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# On locally finite groups whose cyclic subgroups are GNA-subgroups

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Abstract. In this paper we obtain the description of locally finite groups whose cyclic subgroups are GNA-subgroups.

## Introduction

The investigation of influence of some systems of subgroups on the structure of the group is one of the classical problem in group theory. For example, normal subgroups and their natural generalizations have a very strong influence on the group structure. However, there are many others important types of subgroups that have a significant effect on the group structure. We have in mind some antipodes of (generalized) normal subgroups (for example, abnormal subgroups, self-normalizing subgroups, contranormal subgroups, malnormal subgroups and others), i.e. subgroups that have diametrically opposite properties with respect to the original subgroups. Recall that a subgroup $H$ of a group $G$ is called *abnormal* in $G$ if $g \in \langle H, H^g \rangle$ for each element $g \in G$. Recall also that a subgroup $H$ of a group $G$ is *self-normalizing* in $G$ if $N_G(H) = H$. Note that every abnormal subgroup of $G$ is self-normalizing in $G$.

On the other hand, there are subgroups that combine the concepts of normality and abnormality. One of the typical examples of such subgroups

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are pronormal subgroups. A subgroup $H$ of a group $G$ is said to be *pronormal* in $G$ if for each element $g \in G$ the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$, i.e. $H^g = H^u$ for some element $u \in \langle H, H^g \rangle$. Note the following very useful property of pronormal subgroups: if $H$ is a pronormal subgroup of $G$, then the normalizer $N_G(H)$ is an abnormal subgroup of $G$ (see, for example, [1]), and hence self-normalizing in $G$.

The first paper, devoted to the study of the influence of certain systems of subgroups on the structure of the group, is a classical article of R. Dedekind [6], in which he described the structure of finite groups whose all subgroups are normal. Later this result was extended to the case of infinite groups (see, for example, [2]). A group $G$ (not necessary finite) is called a *Dedekind group*, if its all subgroups are normal. By [2], if $G$ is a Dedekind group, then $G$ is either abelian or $G = Q \times D \times B$, where $Q$ is a quaternion group of order 8, $D$ is an elementary abelian 2-subgroup and $B$ is a periodic abelian $2'$-subgroup.

Let $\mathcal{P}$ and $\mathcal{AP}$ are subgroup properties. Moreover, suppose that all $\mathcal{AP}$-subgroups are antipodes (in some sense) to all $\mathcal{P}$-subgroups. There are many papers devoted to the study of the structure of groups whose subgroups are either $\mathcal{P}$-subgroups or $\mathcal{AP}$-subgroups (for example, normal or abnormal subgroups). In the present paper we consider the local (in some sense) version of this situation. Taking into account the above remarks on abnormal, self-normalizing and pronormal subgroups, we naturally obtain the following concept.

**Definition 1.** Let $G$ be a group. A subgroup $H$ of $G$ is said to be a GNA-*subgroup* (generalized normal and abnormal) of $G$ if for every element $g \in G$ either $H^g = H$ or $N_K(N_K(H)) = N_K(H)$, where $K = \langle H, g \rangle$.

In the paper [14], it has been obtained the description of locally finite groups whose all subgroups are GNA-subgroups.

Consider some relationships between the GNA-subgroups and some another types of subgroups. We note at once that from definition of GNA-subgroups we obtain that every pronormal subgroup is a GNA-subgroup. Moreover, example from [14] shows that this generalization is non-trivial. That is there are GNA-subgroups, which are not pronormal.

We recall that a subgroup $H$ of a group $G$ is called *self conjugate-permutable* in $G$ if $H$ satisfies the following condition: if $HH^g = H^gH$ then $H^g = H$ for all corresponding elements $g \in G$ [11]. It is well known that every pronormal subgroup is self conjugate-permutable [9]. Hence, very natural to consider the relationships between the classes of self conjugate-permutable subgroups and GNA-subgroups. Actually, if $H$ is a self conjugate-permutable subgroup of $G$, then $H$ is a GNA-subgroup of $G$. Indeed, suppose that a subgroup $H$ of $G$ is self conjugate-permutable in $G$. In fact, we can consider a group $G$ as a union $X \cup Y$, where $X$ is the set of elements $x \in G$ such that $HH^x = H^xH$, and $Y$ is the set of elements $y \in G$ such that $HH^y \neq H^yH$. In the first case we obtain that $H^x = H$, because $H$ is self conjugate-permutable in $G$. In other words, we obtain the first variant in the definition of GNA-subgroups. Let now $y$ is an element of $G$ such that $HH^y \neq H^yH$. Put $K = \langle H, y\rangle$ and consider

$$N_K(H) = N_{\langle H,y\rangle}(H) = \{t \in \langle H, y\rangle \mid H^t = H\}.$$

Clearly, $H \leqslant N_K(H)$. On the other hand, $N_K(H) \leqslant H$, because in another case this contradicts with the choice of element $y$. Therefore, $N_K(H) = H$. This means that for all elements $y \in G$ such that $HH^y \neq H^yH$ we have $N_K(N_K(H)) = N_K(H)$, where $K = \langle H, y\rangle$. This fact shows that a subgroup $H$ is a GNA-subgroup of $G$.

Note that the converse statement in general is not hold.

**Example 1.** Let $G = S_3 \times S_3$ (in SmallGroup library of GAP – SmallGroup(36,10)). Then $G$ contains 6 subgroups $H_i$ $(i = 1, \dots, 6)$ such that $H_i \cong S_3$ and $H_i$ is a GNA-subgroup, but it is not self conjugate-permutable in $G$, for every $i = 1, \dots, 6$.

In [9] authors obtained the description of locally finite groups whose cyclic subgroups are self conjugate-permutable. Since GNA-subgroups are non-trivial generalization of self conjugate-permutable subgroups, it is natural to consider the structure of locally finite groups whose cyclic subgroups are GNA-subgroups.

In this paper we obtain the description of such groups.

## 1. Preliminary results

**Lemma 1.** *Let $G$ be a group whose cyclic subgroups are* GNA-*subgroups.*

(i) *If $H$ is a subgroup of $G$, then every cyclic subgroup of $H$ is a* GNA-*subgroup.*

(ii) *If $H$ is a normal subgroup of $G$, then every cyclic subgroup of $G/H$ is a* GNA-*subgroup.*

*Proof.* It follows from the definition of GNA-subgroups. □

**Lemma 2.** *Let $G$ be a group and $H$ be an ascendant subgroup of $G$. If $H$ is a* GNA-*subgroup of $G$, then $H$ is normal in $G$.*

*Proof.* Let

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \ldots H_\alpha \trianglelefteq H_{\alpha+1} \trianglelefteq \ldots H_\gamma = G$$

be an ascending series between $H$ and $G$. We will prove that $H$ is normal in each $H_\alpha$ for all $\alpha \leqslant \gamma$. We will use a transfinite induction.

Let $\alpha = 1$. Then we have $H = H_0 \trianglelefteq H_1 = G$, which implies that $H \trianglelefteq G$. Assume now that $H$ is normal in $H_\beta$ for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, then $H_\alpha = \bigcup\limits_{\beta<\alpha} H_\beta$. Let $x$ be an arbitrary element of $H_\alpha$. Then $x \in H_\beta$ for some $\beta < \alpha$. By induction hypothesis, $H$ is normal in $H_\beta$. This means that $H^x = H$ for each $x \in H_\alpha$, which implies that $H \trianglelefteq H_\alpha$.

Suppose now that $\alpha$ is not a limit ordinal. Let $x \in H_\alpha$. Put $K = \langle H, x\rangle$. By induction hypothesis, $H$ is a (proper) normal subgroup of $H_{\alpha-1}$. Thus, we have two possibilities: either $H \leqslant K \leqslant H_{\alpha-1}$ or $H < H_{\alpha-1} \leqslant K$. In the first case $H \trianglelefteq K$. Therefore, $N_K(H) = K \neq H$, which implies that $N_K(N_K(H)) \neq N_K(H)$. In the second case we have

$$N_K(H) = \{k \in K \mid H^k = H\} \geqslant H_{\alpha-1} > H,$$

and we again obtain that $N_K(N_K(H)) \neq N_K(H)$. Thus, in both cases we have equality $H^x = H$ for every $x \in H_\alpha$, which implies that $H \trianglelefteq H_\alpha$. For $\alpha = \gamma$ we obtain that $H$ is normal in $H_\gamma = G$. □

**Corollary 1.** *Let $G$ be a group and $H$ be a subnormal subgroup of $G$. If $H$ is a* GNA*-subgroup, then $H$ is normal in $G$.*

**Corollary 2.** *Let $G$ be a nilpotent group. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then every subgroup of $G$ is normal in $G$.*

*Proof.* It follows from the fact that every subgroup of a nilpotent group is subnormal. □

**Corollary 3.** *Let $G$ be a nilpotent group. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $G$ is a Dedekind group.*

*Proof.* It follows from Corollary 2 and the definition of Dedekind groups. □

**Corollary 4.** *Let $G$ be a locally nilpotent group. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $G$ is a Dedekind group.*

*Proof.* Let $x, y$ are arbitrary elements of $G$. Put $K = \langle x, y\rangle$. Then $K$ is a nilpotent subgroup of $G$. Since $\langle x\rangle$ is a GNA-subgroup of $K$, by Corollary 2, $\langle x\rangle$ is normal in $K$. Therefore, $\langle x\rangle^y = \langle x\rangle$. This is valid for every element $y \in G$, which implies that $\langle x\rangle$ is normal in $G$. Since every cyclic subgroup of $G$ is normal in $G$, then every subgroup of $G$ is normal in $G$. Thus, $G$ is a Dedekind group. □

If $G$ is a group then we let $\Pi(G)$ denote the set of prime divisors of the orders of the elements of $G$.

**Lemma 3.** *Let $G$ be a group and $K$ be a finite subgroup of $G$. Suppose that every cyclic subgroup of $G$ is a* GNA*-subgroup. Let $p$ be the least prime of $\Pi(K)$. Then $K = R \leftthreetimes P$ where $P$ (respectively $R$) is a Sylow $p$-subgroup (respectively $p'$-subgroup) of $K$.*

*Proof.* Let $P$ be a Sylow $p$-subgroup of $K$. By Corollary 3, $P$ is a Dedekind group. Put $T = N_K(P)$. Then every cyclic subgroup of $P$ is subnormal in $T$, and by Corollary 1, it is normal in $T$. It follows that $P$ has a $T$-chief series whose factors have order $p$. Let $U, V$ are $T$-invariant subgroups of $P$ such that $U \leqslant V$ and $V/U$ is a $T$-chief factor. By the proven above, $|V/U| = p$. Then $|T/C_T(V/U)|$ divides $p-1$, and the choice of $p$ implies that $T = C_T(V/U)$. In other words, every $T$-chief factor of $P$ is central in $T$. Hence, $P$ has a $T$-central series. It follows that $T = P \times S$ where $S$ is a Sylow $p'$-subgroup of $T$. Suppose first that $p = 2$. Using now [3, Theorem 1] we obtain a following semidirect decomposition $K = R \leftthreetimes P$ where $R$ is a Sylow $2'$-subgroup of $K$. If $p \neq 2$ then the description of Dedekind groups shows, that $P$ is abelian. Using now a Burnside's theorem (see, for example, [15, Theorem 10.21]), we obtain that $K = R \leftthreetimes P$ where $R$ is a Sylow $p'$-subgroup of $K$. □

**Corollary 5.** *Let $G$ be a group, $K$ be a finite subgroup of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $K$ is soluble.*

*Proof.* Let $D$ be a Sylow 2-subgroup of $K$. By Lemma 3, we have $K = R \leftthreetimes D$, where $R$ is a Sylow $2'$-subgroup of $K$. The subgroup $R$ is soluble [8], therefore $K$ is also soluble. □

We recall that a finite group $G$ has a *Sylow tower* when every non-trivial homomorphic image of $G$ has a non-trivial normal Sylow subgroup, that is, $G$ has a normal series

$$\langle 1\rangle = P_0 \leqslant P_1 \leqslant \ldots \leqslant P_n = G$$

such that each $P_{i+1}/P_i$, $0 \leqslant i \leqslant n-1$, is isomorphic to a Sylow $p$-subgroup of $G$, where $p \in \Pi(G)$.

**Lemma 4.** *Let $G$ be a group, $K$ be a finite subgroup of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $K$ has a Sylow tower.*

*Proof.* Let $\Pi(K) = \{p_1, p_2, \dots, p_k\}$ and suppose that $p_1 < p_2 < \dots < p_k$. We will prove this assertion using induction in $|\Pi(K)| = k$. If $k = 1$, then $K$ is a $p_1$-subgroup, and all is proved. Assume now that $k > 1$. By Lemma 3, $K = R \leftthreetimes P$, where $P$ is a Sylow $p_1$-subgroup of $K$ and $R$ is a normal Sylow $p_1'$-subgroup of $K$. We have now $\Pi(R) = \{p_2, \dots, p_k\}$, and therefore by the induction hypothesis $R$ has a Sylow tower. Since $R$ is normal in $K$, every term of this Sylow tower is $K$-invariant, which proves the assertion. □

**Corollary 6.** *Let $G$ be a group, $K$ be a finite subgroup of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $K$ is supersoluble.*

*Proof.* Let $\Pi(K) = \{p_1, p_2, \dots, p_k\}$ and suppose that $p_1 < p_2 < \dots < p_k$. By Lemma 4, $K$ has a series of normal subgroups

$$K = S_0 > S_1 > \dots > S_{k-1} > S_k = \langle 1 \rangle$$

such that $S_{j-1} = S_j \leftthreetimes P_j$ where $P_j$ is a Sylow $p_j$-subgroup of $K$ and $S_j$ is a normal Sylow $p_j'$-subgroup of $K$. We will prove this assertion using induction in $|\Pi(K)| = k$. If $k = 1$, then $K$ is a $p_1$-subgroup and all is proved. Assume now that $k > 1$. The subgroup $S_{k-1}$ is the normal Sylow $p_k$-subgroup of $K$. Then every its cyclic subgroup is subnormal in $K$, and by Corollary 1, every cyclic subgroup of $S_{k-1}$ is normal in $K$. We have now $\Pi(K/S_{k-1}) = \{p_1, p_2, \dots, p_{k-1}\}$, and therefore by the induction hypothesis $K/S_{k-1}$ is supersoluble, which proves the result. □

**Corollary 7.** *Let $G$ be a group, $K$ be a locally finite subgroup of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $K$ is locally supersoluble.*

**Corollary 8.** *Let $G$ be a locally finite group. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then any Sylow $2'$-subgroup of $G$ is normal.*

*Proof.* Let $\mathfrak{L}$ be a local family of $G$ consisting of finite subgroups. If $L \in \mathfrak{L}$, then by Lemma 3, a Sylow $2'$-subgroup of $L$ is normal in $L$. Since it is valid for each $L \in \mathfrak{L}$, then the Sylow $2'$-subgroup of $G$ is normal in $G$. □

Let $G$ be a group and $\mathfrak{R}^{\mathfrak{LN}}$ be a family of all normal subgroups $H$ of $G$ such that $G/H$ is locally nilpotent. Then the intersection

$$\bigcap \mathfrak{R}^{\mathfrak{LN}} = R^{\mathfrak{LN}}$$

is called the *locally nilpotent residual* of $G$. It is not difficult to prove that if $G$ is locally finite, then $G/R^{\mathfrak{LN}}$ is locally nilpotent.

**Corollary 9.** *Let $G$ be a locally finite group and $L$ be a locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA-*subgroup, then* $2 \notin \Pi(L)$.

*Proof.* Let $D$ be a Sylow $2'$-subgroup of $G$. By Corollary 8, $D$ is normal in $G$. The factor-group $G/D$ is a locally finite 2-group, in particular, it is locally nilpotent. It follows that $L \leqslant D$. □

**Lemma 5.** *Let $G$ be a locally finite group. If every cyclic subgroup of $G$ is a* GNA-*subgroup, then the derived subgroup $[G, G]$ is locally nilpotent.*

*Proof.* Indeed, by Corollary 7, $G$ is locally supersoluble. In particular, $G$ is locally soluble and therefore $G$ has a chief series $\mathfrak{S}$ whose factors are abelian (see, e.g., [10, §58]). The fact that $G$ is locally supersoluble implies that all factors of the chief series $\mathfrak{S}$ have prime orders [12, Lemma 7]. Let

$$\mathfrak{H} = \{H \in \mathfrak{S} \mid \exists\ H^{\nabla} \in \mathfrak{S} \text{ such that } H/H^{\nabla} \text{ is a } G\text{-chief factor}\}.$$

Since the factor $H/H^{\nabla}$ is cyclic, $G/C_G(H/H^{\nabla})$ is abelian for every subgroup $H \in \mathfrak{H}$. Let

$$K = \bigcap_{H \in \mathfrak{H}} C_G(H/H^{\nabla}).$$

By Remak's theorem we obtain an imbedding

$$G/K \hookrightarrow \mathbf{Cr}_{H \in \mathfrak{H}} G/C_G(H/H^{\nabla}),$$

which shows that $G/K$ is abelian. It follows that $[G, G] \leqslant K$. This inclusion shows that each factor $H/H^{\nabla}$ is central in $[G, G]$ for each subgroup $H \in \mathfrak{H}$, $H \leqslant [G, G]$. Thus $[G, G]$ has a central series. Being locally finite, $[G, G]$ is locally nilpotent. □

**Corollary 10.** *Let $G$ be a locally finite group and $L$ be a locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA-*subgroup, then $L$ is locally nilpotent.*

**Corollary 11.** *Let $G$ be a locally finite group and $L$ be a locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $L$ is abelian and every subgroup of $L$ is $G$-invariant.*

*Proof.* Indeed, by Corollary 9, $L$ is a $2'$-subgroup. Using Corollaries 3, 4 and 10 we obtain that $L$ is abelian. Finally, Corollary 1 proves that every subgroup of $L$ is normal in $G$. □

**Corollary 12.** *Let $G$ be a locally finite group and $L$ be a locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $G/C_G(L)$ is abelian.*

*Proof.* By Corollary 11, every subgroup of $L$ is $G$-invariant. It follows that $G/C_G(L)$ is abelian (see, for example [16, Theorem 1.5.1]). □

**Lemma 6.** *Let $G$ be a locally finite group and $L$ be the locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $G/L$ is a Dedekind group.*

*Proof.* Let $p \in \Pi(G/L)$ and let $P/L$ be a Sylow $p$-subgroup of $G/L$. Choose two arbitrary elements $xL, yL \in P/L$. Since $G$ is locally finite, $G/L$ is locally nilpotent, so that $\langle xL, yL\rangle = K/L$ is a finite $p$-subgroup. Then there exists a finite subgroup $F$ such that $K = FL$. Choose a Sylow $p$-subgroup $V$ of $F$. Then $V(F \cap L)/(F \cap L)$ is a Sylow $p$-subgroup of $F/(F \cap L)$. Since $F/(F \cap L) \cong FL/L$ is a $p$-group, $V(F \cap L) = F$, and $VL = K$. By Corollary 3, $V$ is a Dedekind group. It follows that $VL/L = K/L$ is also a Dedekind group. In turn, it follows that $\langle xL\rangle^{yL} = \langle xL\rangle$. Since this is true for each element $yL \in P/L$, $\langle xL\rangle$ is normal in $P/L$. Hence $P/L$ is a Dedekind group. □

**Lemma 7.** *Let $G$ be a locally finite group, $p$ be an odd prime and $P$ be a $p$-subgroup of $G$. Suppose that $N_G(P)$ contains a $p'$-element $x$ such that $[P, x] \neq \langle 1\rangle$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then every subgroup of $P$ is $\langle x\rangle$-invariant, $P = [P, x]$, and $C_P(x) = \langle 1\rangle$.*

*Proof.* By Corollaries 3 and 4, $P$ is abelian. By [4, Proposition 2.12], $P = [P, x] \times C_P(x)$. Suppose that $C_P(x) \neq \langle 1\rangle$, and choose in $C_P(x)$ an element $c$ of order $p$. Let now $a$ be an element of $[P, x]$ having order $p$. Lemma 2 shows that every subgroup of $P$ is $\langle x\rangle$-invariant. Since $a \notin C_P(x)$, $a^x = a^d$, where $d$ is a $p'$-number. Moreover, $d$ is not congruent to $1 (\mathrm{mod}\ p)$. We have

$$(ac)^x = a^x c^x = a^x c.$$

On the other hand, since $ac \notin C_P(x)$,

$$(ac)^x = (ac)^t$$

where $t$ is also $p'$-number such that $t$ is not congruent with $1 (\mathrm{mod}\ p)$. Hence

$$a^d c = (ac)^t = a^t c^t,$$

and therefore $d \equiv t (\mathrm{mod}\ p)$ and $t \equiv 1 (\mathrm{mod}\ p)$. This contradiction proves that $C_P(x) = \langle 1 \rangle$, and hence $[P, x] = P$. □

**Lemma 8.** *Let $G$ be a locally finite group and $L$ be a locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then $\Pi(L) \cap \Pi(G/L) = \varnothing$.*

*Proof.* Suppose the contrary. Let there exists a prime $p$ such that $p \in \Pi(L) \cap \Pi(G/L)$. The inclusion $p \in \Pi(L)$ together with Corollary 9 shows that $p \neq 2$. Let $P$ be a Sylow $p$-subgroup of $L$ and $K = N_G(P)$. Suppose that $K = G$. The subgroup $L$ is abelian by Corollary 11, so that $L = P \times Q$ where $Q$ is a Sylow $p'$-subgroup of $L$. By Lemma 6, $G/L$ is a Dedekind group, in particular, it is nilpotent. In the factor-group $G/Q$ we have $L/Q = PQ/Q \leqslant \zeta(G/Q)$. It follows that $G/Q$ is nilpotent, that contradicts the choice of $L$. This contradiction shows that $K \neq G$. Since $p \neq 2$, Corollary 4 shows that Sylow $p$-subgroups of $G$ are abelian. Then from $K \neq G$ we obtain, that $K$ contains some $p'$-element $x$. Without loss of generality we can suppose that $x$ is an $r$-element for some prime $r$. Since $G/L$ is a Dedekind group and $p \in \Pi(G/L)$, $G/L$ contains a $p$-element $vL \in \zeta(G/L)$. It follows that $[v, x] \in L$. Without loss of generality we can suppose that $v$ is a $p$-element. Put $H = \langle x, v, P \rangle = P\langle x, v \rangle$. Then the index $|H : P|$ is finite, so that the Sylow $\{p, r\}'$-subgroup $R$ of $H$ is finite. The isomorphism $H/(H \cap L) \cong HL/L = \langle x, v \rangle L/L = \langle xL, vL \rangle$ implies that $\Pi(H/(H \cap L)) = \{p, r\}$, which implies that $R \leqslant H \cap L$. Corollary 11 shows that $R$ is $G$-invariant, in particular, $R$ is normal in $H$. The finiteness of $R$ implies that $H/C_H(R)$ is finite, therefore $H = R \leftthreetimes V$, where $V$ is a Sylow $\{p, r\}$-subgroup of $H$ [7, Theorem 2.4.5]. Clearly, $V \cap L = P$, and $V/P = V/(V \cap L) \cong VL/L = \langle xL \rangle \times \langle vL \rangle$ is an abelian subgroup of order $rp$. Therefore, without loss of generality, we can suppose that $P_1 = \langle P, v \rangle$ is a normal Sylow $p$-subgroup of $V$ and $V = P_1 \leftthreetimes \langle x \rangle$. Moreover, $[V, x] \leqslant P$. On the other hand, the choice of $x$ implies that $[V, x] \neq \langle 1 \rangle$. Then by Lemma 7, $[V, x] = V$, and we obtain a contradiction. This contradiction proves that $P$ is a Sylow $p$-subgroup of the entire group $G$. □

**Corollary 13.** *Let $G$ be a locally finite group and $L$ be the locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then every subgroup of $C_G(L)$ is $G$-invariant.*

*Proof.* By Lemma 8 $L$ is the Sylow $\pi$-subgroup of $G$, where $\pi = \Pi(L)$. By [5, Theorem 7] $C_G(L) = L \times V$, where $V = \mathbf{O}_{\pi'}(C_G(L))$. By Corollary 11, every subgroup of $L$ is $G$-invariant. Therefore, it is enough to prove that every subgroup of $V$ is $G$-invariant. Let $U$ be an arbitrary subgroup of $V$. Since $V$ is $G$-invariant, $[U, G] \leqslant V$. On the other hand, by Lemma 6, $G/L$ is a Dedekind group, so that $[U, G] \leqslant UL$. Thus we have

$$[U, G] \leqslant V \cap UL = U(V \cap L) = U. \qquad \square$$

## 2. Proof of main result

**Theorem 1.** *Let $G$ be a locally finite group and $L$ be a locally nilpotent residual of $G$. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then the following conditions hold:*

(i) *$L$ is abelian;*

(ii) *$2 \notin \Pi(L)$ and $\Pi(L) \cap \Pi(G/L) = \varnothing$;*

(iii) *$G/L$ is a Dedekind group;*

(iv) *every subgroup of $C_G(L)$ is $G$-invariant.*

*Conversely, if a group $G$ satisfies conditions (i)-(iv), every cyclic subgroup of $G$ is a* GNA*-subgroup.*

*Proof.* Condition (i) follows from Corollary 11. Condition (ii) follows from Corollary 9 and Lemma 8. Condition (iii) follows from Lemma 6. Condition (iv) follows from Corollary 13.

Conversely, suppose that a group $G$ satisfies conditions (i)-(iv). Let $H$ be an arbitrary cyclic (and hence finite cyclic) subgroup of $G$. Put $C = C_G(L)$. By condition (iv) the intersection $H \cap C$ is $G$-invariant. In particular, if $H \leqslant C$, then $H$ is normal in $G$. In particular, $H$ is a GNA-subgroup of $G$. Suppose now that $H \neq H \cap C$. Clearly, it is enough to prove now that $H/(H \cap C)$ is a GNA-subgroup of $G/(H \cap C)$. This shows that without loss of generality we can suppose that $H \cap C = \langle 1 \rangle$. By condition (i) and (iv), every subgroup of $L$ is $G$-invariant. It follows that $G/C_G(L) = G/C$ is abelian (see, for example [16, Theorem 1.5.1]). Then $H \cong H/(H \cap C) \cong HC/C$ is abelian. From condition (ii) we obtain that $\Pi(L) \cap \Pi(H) = \varnothing$.

Let $x$ be an arbitrary element of $G$. Put $K = \langle H, x \rangle$ and $\pi = \Pi(H)$. Let $x \in L$. Then $K = \langle x \rangle^H \leftthreetimes H$. This means that $H$ is a Sylow $\pi$-subgroup of $K$. Moreover, $H$ is not normal in $K$, which implies that $H$ is pronormal in $K$. This fact shows that $N_K(H)$ is abnormal in $K$. Thus, $N_K(N_K(H)) = N_K(H)$.

Let $x \notin L$. Put $L_1 = K \cap L$ and $\pi = \Pi(H)$. By condition (iii) $L_1H$ is normal in $K$. The equation $\Pi(L) \cap \Pi(H) = \varnothing$ implies that $H$ is a Sylow $\pi$-subgroup of $L_1H$. This means that $H$ is pronormal in $L_1H$. Since $K = L_1HN_K(H) = L_1N_K(H)$, $H$ is pronormal in $K$, and we again obtain that $N_K(N_K(H)) = N_K(H)$.

Hence in any case, $H$ is a GNA-subgroup of $G$. $\square$

**Corollary 14.** *Let $G$ be a locally finite group. If every cyclic subgroup of $G$ is a* GNA*-subgroup, then the derived subgroup of $G$ is abelian.*

*Proof.* Let $L$ be the locally nilpotent residual of $G$. Since $L \leqslant [G, G]$, $G/[G, G] \cong (G/L)/([G, G]/L)$. Since $G/L$ is locally nilpotent,

$$G/L = \mathbf{Dr}_{p\in\Pi(G/L)} S_p/L$$

where $S_p/L$ is a Sylow $p$-subgroup of $G/L$. Then

$$[G, G]/L = [G/L, G/L] = \mathbf{Dr}_{p\in\Pi(G/L)}[S_p/L, S_p/L].$$

Put $D_p/L = [S_p/L, S_p/L]$. By Theorem 1, $G/L$ is a Dedekind group. It follows that $D_p/L$ is abelian for each $p \in \Pi(G/L)$. By Corollary 12, $[G, G] \leqslant C_G(L)$, in particular, $[G, G]$ is nilpotent. Let $p \in \Pi([G, G])\setminus\Pi(L)$. Choose in $[G, G]$ a Sylow $p$-subgroup $P$. By Lemma 8, $P \cap L = \langle 1\rangle$, thus $P \cong P/(P \cap L) \cong PL/L \cong D_p/L$. Therefore, $P$ is abelian. Since by Corollary 11, $L$ is abelian, $[G, G]$ is abelian too. $\square$

Note that Theorem 1 can not be generalized to the case of arbitrary periodic groups. The following example shows this.

**Example 2.** Let $G$ be a Tarski monster group, i.e. $G$ is an infinite group such that every proper non-identity subgroup is cyclic of prime order $p$ for some fixed $p$. Note that A.Yu. Ol'shanskii proved the existence of such groups for any prime number $p > 10^{75}$ [13, §28]. Every element of $G$ generates cyclic subgroup of prime order $p$ and every two non-commuting elements generate the whole group $G$. If $y \in \langle x\rangle$ then $\langle x\rangle^y = \langle x\rangle$. If $y \notin \langle x\rangle$ then $\langle x, y\rangle = G$ and $\langle x\rangle = N_G(\langle x\rangle)$, which implies $N_G(\langle x\rangle) = N_G(N_G(\langle x\rangle))$. Therefore, $G$ is a periodic group whose cyclic subgroups are GNA-subgroups. But $G$ is a simple group.

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