exists a maximal subgroup $M$ of $G$ containing $G'$, and $M$ is abelian since it is normal in $G$. Put $|G : M| = p$, and let $x$ be an element of $G$ with order $p^n$ (for some $n > 0$) such that $G = \langle x, M \rangle$. Write $M = A \times B$, where $A$ is a $p'$-group and $B$ is a $p$-group, and assume that the subgroup $\langle x, A \rangle$ is properly contained in $G$. Let $g$ be an element of $G \setminus \langle x, A \rangle$. As $\langle x, g, A \rangle / A$ is a finite $p$-group, the subgroup $\langle x, A \rangle$ is properly contained in its normalizer and hence it is abelian. It follows that $A$ is contained in the centre of $G$. Thus $[x, B] \neq \{1\}$ and so the non-abelian $p$-subgroup $\langle x, B \rangle$ must be self-normalizing. On the other hand, $A$ is contained in $N_G(\langle x, B \rangle)$, so that $A = \{1\}$ and $G$ is a $p$-group, which is impossible by Lemma 1.4. This contradiction shows that

$$G = \langle x, A \rangle = \langle x \rangle \ltimes A.$$

Again Lemma 1.4 yields that the subgroup $[A, x]$ is not contained in $Z(G)$, so that $[A, x, x] \neq \{1\}$ and hence $\langle [A, x], x \rangle$ is not abelian. On the other hand, $\langle [A, x], x \rangle$ is a normal subgroup of $G$, so that $\langle [A, x], x \rangle = G$ and hence $A = [A, x] = G'$, with $x^p \in M = C_G(G')$.

Conversely, assume that $G = \langle x\rangle \ltimes G'$ has the structure described in the statement, and let $X$ be any non-abelian subgroup of $G$. Obviously, $X$ is not contained in $\langle x^p, G'\rangle$, and so there is a non-trivial element $y$ of $X$ with order a power of $p$ such that $\langle y, G'\rangle = G$. Then $[G', y] = G'$. Consider the normal subgroup $N = G' \cap X$ of $G$, and put $\bar{G} = G/N$. Then $[\bar{G}', \bar{y}] = \bar{G}'$ and so also $C_{\bar{G}'}(\bar{y}) = \{1\}$ by Lemma 1.6. On the other hand,

$$[y, G' \cap N_G(X)] \leq G' \cap X = N,$$

and hence it follows that $G' \cap N_G(X) = G' \cap X$. Therefore we have

$$N_G(X) = X(G' \cap N_G(X)) = X,$$

ando so $G$ has the property $\mathcal{H}$. □

## 2. Self-normalizing infinite non-abelian subgroups

The first lemma of this section extends Lemma 1.4 to the class $\mathcal{H}_\infty$.

**Lemma 2.1.** Let $G$ be a soluble locally nilpotent $\mathcal{H}_\infty$-group. Then all infinite proper subgroups of $G$ are abelian.

*Proof.* As the statement is obvious if the group $G$ is nilpotent, we may suppose that $G$ is not nilpotent, so that in particular $G'$ is infinite and $G$ is not finitely generated. Consider any infinite non-abelian subgroup $X$ of $G$, and let $x$ and $y$ be elements of $X$ such that $xy \neq yx$. Then $\langle x, y, G'\rangle = G$ and $G/G'$ is finitely generated. For each element $z$ of $G$, the non-abelian subgroup $\langle x, y, z\rangle$ is not self-normalizing and hence it is finite. Therefore $G$ is periodic and $G/G'$ is finite. Since $G'$ is abelian, the group $G$ is hypercentral, so that every proper subgroup of $G$ is properly contained in its normalizer; thus all infinite proper subgroups of $G$ are abelian. □

**Lemma 2.2.** Let $G$ be an infinite soluble non-abelian group whose infinite proper subgroups are abelian. Then $G$ is a Černikov group and its finite residual has no infinite proper $G$-invariant subgroups.

*Proof.* Clearly $G$ satisfies the minimal condition on non-abelian subgroups, and hence it is a Černikov group by Lemma 1.1. Let $J$ be the finite residual of $G$, and let $N$ be any infinite $G$-invariant subgroup of $J$. Consider two elements $x, y$ of $G$ such that $xy \neq yx$. Then $\langle x, y, N\rangle$ is an infinite non-abelian subgroup of $G$, so that $\langle x, y, N\rangle = G$ and the index $|G : N|$ is finite. Therefore $N = J$ and $J$ has no infinite proper $G$-invariant subgroups. □

**Lemma 2.3.** Let $G$ be an infinite non-abelian $\mathcal{H}_\infty$-group with finite commutator subgroup. Then $G$ is a finite extension of a group of type $p^\infty$ for some prime number $p$.

*Proof.* The group $G$ is periodic by Lemma 1.2, and so it contains a finite non-abelian subgroup $E$. Then every proper subgroup of the abelian group $G/EG'$ must be finite, and hence $G/EG'$ is a group of type $p^\infty$ for some prime number $p$. As $EG'$ is finite, it follows that $G$ is a finite extension of a group of type $p^\infty$. □

**Lemma 2.4.** Let $G$ be a periodic soluble $\mathcal{H}_\infty$-group with infinite commutator subgroup. If $N$ is an infinite $G$-invariant subgroup of $G'$ such that $G'/N$ is infinite, then $G/G'$ is a cyclic $p$-group for some prime number $p$ and $N$ contains the $p$-component of $G'$.

*Proof.* As $G'/N$ is infinite, the factor group $\bar{G} = G/N$ neither is abelian nor minimal non-abelian. Moreover, $\bar{G}$ has the property $\mathcal{H}$, so that it follows from Theorem 1.7 that $\bar{G}/\bar{G}'$ is a cyclic $p$-group while $\bar{G}'$ is a $p'$-group for some prime number $p$. Therefore $G/G'$ is a cyclic $p$-group and the $p$-component of $G'$ is contained in $N$. □

**Lemma 2.5.** Let $G$ be a soluble $p$-group with the property $\mathcal{H}_\infty$. If $G'$ is infinite, then $G$ is a Černikov group.

*Proof.* Assume first that $G$ is nilpotent. Then all infinite proper subgroups of $G$ are abelian and hence $G$ is a Černikov group by Lemma 1.1. Suppose now that $G$ is not nilpotent, so that in particular we may consider an element $x \in G \setminus C_G(G')$ and $G = \langle x, G' \rangle$. Thus $G'$ has finite index in $G$ and so it contains the finite residual $J$ of $G$. Assume for a contradiction that $\bar{G} = G/J$ is infinite. Since $\bar{G}/\bar{G}'$ is finite, there exist in $\bar{G}$ normal subgroups of finite index $\bar{H}$ and $\bar{K}$ such that $\bar{H} < \bar{K} \leq \bar{G}'$ and $\bar{K} \cap \langle \bar{x} \rangle = \{1\}$. As $\bar{G}'$ is infinite, the group $\bar{G}/Z(\bar{G})$ is likewise infinite and hence the subgroup $\langle \bar{x}, \bar{H} \rangle$ cannot be abelian; on the other hand, $\langle \bar{x}, \bar{H} \rangle$ is subnormal in $\bar{G}$ since $\bar{G}/\bar{H}$ is a finite $p$-group. It follows that $\bar{G} = \langle \bar{x}, \bar{H} \rangle$ and so

$$\bar{K} = \langle \langle \bar{x} \cap \bar{K} \rangle, \bar{H} \rangle = \bar{H}.$$

This contradiction shows that $G/J$ is finite, and in particular the abelian subgroup $J$ is divisible. Moreover, Lemma 2.4 yields that the socle of $J$ is finite, so that $J$ satisfies the minimal condition on subgroups and $G$ is a Černikov group. □

**Lemma 2.6.** Let $G$ be an infinite non-abelian $\mathcal{H}_\infty$-group with $G' < G$. Then one of the following conditions holds:

(a) $G$ is a Černikov group and its finite residual has no infinite proper $G$-invariant subgroups;

(b) $G = \langle x \rangle \ltimes G'$, where $x$ is an element of prime-power order $p^n$, $x^p$ belongs to $C_G(G')$ and the commutator subgroup $G'$ is a periodic abelian group with no elements of order $p$.

*Proof.* It follows from Lemma 1.2 that $G$ is periodic. Moreover, by Lemma 2.3 we may assume that $G'$ is infinite and so abelian, and Lemma 2.2 allows to suppose that $G$ contains some infinite proper non-abelian subgroup, so that in particular $G$ is not nilpotent and hence $C_G(G')$ is properly contained in $G$. Let $x$ be an element of prime-power order $p^n$ in the set $G \setminus C_G(G')$. As $\langle x, G' \rangle$ is an infinite non-abelian normal subgroup of $G$, we have $G = \langle x, G' \rangle$, so that $G/G'$ is a cyclic $p$-group and $G' = [G', x]$. Write $G' = P \times Q$, where $P$ is a $p$-group and $Q$ is a $p'$-group. It follows from Lemma 2.4 that one of the subgroups $P$ and $Q$ must be finite.

Suppose first that $Q$ is finite, so that the $p$-subgroup $\langle x, P \rangle$ has finite index in $G$. On the other hand, $\langle x, P \rangle \simeq G/Q$ has infinite commutator subgroup and hence it is a Černikov group by Lemma 2.5. Thus $G$ itself is a Černikov group. The finite residual $J$ of $G$ is contained in $P$ and has no proper subgroups of finite index, so that $J$ does not contain infinite proper $G$-invariant subgroups by Lemma 2.4.

Assume now that $P$ is finite. Thus $G/Q$ is a finite $p$-group and so the infinite subgroup $\langle x, Q \rangle$ is subnormal in $G$. Moreover, $\langle x, Q \rangle$ is not abelian as the commutator subgroup of $G$ is infinite, and then $G = \langle x, Q \rangle$. It follows that $G' = Q$ has no elements of order $p$. Finally, the normal subgroup $\langle x^p, G' \rangle$ is properly contained in $G$, so that it is abelian and hence $C_{\langle x \rangle}(G') = \langle x^p \rangle$. The lemma is proved. □

It follows from Lemma 2.6 that any soluble non-abelian $\mathcal{H}_\infty$-group either has the property $\mathcal{H}$ or is a Černikov group, and hence it allows to restrict our attention to the case of Černikov groups. In fact:

**Corollary 2.7.** Let $G$ be a group such that $G' < G$. If $G$ belongs to $\mathcal{H}_\infty \setminus \mathcal{H}$, then it is a Černikov group.

**Lemma 2.8.** Let $G$ be a Černikov group whose finite residual $J$ has no infinite proper $G$-invariant subgroups. If the centralizer $C_G(J)$ is abelian and $G/C_G(J)$ has prime order, then every infinite non-abelian subgroup of $G$ contains $J$.

*Proof.* Let $X$ be any infinite non-abelian subgroup of $G$. Then $X$ is not contained in $C_G(J)$ and hence $G = XC_G(J)$. It follows that the finite residual $Y$ of $X$ is a normal subgroup of $G$, so that $Y = J$ and $J$ is contained in $X$. □

We also need the following information on finite non-nilpotent groups in the class $\mathcal{H}$.

**Lemma 2.9.** Let $G = \langle x \rangle \ltimes G'$ be a finite non-nilpotent $\mathcal{H}$-group, and let $A$ be an abelian subgroup of $G$. Then either $A \leq C_G(G')$ or $A = \langle x^g \rangle$ for some $g \in G$.

*Proof.* Let $p^n$ be the order of the element $x$ (where $p$ is a prime); then by Theorem 1.7 the order of $G'$ is prime to $p$ and $C_G(G') = \langle x^p, G' \rangle$. Suppose that $A$ is not contained in $C_G(G')$, so that there exists an element $g$ of $G$ such that $x^g \in A$. As $C_{G'}(x) = \{1\}$, it follows that $A = \langle x^g \rangle$. □

Our last two results complete the classification of soluble $\mathcal{H}_\infty$-groups. Observe that of course a Černikov group is central-by-finite if and only if it has finite commutator subgroup.

**Theorem 2.10.** Let $G$ be a Černikov group such that $G' < G$ and $G/Z(G)$ is infinite. Then $G$ is a $\mathcal{H}_\infty$-group if and only if it satisfies one of the following conditions:

(a) $G$ has the property $\mathcal{H}$;

(b) all infinite proper subgroups of $G$ are abelian;

(c) $G = \langle x, G' \rangle$ for some element $x$ of prime-power order $p^n$, the finite residual $J$ of $G$ is a $p$-group having no infinite proper $G$-invariant subgroups, and $G' = J \times K$ where $K$ is a finite abelian $p'$-subgroup and $C_{\langle x \rangle}(G') = \langle x^p \rangle$.

*Proof.* Suppose first that $G$ is a group in the class $\mathcal{H}_\infty \setminus \mathcal{H}$. It follows from Lemma 2.6 that the finite residual $J$ of $G$ has no infinite proper $G$-invariant subgroups; in particular, $J$ is a $p$-group for some prime number $p$. Moreover, the centralizer $C_G(J)$ is properly contained in $G$, as $G/Z(G)$ is infinite. Assume that the finite group $G/J$ is nilpotent, and let $g$ be an element of prime-power order $q^k$ in the set $G \setminus C_G(J)$. Then $\langle g, J \rangle$ is a non-abelian subnormal subgroup of $G$, so that $\langle g, J \rangle = G$ and $G/J$ is cyclic. Clearly, $\langle g^q, J \rangle$ is an infinite proper normal subgroup of $G$, so that it is abelian and hence $\langle g^q, J \rangle = C_G(J)$. Thus $G/C_G(J)$ has order $q$. Let $X$ be any infinite non-abelian subgroup of $G$. It follows from Lemma 2.8 that $X$ contains $J$; on the other hand, $X$ is not contained in $C_G(J)$ and hence $X = G$. Therefore in this case all infinite proper subgroups of $G$ are abelian.

Assume now that the finite group $G/J$ is not nilpotent. As $G'$ is infinite, every proper subgroup of $G$ containing $G'$ is abelian; thus $G/G'$ neither is the set-theoretic union of its proper subgroups nor can be generated by two proper subgroups, and hence it is a cyclic $q$-group for some prime number $q$. In particular, $J$ is contained in $G'$. As the finite non-nilpotent group $G/J$ has the property $\mathcal{H}$, it follows from Theorem 1.7 that

$$G/J = \langle xJ\rangle \times G'/J,$$

where $x$ is an element whose order is a power of $q$ and $G'/J$ is a $q'$-group. The subgroup $\langle x^q, G'\rangle$ must be abelian, and so $C_{\langle x\rangle}(G') = \langle x^q\rangle$. Moreover, $[G', x] = G'$ and hence it follows from Lemma 1.6 and Theorem 1.7 that $G'$ contains elements of order $q$. Therefore $q = p$ and $G' = J \times K$ for some finite abelian $p'$-subgroup $K$.

Conversely, suppose that $G$ satisfies condition (c) of the statement. Then

$$G' = [G', x] = [J, x] \times [K, x],$$

so that $[J, x] = J$ and $[K, x] = K$; as $C_{\langle x\rangle}(K) = \langle x^p\rangle$, it follows from Lemma 1.6 that $C_K(x) = \{1\}$. Write $C/J = C_{G'/J}(xJ)$. We have

$$[K \cap C, x] \le J \cap K = \{1\},$$

so that $K \cap C = \{1\}$ and hence

$$C = JK \cap C = J \times (K \cap C) = J.$$

Therefore $G/J$ is a finite non-nilpotent $\mathcal{H}$-group. Moreover, as $G/Z(G)$ is infinite, the centralizer $C_G(J)$ is properly contained in $G$, so that $C_G(J) = \langle x^p, G'\rangle$; in particular, $C_G(J)$ is abelian and $G/C_G(J)$ has order $p$. Let $X$ be any infinite non-abelian subgroup of $G$. Then $X$ contains $J$ by Lemma 2.8, and so $N_G(X)$ if $X/J$ is not abelian. Assume now that $X/J$ is abelian; as $X$ is not contained in $\langle x^p, G'\rangle$, Lemma 2.9 yields that $X = \langle x^g, J\rangle$ for some element $g \in G$. Finally,

$$[K \cap N_G(X), X] \le K \cap X = \{1\},$$

so that $K \cap N_G(X) \le C_K(x^g) = \{1\}$ and $N_G(X) = X$. Therefore the group $G$ has the property $\mathcal{H}_\infty$. □

In the case of central-by-finite groups, the description of the property $\mathcal{H}_\infty$ must be different. To see this, it is enough to consider a direct product

$$Alt(5) \times P,$$

where $Alt(5)$ is the alternating group of degree 5 and $P$ is a group of type $p^\infty$ for some prime number $p > 5$.

**Theorem 2.11.** Let $G$ be an infinite soluble Černikov group such that $G/Z(G)$ is finite. Then $G$ is a $\mathcal{H}_\infty$-group if and only if it satisfies one of the following conditions:

(a) all infinite proper subgroups of $G$ are abelian;

(b) the finite residual $J$ of $G$ is a group of type $q^\infty$ for some prime number $q$, $G/J = \langle xJ\rangle \ltimes G'J/J$ is a (finite) non-nilpotent $\mathcal{H}$-group and $\langle x^p, G', J\rangle$ is abelian (where $p$ is the prime number such that the coset $xJ$ has $p$-power order).

*Proof.* Suppose first that $G$ has the property $\mathcal{H}_\infty$. If $G$ is nilpotent, all infinite proper subgroups of $G$ are abelian, so that we may assume that $G$ is not nilpotent, and hence the finite group $G/J$ is likewise non-nilpotent since $J \le Z(G)$. Of course, $G/J$ is a $\mathcal{H}$-group and so it follows from Theorem 1.7 that

$$G/J = \langle xJ\rangle \ltimes G'J/J,$$

where the order of the coset $xJ$ is a power of a prime number $p$. Finally, $J$ is a group of type $q^\infty$ for some prime $q$ by Lemma 2.3.

Conversely, suppose that $G$ satisfies condition (b) of the statement, and let $X$ be any infinite non-abelian subgroup of $G$. Then $X$ contains $J$ and $X/J$ cannot be cyclic as $J \le Z(G)$. Assume that $X/J$ is abelian, so that it follows from Lemma 2.9 that $X/J$ is contained in

$$C_{G/J}(G'J/J) = \langle x^p, G', J\rangle/J,$$

contradicting the assumption that $\langle x^p, G', J\rangle$ is abelian. Thus $X/J$ is not abelian and hence $N_G(X) = X$. Therefore the group $G$ belongs to the class $\mathcal{H}_\infty$. □

## References


[1] B. Amberg, S. Franciosi, F. de Giovanni, *Products of Groups*, Clarendon Press, 1992.

[2] S.N. Černikov, *Investigation of groups with given properties of the subgroups*, Ukrain. Math. J. 21 (1969), 160–172.

[3] S. Franciosi, F. de Giovanni, *On torsion groups with nilpotent automorphism groups*, Comm. Algebra 14 (1986), 1909–1935.

[4] N.F. Kuzennyi, N.N. Semko, *Structure of solvable nonnilpotent metahamiltonian groups*, Math. Notes 34 (1983), 572–577.

[5] V.N. Obraztsov, *An embedding theorem for groups and its corollaries*, USSR Sb. 66 (1990), 541–553.

[6] R.E. Phillips, J.S. Wilson, *On certain minimal conditions for infinite groups*, J. Algebra 51 (1978), 41–68.

[7] D.J.S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer, 1972.

[8] G.M. Romalis, N.F. Sesekin, *Metahamiltonian groups*, Ural. Gos. Univ. Mat. Zap. 5 (1966), 101–106.

[9] G.M. Romalis, N.F. Sesekin, *Metahamiltonian groups II*, Ural. Gos. Univ. Mat. Zap. 6 (1968), 52–58.

[10] G.M. Romalis, N.F. Sesekin, *Metahamiltonian groups III*, Ural. Gos. Univ. Mat. Zap. 7 (1969/70), 195–199.g

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