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ANALYTICAL CONSTRUCTION OF THE HIERARCHICAL MATRIX LYAPUNOV FUNCTION FOR IMPULSE SYSTEMS *

АНАЛІТИЧНА ПОБУДОВА ІЄРАРХІЧНОЇ МАТРИЧНОЇ ФУНКЦІЇ ЛЯПУНОВА ДЛЯ ІМПУЛЬСНИХ СИСТЕМ

For impulse systems, we develop a method for the construction of the hierarchical matrix Lyapunov function.

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Introduction. In this paper, we consider systems of differential equations that simulate evolutionary processes in techniques and engineering under impulse perturbations.

In the last years, many phenomena with manifested impulse effect were described in biology, medicine, the theory of optimal control, economics, pharmacokinetics, and the theory of systems with modulation.

Numerous problems in the theory of differential equations with impulse perturbation are already solved (see [1–5]). Nevertheless, in general, the development of constructive methods for the investigation of various dynamical properties of such systems is still of considerable interest.

The aim of this paper is to develop a method of matrix-valued Lyapunov functions for impulse systems.

Hierarchical impulse systems. We consider the impulse system

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \qquad t \neq \tau_k(x), \\ \Delta x &= x(t+0) - x(t) = I^k(x), \\ x(t_0^+) &= x_0, \qquad k \in Z. \end{aligned} \tag{1}$$

Here, $x \in (x_1, \ldots, x_n)^T \in R^n$, $f \in C(R_+ \times \Omega(\rho), R^n)$, $I^k \in C(\Omega(\rho), R^n)$, and $\Omega(\rho) = \{x \in R^n \colon \|x\| < \rho\}$, $\rho > 0$, $\Omega(\rho) \subseteq R^n$.

We assume that system (1) satisfies some general conditions.

A_1. The functions $\tau_k \colon \Omega \to R_+$ are continuous in x.

A_2. The following relations hold:

$$0 < \tau_1(x) < \tau_2(x) < \ldots < \tau_k(x) < \ldots \quad \text{for} \quad x \in \Omega,$$

$$\inf\{\tau_k(x) - \tau_{k-1}(x) \colon k \geq 2,\ x \in R^n\} > 0,$$

and $\lim_{k\to\infty} \tau_k(x) = \infty$ uniformly in $x \in \Omega$.

A_3. The integral curve of any solution of system (1) intersects any hypersurface $S_k \colon t = \tau_k(x)$, $k \in Z$, not more than once.

A_4. There exists a constant $\mu \in (0, \rho)$ such that $x + I^k(x) \in \Omega(\rho)$ $\forall\, k \in Z$ whenever $x \in \Omega(\mu)$.

A_5. For any decomposition of a vector $x \in R^n$ into subvectors $x_s \in R^{n_s}$, $n_1 + \ldots + n_N = n$, the values $\tau_k(x_s)$, $k \in Z$, satisfy conditions $A_1 - A_4$.

Assume that system (1) consists of N independent subsystems

$$\frac{dx_i}{dt} = g_i(t, x_i), \qquad t \neq \tau_k(x_i), \quad i = 1, 2, \ldots, N,$$

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$$\Delta x_i = \Phi_i^k(x_i), \quad t \in \tau_k(x), \quad k \in Z, \tag{2}$$
$$x_i(t_0^+) = x_{i0},$$

where $x_i \in R^{n_i}$, $g_i \in C(R_+ \times R^{n_i}, R^{n_i})$, $g_i(t, 0) = 0 \quad \forall\, t \in R_+$ and link functions

$$h_i(t, x_1, \ldots, x_N)\colon\ h \in C(R_+ \times R^{n_1} \times \ldots \times R^{n_N}, R^{n_i}),\ t \neq \tau_k(x_i),$$
$$J_i^k(x) = I_i^k(x) - \Phi_i^k(x_i), \quad t = \tau_k(x_i). \tag{3}$$

Thus, system (1) can be transformed as follows:

$$\frac{dx_i}{dt} = g_i(t, x_i) + h_i(t, x_1, \ldots, x_N), \quad t \neq \tau_k(x_i),$$
$$\Delta x_i = \Phi_i^k(x_i) + J_i^k(x), \quad t \in \tau_k(x_i). \tag{4}$$

Furthermore, we assume that subsystems (2) are disconnected and

$$X = X_1 \oplus X_2 \oplus \ldots \oplus X_N, \tag{5}$$

where X and X_i are state spaces of systems (1) and (2), respectively.

Further, we assume that each subsystem (2) admits a decomposition into M_i components defined by

$$\frac{dx_{ij}}{dt} = p_{ij}(t, x_{ij}), \quad t \neq \tau_k(x_{ij}), \quad j = 1, 2, \ldots, M_i,$$
$$\Delta x_{ij} = \Phi_{ij}^k(x_{ij}), \quad t = \tau_k(x_{ij}), \quad k \in Z, \tag{6}$$
$$x_{ij}(t_0^+) = x_{ij0},$$

which after interaction form the subsystems

$$\frac{dx_{ij}}{dt} = p_{ij}(t, x_{ij}) + q_{ij}(t, x_i), \quad t \neq \tau_k(x_{ij}), \quad j = 1, 2, \ldots, M_i,$$
$$\Delta x_{ij} = I_{ij}^k(x_{ij}) + J_{ij}^k(x_i), \quad t = \tau_k(x_{ij}), \quad k \in Z, \tag{7}$$
$$x_{ij}(t_0^+) = x_{ij0},$$

where $x_{ij} \in R^{n_{ij}}$, $p_{ij} \in C(R_+ \times R^{n_{ij}}, R^{n_{ij}})$, $q_{ij} \in C(R_+ \times R^{n_i}, R^{n_{ij}})$, and $x_{ij} = 0$ is the only equilibrium state of subsystems (6). Assume that subsystems (6) are disconnected, i. e.,

$$X_i = X_{i1} \oplus X_{i2} \oplus \ldots \oplus X_{iM_i}, \quad i = 1, 2, \ldots, N, \tag{8}$$

where X_i and X_{ij} are state spaces of subsystems (7) and (6), respectively.

Impulse systems simulated by equations (1) and admitting the first-level decomposition (2) – (4) and second-level one (6) – (8) have *a multilevel hierarchical structure* (see [6]).

The structure of a hierarchical matrix function. According to the two levels of decomposition (2) – (4) and (6) – (8) of system (6), we consider a two-level construction of a submatrix of a matrix-valued function (see [7])

$$U(t, x) = [\,U_{ij}(t, \cdot)\,], \quad U_{ij} = U_{ji}, \tag{9}$$

where $U_{ii}\colon R_+ \times R^{n_i} \to R_+$, $i = 1, 2, \ldots, N$, $U_{ij}\colon R_+ \times R^{n_i} \times R^{n_j} \to R$, $i \neq j$, $j = 1, 2, \ldots, N$, for system (6).

Further, we need the class U_0 of piecewise continuous matrix-valued functions. Let $\tau_0(x) \equiv 0$ for all $x \in R^n$. We introduce the sets

$$G_k = \{(t,x):\ R_+ \times \Omega(\rho):\ \tau_{k-1}(x) < t < \tau_k(x)\},$$

$$G = \bigcup_{k=1}^{\infty} G_k, \quad k \in Z,$$

$$\sigma_k = \{(t,x):\ R_+ \times \Omega:\ t = \tau_k(x)\}, \quad k \in Z.$$

Definition 1. *A matrix-valued function* $U: R_+ \times \Omega(\rho) \to R^{N\times N}$ *belongs to the class* W_0 *if* $U(t,x)$ *is continuous on every set* $\{G_k\}$ *and for* $(t_0, x_0) \in \sigma_k \cap D \subset \subset R_+ \times \Omega(\rho)$, $k \in Z$, *there exist the limits*

$$\lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in G_k}} U(t,x) = U(t_0^-, x_0),$$

$$\lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in G_{k+1}}} U(t,x) = U(t_0^+, x_0),$$

and

$$U(t_0^-, x_0) = U(t_0^+, x_0) \in R^{N\times N}.$$

The matrix-valued function (9) has the following structure: The functions $U_{ii}(t,\cdot)$ are constructed for subsystems (2), and the functions $U_{ij}(t,\cdot)$ $(i \neq j)$ take into account the links $h_i(t, x_1, \ldots, x_N)$ between subsystems (2).

The functions $U_{ii}(t,\cdot)$ have the explicite form

$$U_{ii}(t,\cdot) = \xi_i^T B_i(t,\cdot)\,\xi_i, \quad i = 1, 2, \ldots, N, \tag{10}$$

where $\xi_i \in R_+^{n_i}$, $\xi_i > 0$, and the submatrix functions $B_i(t,\cdot) = [u_{pq}^{(i)}(t,\cdot)]$, $p, q = = 1, 2, \ldots, M_i$, have the elements

$$u_{pp}^{(i)}: R_+ \times R^{n_{ip}} \to R, \quad j = 1, 2, \ldots, N,$$

$$u_{pq}^{(i)}: R_+ \times R^{n_{ip}} \times R^{n_{iq}} \to R, \quad u_{pq}^{(i)} = u_{qp}^{(i)}.$$

The functions $u_{pp}^{(i)}(t,\cdot)$ are constructed for subsystems (6), and $u_{pq}^{(i)}$ $(p \neq q)$ take into account the influence of link functions $q_{ij}(t, x_i)$ between subsystems (6).

Similarly to Definition 1, for the elements $u_{pq}^{(i)}$ with $i = 1, 2, \ldots, N$ and $p, q = = 1, 2, \ldots, M_i$, we consider the classes w_0 of piecewise continuous functions.

In order to establish conditions for the function

$$V(t, x, \eta) = \eta^T U(t,x)\eta, \quad \eta \in R_+^n, \quad \eta > 0, \tag{11}$$

of fixed sign, we need several assumptions.

Assumption 1. There exist

1) open connected time-invariant neighborhoods $\mathcal{N}_{ip} \subseteq R^{n_{ip}}$, $i = 1, 2, \ldots, N$, $p = = 1, 2, \ldots, M_i$, of states $x_{ip} = 0$;

2) functions $\varphi_{ip}, \overline{\varphi}_{ip} \in K(KR)$;

3) constants $\underline{\alpha}_{pp}^{(i)} > 0$, $\overline{\alpha}_{pp}^{(i)} > 0$, $\underline{\alpha}_{pq}^{(i)} = \underline{\alpha}_{qp}^{(i)}$, $\overline{\alpha}_{pq}^{(i)} = \overline{\alpha}_{qp}^{(i)}$, $q = 1, 2, \ldots \ldots, M_i$, and functions $u_{pp}^{(i)} \in w_0$, $u_{pq}^{(i)} \in w_0$, $p \neq q$, $p, q = 1, 2, \ldots, M_i$, satisfying the estimates

a) $\underline{\alpha}_{pp}^{(i)} \varphi_{ip}^2(\|x_{ip}\|) \le u_{pp}^{(i)}(t, x_{ip}) \le \overline{\alpha}_{pp}^{(i)} \varphi_{ip}^2(\|x_{ip}\|)$ for all $t \neq \tau_k(\cdot)$, $k \in Z$, and $x_{ip} \in N_{ip}$;

b) $\underline{\alpha}_{pq}^{(i)} \varphi_{ip}(\|x_{ip}\|) \varphi_{iq}(\|x_{iq}\|) \le u_{pq}^{(i)}(t, x_{ip}, x_{iq}) \le \overline{\alpha}_{pq}^{(i)} \varphi_{ip}(\|x_{ip}\|) \varphi_{iq}(\|x_{iq}\|)$ for all $t \ne \tau_k(\cdot)$, $k \in Z$, and $(x_{ip}, x_{iq}) \in \mathcal{N}_{ip} \times \mathcal{N}_{iq}$.

Proposition 1. *If all conditions of Assumption* 1 *are satisfied, then the functions* $U_{ii}(t, x_i)$ *satisfy the estimates*

$$w_i^T \Phi_i^T A_{ii} \Phi_i w_i \le U_{ii}(t, x_i) \le \overline{w}_i^T \Phi_i^T B_{ii} \Phi_i \overline{w}_i \tag{12}$$

for all $t \ne \tau_k(\cdot)$, $k \in Z$, *and all* $x_i \in \mathcal{N}_i$, $\mathcal{N}_i = \mathcal{N}_{i1} \times \mathcal{N}_{i2} \times \ldots \times \mathcal{N}_{iM_i}$, $i = 1, 2, \ldots, N$.

Here, we have used the following notation:

$$w_i^T = (\varphi_{i1}(\|x_{i1}\|), \varphi_{i2}(\|x_{i2}\|), \ldots, \varphi_{iM_i}(\|x_{iM_i}\|)),$$

$$\overline{w}_i^T = (\overline{\varphi}_{i1}(\|x_{i1}\|), \overline{\varphi}_{i2}(\|x_{i2}\|), \ldots, \overline{\varphi}_{iM_i}(\|x_{iM_i}\|)),$$

$$\Phi_i^T = \Phi_i = \operatorname{diag}(\xi_{i1}, \xi_{i2}, \ldots, \xi_{iM_i}),$$

$$A_{ii} = [\underline{\alpha}_{pq}^{(i)}], \quad B_{ii} = [\overline{\alpha}_{pq}^{(i)}], \quad i = 1, 2, \ldots, M_i.$$

Proposition 1 can be proved by direct substitution of estimates 3a) and 3b) into the quadratic forms $\xi_i^T B_i(t, \cdot) \xi_i$, $i = 1, 2, \ldots, N$, provided that the other conditions of Assumption 1 are satisfied.

Assumption 2. There exist

1) open connected time-invariant neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$, $i = 1, 2, \ldots, N$, of equilibrium states $x_i = 0$;

2) functions $\varphi_i, \overline{\varphi}_i \in K(KR)$;

3) constants β_{ij}, $\overline{\beta}_{ij}$, $\beta_{ij} = \beta_{ji}$, $\overline{\beta}_{ij} = \overline{\beta}_{ji}$ for all $(i \ne j) \in [1, N]$, such that

$$\beta_{ij} \|w_i\| \|w_j\| \le U_{ij}(t, x_i, x_j) \le \overline{\beta}_{ij} \|\overline{w}_i\| \|\overline{w}_j\| \tag{13}$$

for all $t \ne \tau_k(\cdot)$, $k \in Z$, and $(x_i, x_j) \in \mathcal{N}_i \times \mathcal{N}_j$, $(i \ne j) \in [1, N]$.

Proposition 2. *If the conditions of Assumption* 2 *and estimates* a) *and* b) *of condition* 3) *in Assumption* 1 *are satisfied, then functions* (11) *satisfies the bilateral inequality*

$$w^T H^T A H w \le V(t, x, \eta) \le \overline{w}^T H^T B H \overline{w} \tag{14}$$

for all $t \ne \tau_k(\cdot)$, $k \in Z$, *and all* $x \in \mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_N$.

Here,

$$w^T = (w_1, w_2, \ldots, w_N), \quad \overline{w}^T = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_N),$$

$$H^T = H = \operatorname{diag}[\eta_1, \eta_2, \ldots, \eta_N],$$

$$A = [\beta_{ij}], \quad B = [\overline{\beta}_{ij}], \quad i \ne j = 1, 2, \ldots, N,$$

$$\beta_{ii} = \lambda_m(\Phi_i^T A_{ii} \Phi_i), \quad \overline{\beta}_{ii} = \lambda_M(\Phi_i^T B_{ii} \Phi_i).$$

Proof. The direct substitution of estimates (12) and (13) into expression (11) for the function $V(t, x, \eta)$ yields estimate (14).

Structure of the total derivative of the hierarchical matrix function. Further, in order to establish the structure of the total derivative of function (11) along the solution of system (1) admitting two-level decomposition, we introduce some definitions and notations.

Definition 2. *A matrix-valued function* $U: R_+ \times \Omega(\rho) \to R^{N \times N}$ *belongs to the class* W_1 *if the matrix function* $U \in W_0$ *and is continuously differentiable on the set* $\sum_{k=1}^{\infty} G_k \cap D$, $D \subset R_+ \times \Omega(\rho)$.

Definition 3. *A matrix-valued function* $U : R_+ \times \Omega(\rho) \to R^{N\times N}$ *belongs to the class* W_2 *if the matrix function* $U \in W_0$ *and is locally Lipschitzian in the second argument.*

Let $x(t)$ be any solution of (1) defined for $t \in [t_0, t_0 + a) \subset J \subset R_+$, $a = \text{const} > 0$, and such that $x(t) \in \Omega_1$, $\Omega_1 \subset \Omega(\rho)$ for all $t \in J$.

We define the function

$$D^+ U(t,x) = \limsup_{\theta\to 0^+} \{[U(t+\theta, x(t+\theta)) - U(t, x(t))]\,\theta^{-1}\}$$

for all $(t, x(t)) \in \sum_{k=1}^{\infty} G_k \bigcap D$.

The actual computation of $D^+ U(t,x)$ is performed elementwise. An efficient application of the upper right-hand derivative within the framework of the second Lyapunov method is based on the following result [5] which enables the computation of $D^+ U(t,x)$ without immediate use of a solution on (1):

Let a matrix-valued function U *belong to* W_2. *Then*

$$D^+ U(t,x) = \limsup_{\theta\to 0^+} \{[U(t+\theta, x+\theta f(t,x)) - U(t, x(t))]\,\theta^{-1}\}$$

for $(t,x) \in D$.

Assumption 3. There exist

1) open connected time-invariant neighborhoods $\mathcal{N}_{ip}$, $\mathcal{N}_{ip} \subseteq R^{n_{ip}}$, of states $x_{ip} = 0$, $i = 1, 2, \ldots, N$, $p = 1, 2, \ldots, M_i$;

2) functions $u_{pq}^{(i)} \in w_0$, $q = 1, 2, \ldots, M_i$;

3) functions $\beta_{ip} \in K(KR)$;

4) real numbers $\rho_p^{(i)}$, $\mu_p^{(i)}$, $\mu_{pq}^{(i)}$ such that

a) $\sum_{p=1}^{M_i} \xi_{ip}^2 \{D_t^+ u_{pp}^{(i)} + (D_{x_{ip}}^+ u_{pp}^{(i)})^T p_{ip}(t, x_{ip})\} \le \sum_{p=1}^{M_i} \rho_p^{(i)} \beta_{ip}^2(\|x_{ip}\|)$ for all $t \ne \tau_k(\cdot)$, $k \in Z$, and $x_{ip} \in N_{ip}$;

b)
$$\sum_{p=1}^{M_i} \xi_{ip}^2 \{(D_{x_{ip}}^+ u_{pp}^{(i)})^T q_{ip}(t, x_i)\} +$$
$$+ 2\sum_{p=1}^{M_i-1} \sum_{q=p+1}^{M_i} \xi_{ip}\xi_{iq} \{D_t^+ u_{pq}^{(i)} + (D_{x_{ip}}^+ u_{pp}^{(i)})^T (p_{ip}(t, x_{ip}) + q_{ip}(t, x_i)) +$$
$$+ (D_{x_{iq}}^+ u_{pq}^{(i)})^T (p_{iq}(t, x_{iq}) + q_{iq}(t, x_i)\} \le$$
$$\le \sum_{p=1}^{M_i} \mu_p^{(i)} \beta_{ip}^2(\|x_{ip}\|) + 2\sum_{p=1}^{M_i-1} \sum_{q=p+1}^{M_i} \mu_{pq}^{(i)} \beta_{ip}(\|x_{ip}\|)\,\beta_{ip}(\|x_{iq}\|)$$

for all $t \ne \tau_k(\cdot)$, $k \in Z$, and all $(x_{ip}, x_{iq}) \in \mathcal{N}_{ip} \times \mathcal{N}_{iq}$.

Here,

$$D_t^+ u_{pp}^{(i)}(t, x_{ip}) = \limsup \left\{ \frac{u_{pp}^{(i)}(t+\theta, x_{ip}) - u_{pp}^{(i)}(t, x_{ip})}{\theta} : \theta \to 0^+ \right\}, \tag{15}$$

$$D_{x_{ip}}^+ u_{pp}^{(i)}(t, x_{ip}) = \limsup \left\{ \frac{u_{pp}^{(i)}(t+\theta, x_{ip} + I_{i,k+1}\Delta x_{ip}) - u_{pp}^{(i)}(t+\theta, x_{ip} + I_{i,k}\Delta x_{ip})}{\Delta x_{ip}} : \right.$$

$$\left.\theta \to 0^+, \ \|x_{ip}\| \to 0\right\}, \tag{16}$$

where $x_{ip}(t+\theta\,;t, x_{ip}) = x_{ip} + \Delta x_{ip}$,

$$I_{ik} = \operatorname{diag}\left((1-\delta_{1k}), (1-\delta_{2k}), \ldots, (1-\delta_{kk}), 0, \ldots, 0\right) \in R^{n_{ip}\times n_{ip}},$$

and

$$\delta_{ik} = \begin{cases} 0, & i \neq k; \\ 1, & i = k, \end{cases}$$

is the Kronecker symbol.

Proposition 3. *If all conditions of Assumptions* 3 *are satisfied, then the upper right-hand derivative of the functions* $U_{ij}(t, x)$ *along solutions of* (7) *satisfies the estimate*

$$D^+ U_{ii}(t, x_i) \leq \lambda_M(S_{ii}) \| \beta_i \|^2 \tag{17}$$

for all $t \neq \tau_k(\cdot)$, $k \in Z$, *and all* $x_i \in \mathcal{N}_i$.

Here, $\beta_i^T = (\beta_{i1}(\| x_{i1} \|), \beta_{i2}(\| x_{i2} \|), \ldots, \beta_{iM_i}(\| x_{iM_i} \|))$ *and* $\lambda_M(S_{ii})$ *is the maximal eigenvalue of the matrix* S_{ii} *with the elements*

$$\begin{aligned} \sigma_{pp}^{(i)} &= p_p^{(i)} + \mu_p^{(i)}, \quad p = 1, 2, \ldots, M_i, \\ \sigma_{pq}^{(i)} &= \sigma_{qp}^{(i)} = \mu_{pq}^{(i)}, \quad (p \neq q) \in [1, M_i]. \end{aligned} \tag{18}$$

Proof. In view of (15) and (16), we establish that the expressions $D^+ u_{pq}^{(i)}(t, \cdot)$ satisfy the estimates

$$D^+ u_{pq}^{(i)}(t, x_{ip}) \leq D_t^+ u_{pq}(t, x_{ip}) + (D_{x_{ip}}^+ u_{pp}^{(i)}(t, x_{ip}))^T \frac{dx_{pq}}{dt} \tag{19}$$

for all $i = 1, 2, \ldots, N$ and $(p, q) \in [1, M_i]$. In view of (19) and conditions 1) – 4) in Assumption 3, we arrive at estimates (17).

Assumption 4. There exist

1) some constants $\rho_{ip}^0 > 0$ and $\rho_{ip}^0 < \rho$ such that $x_{ip} \in \Omega(\rho_{ip}^0)$ guarantees the inclusion $x_{ip} + J_{ip}^k(x_{ip}) \in \Omega(\rho_{ip})$ for all $k \in Z$;

2) functions $u_{pq}^{(i)} \in w_0$ for all $i = 1, 2, \ldots, N$ and $(p, q) \in [1, M_i]$;

3) functions $\psi_{ip} \in K(KR)$;

4) real numbers $a_{pp}^{(i)}$, $b_{pp}^{(i)}$, and $b_{pq}^{(i)}$ such that

a) $\xi_{ip}^2 \{u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip} + J_{ip}^k(x_{ip})) - u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip})\} \leq a_{pp}^{(i)} \psi_{ip}^2(\|x_{ip}\|)$ for all $x_{ip} \in \mathcal{N}_{ip} \subseteq \Omega(\rho_{ip})$;

b)
$$\sum_{p=1}^{M_i} \xi_{ip}^2 \{u_{pp}^{(i)}(\tau_k(x_i), x_i + J_i^k(x_i)) - u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip} + J_{ip}^k(x_{ip})) +$$
$$+ u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip}) - u_{pp}^{(i)}(\tau_k(x_i), x_i)\} +$$
$$+ 2\sum_{p=1}^{M_i-1} \sum_{q=p+1}^{M_i} \xi_{ip}\xi_{iq} \{u_{pq}^{(i)}(\tau_k(x_i), x_{ip} + J_{ip}^k(x_i), x_{iq} + J_{iq}^k(x_i)) -$$
$$- u_{pq}^{(i)}(\tau_k(x_i), x_{ip}, x_{iq})\} \leq$$

$$\leq \sum_{p=1}^{M_i} b_{pp}^{(i)} \psi_{ip}^2(\|x_{ip}\|) + 2 \sum_{p=1}^{M_i-1} \sum_{q=p+1}^{M_i} b_{pq}^{(i)} \psi_{ip}(\|x_{ip}\|) \psi_{ip}(\|x_{iq}\|)$$

for all $(x_{ip}, x_{iq}) \in \mathcal{N}_{ip} \times \mathcal{N}_{iq}$.

Proposition 4: *If all conditions of Assumption* 4 *are satisfied, then for* $t = \tau_k(x_i)$, $k \in Z$, $i = 1, 2, \ldots, N$, *the estimates*

$$U_{ii}(\tau_k(x_i), x_i + J_i^k(x_i)) - U_{ii}(\tau_k(x_i), x_i) \leq \lambda_M(C_{ii}) \|\psi_i\|^2 \tag{20}$$

are true for the functions $U_{ii}(t, x_i)$.

Here,

$$\psi_i^T = (\psi_{i1}(\|x_{i1}\|), \psi_{i2}(\|x_{i2}\|), \ldots, \psi_{iM_i}(\|x_{iM_i}\|)),$$

$$C_{ii} = [c_{pq}^{(i)}], \quad i = 1, 2, \ldots, N, \quad (p, q) \in [1, M_i],$$

$$c_{pq}^{(i)} = c_{qp}^{(i)}, \quad c_{pp}^{(i)} = a_{pp}^{(i)} + b_{pp}^{(i)},$$

$$c_{pq}^{(i)} = c_{qp}^{(i)} = b_{pq}^{(i)}, \quad i = 1, 2, \ldots, N,$$

and $\lambda_M(C_{ii})$ *is the maximal eigenvalue of the matrix* C_{ii}.

The proof of Proposition 4 is obvious in view of condition 4a) and 4) of Assumption 4.

Assumption 5. There exist

1) functions $u_{pq}^{(i)} \in w_0$ for all $i = 1, 2, \ldots, N$, $(p, q) \in [1, M_i]$;

2) functions $\psi_{ip} \in K(KR)$;

3) real numbers $d_{pp}^{(i)}$, $l_{pq}^{(i)}$, such that, for $t = \tau_k(x_{ip})$, $k \in Z$, and $x_{ip} \in \mathcal{N}_{ip}$ the following conditions are satisfied:

a) $\xi_{ip}^2 u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip} + J_{ip}^k(x_{ip})) \leq d_{pp}^{(i)} \psi_{ip}^2(\|x_{ip}\|)$;

b) $$\sum_{p=1}^{M_i} \xi_{ip}^2 \{u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip} + J_{ip}^k(x_i)) - u_{pp}^{(i)}(\tau_k(x_{ip}), x_{ip} + J_{ip}^k(x_{ip}))\} +$$

$$+ 2 \sum_{p=1}^{M_i-1} \sum_{q=p+1}^{M_i} \xi_{ip} \xi_{iq} u_{pq}^{(i)}(\tau_k(x_i), x_{ip} + J_{ip}^k(x_{ip}), x_{iq} + J_{iq}^k(x_{iq})) \leq$$

$$\leq \sum_{p=1}^{M_i} l_{pp}^{(i)} \psi_{ip}^2(\|x_{ip}\|) + 2 \sum_{p=1}^{M_i-1} \sum_{q=p+1}^{M_i} l_{pq}^{(i)} \psi_{ip}(\|x_{ip}\|) \psi_{iq}(\|x_{iq}\|).$$

Proposition 5. *If all conditions of Assumption 5 are satisfied, then for functions* $U_{ii}(t, \cdot)$ with $t = \tau_k(x_i)$, $k \in Z$, *and* $i = 1, 2, \ldots, N$, *the estimate*

$$U_{ii}(\tau_k(x_i), x_i + J_i^k(x_i)) \leq \lambda_M(C_{ii}^*) \|\psi_i\|^2$$

is valid for all $x_i \in \mathcal{N}_i$, *where* $\lambda_M(\cdot)$ *is the maximal eigenvalue of the matrix* C_{ii}^* *with the elements*

$$c_{pp}^{*(i)} = d_{pp}^{(i)} + l_{pp}^{(i)}, \quad i = 1, 2, \ldots, N,$$

$$c_{pq}^{*(i)} = c_{qp}^{*(i)} = l_{pq}^{(i)}, \quad p, q = 1, 2, \ldots, M_i.$$

The proof of Proposition 5 is similar to that of Proposition 4.

Assumption 6. There exist

1) an open connected neighborhood $\mathcal{N} \subseteq R^n$ of $x = 0$;

2) the functions β_{ip}, $i = 1, 2, \ldots, N$, $p = 1, 2, \ldots, M_i$, mentioned in Assumption 3;

3) functions $U_{ij}(t, \cdot)$ satisfying the conditions of Assumption 2;

4) real number θ_{ik}, $i, k = 1, 2, \ldots, N$, such that

$$\sum_{i=1}^{N} \eta_i^2 (D^+ U_{ii}(t, x_i))^T h_i(t, x) +$$

$$+ 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \eta_i \eta_k \{D_t^+ U_{ik}(t, \cdot) + (D_{x_i}^+ U_{ik}(t, x_i, x_k))^T (g_i(t, x_i) + h_i(t, x)) +$$

$$+ (D_{x_k}^+ U_{ik}(t, x_i, x_k))^T (g_k(t, x_k) + h_k(t, x))\} \leq$$

$$\leq \sum_{i=1}^{N} \theta_{ii} \|\beta_i\|^2 + 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \theta_{ik} \|\beta_i\| \, \|\beta_k\|$$

for all $t \neq \tau_k(x_i)$, $k \in Z$, and $(x_i, x_k) \in \mathcal{N}_i \times \mathcal{N}_k$.

Proposition 6. *If all conditions of Assumptions* 3 *and* 6 *are satisfied, then for the function* $D^+ V(t, x, \eta)$, *the estimate*

$$D^+ V(t, x, \eta) \leq \beta^T S \beta \tag{21}$$

is valid along solutions of (1) *for all* $t \neq \tau_k(x_i)$, *where* $\beta = (\beta_1^T, \beta_2^T, \ldots, \beta_N^T)$ *and the matrix* S *has the elements*

$$s_{ii} = \eta_i^2 \lambda_M(S_{ii}) + \theta_{ii},$$

$$s_{ik} = s_{ki} = \theta_{ik}, \qquad i, k = 1, 2, \ldots, N.$$

Proof. Estimate (21) can be obviously obtained from the conditions of Assumption 6 by using estimate (19).

Proposition 7. *If inequality* (21) *is true, then there exist continuous functions* $H_1, H_2 \colon R_+ \to R_+$, $H_1(0) = H_2(0) = 0$, $H_1(r) > 0$, $H_2(r) > 0$ *for* $r > 0$, *such that*

$$0 < H_1(V(t, x, \eta)) \leq \sum_{i=1}^{N} \beta_i^2(\|x_i\|) \leq H_2(V(t, x, \eta)).$$

If, moreover,

a) $\lambda_M(S) < 0$,

b) $\lambda_M(S) > 0$,

then for all $t \neq \tau_k(x)$, $x \in \mathcal{N}$, $k \in Z$, *the following estimates, respectively, hold:*

a) $D^+ V(t, x, \eta) \leq \lambda_M(S) H_1(V(t, x, \eta))$,

a) $D^+ V(t, x, \eta) \leq \lambda_M(S) H_2(V(t, x, \eta))$.

Here, $\lambda_M(\cdot)$ *is the maximal eigenvalue of the matrix* S.

Proof. Estimates a) and b) follow from the fact that $\beta^T S \beta \leq \lambda_M(S) \beta^T \beta$ under the conditions of Proposition 7.

Assumption 7. There exist

1) open connected time-invariant neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of $x_i = 0$, $i = 1, 2, \ldots, N$, and the neighborhood $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_N$ of $x = 0$;

2) functions U_{im}, $m = 1, 2, \ldots, N$, satisfying the conditions of Assumption 2;

3) functions $\psi_{ip} \in K$;

4) real numbers ν_{ip}, $p = 1, 2, \ldots, N$, such that for all $k \in Z$, the following conditions are satisfied for all $(x_i, x_p) \in \mathcal{N}_i \times \mathcal{N}_p$:

$$\sum_{i=1}^{N} \eta_i^2 \{U_{ii}(\tau_k(x), x_i + J_i^k(x)) - U_{ii}(\tau_k(x), x_i)\} +$$

$$+ 2\sum_{i=1}^{N-1} \sum_{p=i+1}^{N} \eta_i \eta_p \{U_{ip}(\tau_k(x), x_i + J_i^k(x), x_p + J_p(x)) - U_{ip}(\tau_k(x), x_i, x_p)\} \le$$

$$\le \sum_{i=1}^{N} \nu_{ii} \psi_i^2(\|x_i\|) + 2\sum_{i=1}^{N-1} \sum_{p=i+1}^{N} \nu_{ip} \psi_i(\|x_i\|) \psi_p(\|x\|). \tag{22}$$

Proposition 8. *If inequalities* (20) *and* (22) *are satisfied, then the estimate*

$$V(\tau_k(x), x + J^k(x), \eta) - V(\tau_k(x), x, \eta) \le \psi^T C \psi \tag{23}$$

holds for all $t = \tau_k(\cdot)$, $k \in Z$, $x \in \mathcal{N}_x$, *where*

$$\psi^T = (\psi_1(\tau_k(x), \|x_1\|), \ldots, \psi_N(\tau_k(x), \|x_N\|)),$$

and the matrix C *has the elements*

$$c_{ii} = \eta_i^2 \lambda_M(C_{ii}) + \nu_{ii},$$

$$c_{ip} = c_{pi} = \nu_{ip}, \quad i \ne p \in [1, N].$$

Proposition 9. *If inequality* (23) *is satisfied, then there exist continuous functions* Q_1 *and* Q_2 *such that* $Q_1(0) = Q_2(0) = 0$, $Q_1(s) > 0$ *and* $Q_2(s) > 0$ *for* $s > 0$, *and*

$$0 < Q_1(V(\tau_k(x), x)) \le \sum_{i=1}^{N} \psi_i^2(\tau_k, \|x_i\|) \le Q_2(V(\tau_k(x), x)).$$

If, moreover,

a) $\lambda_M(S) < 0$,

b) $\lambda_M(S) > 0$,

then for all $k \in Z$, *the following estimates, respectively, hold*:

a) $V(\tau_k(x), x + J^k(x)) - V(\tau_k(x), x) \le \lambda_M(C) Q_1(V(\tau_k(x), x))$,

b) $V(\tau_k(x), x + J^k(x)) - V(\tau_k(x), x) \le \lambda_M(C) Q_2(V(\tau_k(x), x))$.

Here, $\lambda_M(C)$ *is the maximal eigenvalue of the matrix* C.

Proof. Consider the quadratic form $\psi^T C \psi$. It is known that

$$\psi^T C \psi \le \lambda_M(C) \psi^T \psi = \lambda_M(C) \sum_{i=1}^{N} \psi_i^2(\tau_k(x), \|x\|).$$

Hence,

$$V(\tau_k(x), x + J^k(x)) - V(\tau_k(x), x) \le$$

$$\le \begin{cases} \lambda_M(C) Q_1(V(\tau_k(x), x) & \text{if } \lambda_M(C) < 0; \\ \lambda_M(C) Q_2(V(\tau_k(x), x) & \text{if } \lambda_M(C) > 0. \end{cases}$$

This proves Proposition 9.

Assumption 8. There exist

1) open connected time-invariant neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of $x_i = 0$, $i = 1, 2, \ldots, N$;

2) the functions U_{ip}, $i, p = [1, N]$, mentioned in Assumption 2;

3) functions $\psi_{ip}(\tau_k(x), \|x_i\|)$, $i, p \in [1, N]$, $k \in Z$, continuous in the second argument and such that $\psi_{ip}(\tau_k(x), 0) = 0$ and $\psi_{ip}(\tau_k(x), r) > 0$ for $r > 0$;

4) real numbers μ_{ip} such that, for all $k \in Z$, the estimates

$$\sum_{i=1}^{N} \eta_i^2 \{U_{ii}(\tau_k(x), x_i + J_i^k(x))\} +$$

$$+ 2 \sum_{i=1}^{N-1} \sum_{p=i+1}^{N} \eta_i \eta_p U_{ip}(\tau_k(x), x_i + J_i^k(x), x_p + J_p(x)) \le$$

$$\le \sum_{i=1}^{N} \mu_{ii} \psi_i^2(\tau_k(x), \|x_i\|) + 2 \sum_{i=1}^{N-1} \sum_{p=i+1}^{N} \mu_{ip} \psi_i(\tau_k(x), \|x_i\|) \psi_p(\tau_k(x), \|x_p\|)$$

are satisfied for all $(x_i, x_p) \in \mathcal{N}_i \times \mathcal{N}_p$.

Proposition 10. *If all conditions of Assumption* 8 *are satisfied, then the estimate*

$$V(\tau_k(x), x + J^k(x)) \le \psi^T C^* \psi \tag{24}$$

is true, where $\psi^T = (\psi_1(\tau_k(x), \|x_1\|), \ldots, \psi_N(\tau_k(x), \|x_N\|))$, *and the matrix* C^* *has the elements*

$$c_{ii}^* = \eta_i^2 \lambda_M(C_{ii}^*) + \mu_{ii},$$

$$c_{ip}^* = c_{pi}^* = \mu_{ip}, \quad i \ne p \in [1, s].$$

Proposition 11. *If estimate* (24) *is satisfied, then there exist functions* Q_1, Q_2: $R_+ \to R_+$ *such that* $Q_1(0) = Q_2(0) = 0$, $Q_1(r) > 0$ *and* $Q_2(r) > 0$ *for* $r > 0$, *and*

$$\begin{aligned} &\text{a)}\ V(\tau_k(x), x + J^k(x)) - V(\tau_k(x), x) \le \lambda_M(C^*) Q_1(V(\tau_k(x), x)), \\ &\text{b)}\ V(\tau_k(x), x + J^k(x)) - V(\tau_k(x), x) \le \lambda_M(C^*) Q_2(V(\tau_k(x), x)), \end{aligned} \tag{25}$$

for

a) $\lambda_M(C^*) < 0$,

b) $\lambda_M(C^*) > 0$,

respectively.

Remark. It is easy to see that estimates (21) together with inequalities (24) allow us to establish some conditions for the function $V(t, x, \eta)$ decreasing along solutions of (1) and, furthermore, to apply this function to the investigation of the behaviour of solutions of system (1).

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