Algebra and Discrete Mathematics
Number 4. **(2004).** pp. 79 – 105

RESEARCH ARTICLE

On simple groups of large exponents

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Communicated by A. Yu. Olshanskii

Abstract. It is shown that the set of pairwise non-isomorphic 2-generated simple groups of exponent n ($n \geq 2^{48}$ and n is odd or divisible by 2^9) is of cardinality continuum. As a corollary, for any sufficiently large n the set of pairwise non-isomorphic 2-generated groups of exponent n is of cardinality continuum.

Introduction

B.H. Neumann [9] showed that the set of all pairwise non-isomorphic 2-generated groups is of cardinality continuum. It is known that so is the set of pairwise non-isomorphic 2-generated simple groups. Moreover, for any sufficiently large prime number p the set of pairwise non-isomorphic 2-generated simple groups satisfying the identity $x^p = 1$ is of cardinality continuum (for a detailed discussion see [7] and [2]).

In this paper, we prove that for almost all values of n the same is true about the set of pairwise non-isomorphic 2-generated groups of exponent n.

The present investigation is heavily dependent on the technique exposed in S. V. Ivanov's paper [3], the main result of which is that the free m-generated Burnside group $\mathbf{B}(m,\ n)$ of even exponent n is infinite provided $n \geq 2^{48}$, n is either odd or divisible by 2^9 and $m > 1$ (see also [5], and [1], [6], [7] for the earlier results dealing with odd exponents).

This work was supported in part by the NSF grant 0072307 of A.Yu. Ol'shanskii and M.V. Sapir.

2000 Mathematics Subject Classification: *20F50; 20F05, 20F06.*

Key words and phrases: *Burnside group, van Kampen diagram, graded diagram, graded presentation, non-isomorphic 2-generated torsion simple groups.*

To achieve our goal we introduce a family of torsion groups $\{G_{\mathcal{T}}\}$. We shall start with the absolutely free group F of rank 2. First we shall impose on F some infinite set of relations, left-hand-sides of which are taken from a certain infinite set $\mathcal{T}$ of words in a 2-letter alphabet satisfying strong small cancellation and aperiodicity conditions (see Lemma 0.1). Then, one-by-one we will impose the relations that will guarantee that the presentation we have obtained is a presentation of a member of the Burnside variety of exponent n, where $n \geq 2^{48}$ and n is odd or divisible by 2^9. Constructing the presentations of groups $G_{\mathcal{T}}$ in Section 2 below we constantly refer the reader to [3].

In Theorem A below we collect some properties of groups $G_{\mathcal{T}}$. Note, in particular, that finite subgroups of $G_{\mathcal{T}}$ behave in the same way as those of a free Burnside group of the corresponding exponent (see [3], Theorem A(c)).

Theorem A. *For every set $\mathcal{T}$ (finite or infinite) satisfying conditions (a), (b) and (c) of Lemma 0.1 the group $G_{\mathcal{T}}$ has the following properties:*

(1) $G_{\mathcal{T}}$ is a 2-generated infinite group belonging to the Burnside variety of exponent n.

(2) Let $n = n_1 n_2$, where n_1 is the maximal odd divisor of n. Then every finite subgroup of $G_{\mathcal{T}}$ is isomorphic to a subgroup of $\mathbb{D}(2n_1) \times \mathbb{D}(2n_2)^k$, for some k, where $\mathbb{D}(2m)$ is a dihedral group of order $2m$.

(3) The center of $G_{\mathcal{T}}$ is trivial.

Furthermore, the set of pairwise non-isomorphic groups among $\{G_{\mathcal{T}'}\}$, $\mathcal{T}' \subseteq \mathcal{T}$, is of cardinality continuum provided $\mathcal{T}$ is infinite. If $\mathcal{T}'$ is a recursive subset of $\mathcal{T}$, then the word and conjugacy problems are solvable in $G_{\mathcal{T}'}$.

In Section 3 we pick special sets $\mathcal{T}_\alpha$ and show that the set of pairwise non-isomorphic simple quotients of the groups $G_{\mathcal{T}_\alpha}$ is of cardinality continuum.

Theorem B. *For any sufficiently large exponent n the set of pairwise non-isomorphic 2-generated simple groups satisfying the identity $x^n = 1$ is of cardinality continuum.*

In the proof of this theorem given in Section 3 we assume that $n \geq 2^{48}$ and n is either odd or divisible by 2^9. The statement for any multiple of such n clearly follows.

1. $\mathcal{T}$-relators

We assume that the reader is familiar with the terminology from [7]. For the numerical values of auxiliary parameters (α, β, etc.) we refer to [3].

Certain sets of aperiodic words satisfying small cancellation conditions were constructed in [8] and [10]. We refer to [10] for the proof of the following

Lemma 0.1. *For given n and $\xi > 0$ there exists an infinite set of positive words $\mathcal{T} = \{B_1, B_2, \dots\}$ in the alphabet $\{a_1,\ a_2\}$, satisfying the following properties:*

(a) Suppose a cyclic shift of some word $B_i^{\pm 1}$ contains a B-periodic subword U of length greater than $(1+\xi)|B|$. Then $|B| < \xi^{-3/2}$ and $|U| < 11|B| < 11\xi^{-3/2}$.

(b) The symmetrized set obtained from the set $\mathcal{T}$ satisfies the small cancellation condition $C'(\frac{\xi}{10})$.

(c) $|B_i| \geq n^2, i = 1, 2, \dots$.

According to the choice of the auxiliary parameters ($\beta n^2 > 11\xi^{-3/2}$, $\beta > \xi$) the next lemma immediately follows from Lemma 0.1.

Lemma 0.2. *Let U be a B-periodic subword of a cyclic shift of some word $B_i^{\pm 1}$ of length $|U| > \beta|B_i|$. Then $|U| < (1+\xi)|B|$ and $|B_i| < (1+\xi)\beta^{-1}|B|$.*

Define $G(0) = F(a_1,\ a_2)$ to be absolutely free group and set

$$G(1/2) = \langle a_1,\ a_2 \mid B = 1,\ B \in \mathcal{T} \rangle \ . \tag{1}$$

Arguing as for $C'(1/8)$-groups (see [4], Chapter V, Theorem 10.1), in view of Lemma 0.1 we obtain the following

Lemma 0.3. *The group $G(1/2)$ is torsion free.*

2. Inductive construction of group $G_{\mathcal{T}}$

For every i, $i = 0,\ 1/2,\ 1,\ 2,\ \dots$, we shall define the group $G(i)$ of rank i. The groups $G(0)$ and $G(1/2)$ are already defined. Following [3], the elements of $F(a_1,\ a_2)$ and of its quotients are referred to as words (in the alphabet $\{a_1^{\pm 1},\ a_2^{\pm 1}\}$). Let '$\prec$' be a total order on the set of words over $\{a_1^{\pm 1},\ a_2^{\pm 1}\}$, such that $|X| < |Y|$ implies $|X| \prec |Y|$, where $|X|$ is the length of the word X.

Dealing with ranks, we agree that $i-1$ is equal to $1/2$ (resp. 0) if $i = 1$ (resp. $(i = 1/2)$), and $i+1$ is equal to 1 (resp. $1/2$) provided $i = 1/2$ (resp. $i = 0$).

Assuming that the group $G(i-1)$ for $i \geq 1$ is defined, by the period A_i of rank i we mean the first (relative to the imposed order) of the words that have infinite order in $G(i-1)$. Then the group $G(i)$ ($i \geq 1$) is defined by imposing the relation $A_i^n = 1$ on $G(i-1)$:

$$G(i) = \langle a_1,\ a_2 \mid \{B = 1,\ B \in \mathcal{T}\} \cup \{A_1^n = 1,\ \ldots,\ A_i^n = 1\}\rangle\ . \qquad (2)$$

A planar diagram over the presentation (1) (resp. (2)) is called a diagram of rank $1/2$ (resp. i). A cell Π of a diagram Δ of rank i has rank $1/2$ provided the label of its contour is a cyclic shift of $B^{\pm 1}$ for some $B \in \mathcal{T}$. Following [7] and [8] any such cell is referred to as $1/2$-cell or a $\mathcal{T}$-cell while cells of rank j, $j \geq 1$, are called $\mathcal{R}$-cells. A section q of a boundary of Δ is called $\mathcal{T}$-section if $\phi(q)$ is freely equal to a subword of a cyclic shift of $B^{\pm 1}$ for some $B \in \mathcal{T}$. For the definitions of a cell of rank $j \geq 1$ and the strict rank of a diagram we refer to [3]. Similarly to [3] we define the type $\tau(\Delta)$ of a diagram Δ of rank i to be the sequence $(n_i, \ldots, n_{1/2})$, where n_j is the number of cells of rank j in Δ.

As in [3] we define the concepts of (weak) j-compatibility, j-pair, j-cell for $j \geq 1$. The definition of compatibility of two $1/2$-cells (or $1/2$-cell and a $\mathcal{T}$-section) is analogous to the definition of compatibility of two T-cells (resp. T-cell and an H-section) given in Section 4.2 of [8]. We only require that $\phi(t) = 1$ in the free group and the label of the path $q_1 t^{-1} p t q_2$ is freely equal to (a subword of) a cyclic shift of $B^{\pm 1}$ for some $B \in \mathcal{T}$. Note that a pair of two compatible $\mathcal{T}$-cells (following [3] and [8], a pair of such cells is called a $1/2$-pair) can be substituted by a diagram without cells.

The definition of a reduced diagram of rank i is preserved. Defining a strictly reduced diagram of rank i (disk or non-disk) we require in addition the absence of $1/2$-pairs. The definitions of j-bonds and j-contiguity subdiagrams read the same as in [3]. Notice that now j takes on values $0,\ 1/2,\ 1,\ 2,\ \ldots$. The definitions of a (cyclically) reduced in rank i and a simple in rank i word are preserved. In the definition of a smooth section we add one more possibility:

(C3) A $\mathcal{T}$-section s of a contour of a diagram Δ of rank i is called smooth provided there are no $\mathcal{T}$-cells in Δ which are compatible with s.

Such section s is called smooth of rank $1/2$. Strictly smooth sections of rank j are defined for $j \geq 1$ only. The definition of a tame diagram need not be changed. The definitions of the subgroups $\mathcal{F}(A_j)$, $\mathcal{G}(A_j)$, $\mathcal{K}(A_j)$ associated with the periods A_j and the definition of $\mathcal{F}(A_j)$-involutions read the same as in [3].

The formulations and proofs of Lemmas 1.1 - 1.7 from [3] are preserved. Lemmas 2.1 - 2.5 together with Lemmas 3.1 - 20.3 are proved by simultaneous induction on the parameter i, $i = 0,\ 1/2,\ 1,\ 2, \ldots$. Fixing i and assuming that Lemmas 2.1 - 20.3 hold in rank $i-1$, we induct on the type $\tau(\Delta)$ of a diagram Δ of rank i.

In Lemmas 2.1 - 2.5 we establish some properties of bonds and contiguity subdiagrams in reduced diagrams of rank i.

Lemma 2.1. *Let Γ be a contiguity subdiagram of a cell π to a $\mathcal{T}$-cell Π in a reduced diagram Δ of rank i. Then $r(\Gamma) = 0$ and the contiguity degree of π to Π is less than β.*

Proof. We prove this lemma by contradiction. Assume that the triple $(\pi,\ \Gamma,\ \Pi)$ is a counterexample with contiguity subdiagram Γ of minimal type. Let the standard contour of Γ be $\partial\Gamma = d_1pd_2q$, where $p = \Gamma \wedge \Pi$, $q = \Gamma \wedge \pi$. The bonds defining Γ are 0-bonds (otherwise $(\pi,\ \Gamma,\ \Pi)$ is not minimal). That means that $|d_1| = |d_2| = 0$. Assuming that Γ has cells, by Lemma 5.7 ($\tau(\Gamma) < \tau(\Delta)$) there is a θ-cell. However, by Lemma 2.2 applied to Γ ($\tau(\Gamma) < \tau(\Delta)$), the degree of contiguity of any cell from Γ to p is less than β and the contiguity degree of any cell from Γ to q is less than α by Lemmas 2.5 and 3.4 (again, $\tau(\Gamma) < \tau(\Delta)$). The fact that $\theta > \alpha + \beta$ implies that Γ does not have cells. Suppose that the contiguity degree of π to Π is greater than or equal to β. Lemma 0.1(b) and the fact that Δ is reduced mean that π can not be a $\mathcal{T}$-cell. If π is an $\mathcal{R}$-cell, then we obtain a contradiction with Lemma 0.1(a) since $\beta n^2 > 11$. □

Lemma 2.2. *Let Γ be a contiguity subdiagram of a cell π to a $\mathcal{T}$-section q of the boundary of a reduced diagram Δ of rank i. Assume there are no $\mathcal{T}$-cells compatible with q in Δ. Then $r(\Gamma) = 0$ and the contiguity degree of any cell from Δ to q is less than β (q is a β-section of $\partial\Delta$ in the terminology of [8]).*

Proof. The proof is similar to the proof of Lemma 2.1. □

Lemma 2.3. *Let Γ be a contiguity subdiagram of a $\mathcal{T}$-cell π to a geodesic section q in a reduced diagram Δ of rank i. Then*

(1) $r(\Gamma) = 0$;

(2) The contiguity degree of π to q does not exceed $1/2 < \alpha$.

Proof. **1.** Denote the standard contour of Γ by $\partial\Gamma = d_1 p d_2 q_1$, where $p = \Gamma \wedge \pi$, $q_1 = \Gamma \wedge q$. By Lemma 2.1 the bonds defining Γ are 0-bonds, and therefore $|d_1| = |d_2| = 0$. Suppose $r(\Gamma) \neq 0$. Then Lemma 5.7 may be applied to Γ since $\tau(\Gamma) < \tau(\Delta)$, which guarantees that there is a cell $\Pi \in \Gamma$, such that the sum of the contiguity degrees of Π to p and to q_1 is greater than θ. Note that there are no $\mathcal{T}$-cells compatible with the section q_1 in Γ, since otherwise Δ would not be reduced. Thus, by Lemma 2.2 the degree of contiguity of Π to p is less than β. Lemmas 2.3 and 3.3 applied to Γ ($\tau(\Gamma) < \tau(\Delta)$) mean that the degree of contiguity of Π to q is less than α. Finally, the inequality $\theta > \alpha + \beta$ provides a contradiction to the assumption that Γ has cells.

2. If the statement of part (2) was not true, by proven part (1) the section q would not be geodesic in Δ. □

Lemma 2.4. *Let Γ be a contiguity subdiagram of a $\mathcal{T}$-cell π to an $\mathcal{R}$-cell Π in a reduced diagram Δ of rank i. Then*

(1) $r(\Gamma) = 0$*;*

(2) The contiguity degree of π to Π does not exceed α.

Proof. **1.** Denote the standard contour of Γ by $\partial\Gamma = d_1 p d_2 q$, where $p = \Gamma \wedge \Pi$, $q = \Gamma \wedge \pi$. By Lemma 2.1 the bonds defining Γ are 0-bonds. Therefore $|d_1| = |d_2| = 0$. Assume that Γ has cells. By Lemma 5.7 applied to Γ ($\tau(\Gamma) < \tau(\Delta)$) there is a θ-cell in Γ. There are no $\mathcal{T}$-cells compatible with the section q in Γ since Δ is reduced. Therefore, by Lemma 2.2 applied to Γ, the degree of contiguity of any cell from Γ to q is less than β; by Lemmas 2.5 and 3.4 applied to Γ ($\tau(\Gamma) < \tau(\Delta)$) the degree of contiguity of any cell from Γ to p is less than α. Since $\theta > \alpha + \beta$, the assumption that Γ has cells is wrong. Part (1) is proved.

2. Assume that the degree of contiguity of a $\mathcal{T}$-cell π to an $\mathcal{R}$-cell Π of rank $j \geq 1$ via Γ is greater than or equal to α. By proven part (1) the subdiagram Γ does not have cells. This means that there is a common subpath u of $\partial\pi$ and $\partial\Pi$ of length at least $\alpha|\partial\pi|$. By Lemma 0.2 ($\alpha > \beta$) we conclude that $|u| < (1+\xi)|A_j|$. Denote $\partial\pi = uv$. If $|u| \leq |A_j|$, then $A' \equiv \phi(u)A'' \overset{1/2}{=} \phi(v)^{-1}A''$ for some cyclic shift A' of $A_j^{\pm 1}$.

Consequently, the inequality $|v| < \alpha^{-1}(1-\alpha)|u| < |u|$ implies that the word A_j is conjugated in rank $1/2 < j$ to a shorter word, contrary to the definition of A_j. Let now $|A_j| < |u| < (1+\xi)|A_j|$. Denote $u = u_1 u_2$, where $|u_1| = |A_j|$. Then for some cyclic shift A' of A_j one has

$$A' \equiv (\phi(u_1))^{\pm 1} \overset{1/2}{=} (\phi(u_2 v))^{\mp 1}.$$

But

$$|(\phi(u_2v))| = |v| + |u_2| < (1-\alpha)\alpha^{-1}|u| + \xi|u_1| <$$
$$< ((1-\alpha)(1+\xi)\alpha^{-1} + \xi)|u_1| < |u_1|,$$

which means that A_j is conjugated in $G(1/2)$ to a shorter word, contrary to the definition of A_j. Lemma 2.4 is proved. □

Lemma 2.5. *Let Γ be a contiguity subdiagram of a $\mathcal{T}$-cell π to a smooth section q in a reduced diagram Δ of rank i. Then*

(1) $r(\Gamma) = 0$;

(2) The contiguity degree of π to q does not exceed α.

Proof. The case when q is a smooth section of rank $1/2$ is taken care of in Lemma 2.2. If q is a smooth A_j-periodic section, where A_j is the period of rank $j \leq i$, than the proof is similar to the proof of Lemma 2.4. In the case when q is a smooth A-periodic section with a simple in rank i word A the proof also proceeds as in Lemma 2.4 except for instead of the definition of a period of rank i we now use the definition of a simple in rank i word. □

Below we discuss the changes needed be done in Lemmas 3.1 - 20.3 from [3] for the purpose of present paper. The numeration is preserved.

In the formulation of Lemma 3.1 the section p_1 (q_1) is considered to be a smooth $\mathcal{T}$-section of ∂E provided p (q) is either the contour of a cell Π_1 (Π_2) of rank $j_1 = r(\Pi_1) = 1/2$ ($j_2 = r(\Pi_2) = 1/2$), or a $\mathcal{T}$-section of a contour of the diagram Δ. The conclusion of Lemma 3.1 does not change if none of p_1, q_1 is a $\mathcal{T}$-section. Otherwise $\max(|d_1|,\ |d_2|) = 0$.

Proving Lemma 3.1, we first note that the case when at least one of p_1, q_1 is a $\mathcal{T}$-section is taken care of in Lemmas 2.1 and 2.2: the bond E between p_1 and q_1 is a 0-bond, and therefore $\max(|d_1|,\ |d_2|) = 0$. Now assume that none of p_1, q_1 is a $\mathcal{T}$-section and let p_1 (q_1) be a smooth A- (B-) periodic section of ∂E. Consider the case when the principal cell π of the bond E is a $\mathcal{T}$-cell. By Lemmas 2.4, 2.5 $r(\Gamma_1) = r(\Gamma_2) = 0$, where Γ_1 and Γ_2 are the contiguity subdiagrams of π to p_1 and q_1 respectively, such that π together with Γ_1 and Γ_2 form the bond E. Therefore there are common subpaths of $\partial\pi$ and each of the sections p_1 and q_1, and, by the definition of a bond, the lengths of those subpaths are greater than $\beta|\partial\pi|$. Consequently, by Lemma 0.2, $|\partial\pi| < (1+\xi)\beta^{-1}\min(|A|,\ |B|)$. Finally,

$$\max(|d_1|,\ |d_2|) < |\partial\pi| < 2\beta^{-1}\min(|A|,\ |B|) < \gamma\min(n|A|,\ n|B|).$$

In the formulation of Lemma 3.2 we require in addition that $l = r(q) > 1/2$ (indeed, by Lemmas 2.1, 2.2 the degree of contiguity of any cell to a $\mathcal{T}$-cell or to a $\mathcal{T}$-section of the boundary is less than β in a reduced diagram of rank i). In the conclusion of Lemma 3.2 $|A_k|$ is substituted by $n^{-1}|\partial\Pi|$ since we also have to consider the case $r(\Pi) = 1/2$. The proof of Lemma 3.2 does not change if $r(\Pi) > 1/2$. In the case $r(\Pi) = 1/2$ Lemma 3.2 is straightforward from Lemmas 2.4, 2.5 and 0.2.

All definitions from Section 4 of [3] are preserved. The formulations and proofs of Lemmas 4.1 - 4.5 need not be changed. All estimates from the Lemmas 5.1 - 5.7 hold in view of Lemmas 2.1 - 2.5. In the proof of Lemma 5.1 notice that $r(\Pi_2) \neq 1/2$ by Lemma 2.1. If $r(\Pi_1) = 1/2$, then $|q_\Gamma| = |p^\Gamma|$ since $r(\Gamma) = 0$ by Lemma 2.4, and, therefore, the inequality (5.2) is valid. In the proof of Lemma 5.4, if $r(\Pi_k) = 1/2$, where $k = 1$ and/or 2, then $r(\Gamma) = 0$ by Lemma 2.4. Hence $|q_\Gamma| = |p^\Gamma|$ and the inequality (5.5) follows. Proving Lemma 5.5, notice that case 1 is impossible by Lemma 2.1 if the considered ordinary cell Π is a $\mathcal{T}$-cell.

The formulation of Lemma 6.1 does not change. Proving this lemma in the case $r(q) = 1/2$, notice that a diagram Δ' with contour $p'q$ with p' being geodesic does not have cells (otherwise there must be a θ-cell, but the degree of contiguity of any cell from Δ' to q is less than β by Lemma 2.2 applied to Δ'; the degree of contiguity of any cell from Δ' to p' is less than α by Lemmas 2.3 and 3.3, but $\theta > \alpha + \beta$). Then $\rho|q| < |q| = |p'| \leq |p|$.

Now consider the case when q is a smooth B-periodic section, where $B \equiv A_l^{\pm 1}$ for some $l \leq i$ or B is a simple word in rank i. The same proof as in [3] works if rank of the θ-cell Π is greater than $1/2$. We need to consider the situation when the θ-cell Π is of rank $1/2$. In the notation used in the proof of Lemma 6.1 we have $|c_1| = |c_2| = |d_1| = |d_2| = 0$, $v_1 = q_2^{-1}$, $u_1 = p_2^{-1}$ by Lemmas 2.3 and 2.5 applied to Δ. The inequality

$$\rho(|q_1| + |q_3|) < |p_1| + |p_3| + (1-\theta)|\partial\Pi| \tag{3}$$

can be obtained in the same way as in [3].

By Lemma 2.5 the contiguity subdiagram Γ_q does not have cells. By Lemma 2.3 applied to the contiguity subdiagram Γ_p of Π to p and by the definition of a θ-cell the degree of contiguity of Π to q is greater than $\theta - \alpha > \beta$. Consequently, by Lemma 0.2 applied to $\phi(q_2)$ one has $|q_2| < (1+\xi)|B|$.

Assume first that $|q_2| \leq |B|$. It follows that $|q_2| \leq 1/2|\partial\Pi| \leq |u_2| + |v_2| + |p_2|$ since B is cyclically reduced in rank 1/2 and $|p_2| > \theta|\partial\Pi| - |q_2| \geq (\theta - 1/2)|\partial\Pi|$. Therefore

$$\rho|q_2| < \rho|p_2| + \rho(1-\theta)|\partial\Pi| < |p_2| + ((\rho-1)(\theta-1/2) + \rho(1-\theta))|\partial\Pi| \tag{4}$$

Let now $|q_2| > |B|$. Denote $q_2 = q'q''$, where $|q'| = |B|$. Then $|q''| < \xi|q'|$. The fact that B is cyclically reduced implies that $|q'| \leq 1/2|\partial\Pi| \leq |q''| + |u_2| + |v_2| + |p_2|$. Therefore $|q'| - |q''| < |u_2| + |v_2| + |p_2|$. Consequently,

$$\frac{1-\xi}{1+\xi}|q_2| < (1-\xi)|B| < |q'| - |q''| < |u_2| + |v_2| + |p_2| < |p_2| + (1-\theta)|\partial\Pi|.$$

Using the fact that

$$|p_2| > \theta|\partial\Pi| - |q'| - |q''| > \left(\theta - \frac{1+\xi}{2}\right)|\partial\Pi|,$$

we obtain

$$\rho|q_2| < \frac{\rho(1+\xi)}{1-\xi}|p_2| + \frac{\rho(1+\xi)}{1-\xi}(1-\theta)|\partial\Pi| <$$

$$< |p_2| + \left(\left(\frac{\rho(1+\xi)}{1-\xi} - 1\right)\left(\theta - \frac{1+\xi}{2}\right) + \frac{\rho(1+\xi)}{1-\xi}(1-\theta)\right)|\partial\Pi|. \quad (5)$$

Combining the inequalities (3) and (4) if $|q_2| \leq |B|$, and (3) and (5) if $|q_2| > |B|$, we complete the proof in the same way as in [3]. Lemma 6.1 is proved.

The formulations of Lemmas 6.2 - 6.5 do not change. Proving Lemma 6.3 in the case when θ-cell Π is of rank 1/2, note that the bonds defining the contiguity subdiagrams Γ_p, Γ_q are 0-bonds by Lemma 2.1. Analogously, if $r(\Pi) = 1/2$ in the proof of Lemma 6.5, then the bonds defining Γ_b, Γ_c are 0-bonds, while the contiguity subdiagrams Γ_p, Γ_q do not contain cells by Lemmas 2.3 and 2.5. Thus, proofs of both of these lemmas in the case $r(\Pi) = 1/2$ proceed in the same way as in [3]. We only need to replace γ by 0 in the estimates and refer to Lemmas 2.5 and 2.3 instead of Lemmas 3.4 and 3.3.

it is convenient to state Lemma 6.2 as follows.

Lemma 6.2. *Let Δ be a disk tame diagram of rank i. Then $|\partial\Pi| \leq \rho^{-1}|\partial\Delta|$ for any cell $\Pi \in \Delta$. In particular, if $|\partial\Delta| < \rho n|A_k|$ for some $k \leq i$, then $r(\Delta) < k$.*

All lemmas and estimates of Section 7 from [3] remain valid due to Lemmas 2.1 – 2.5. However, the presence of $\mathcal{T}$-cells bring the need to consider several more diagrams of special sort in order to understand the bond structure of contiguity subdiagrams of rank i. We need to consider *degenerate special 8-gons of rank i* and *degenerate special $8'$-gons of rank i*. The definitions of these new types of diagrams follow.

A disk tame diagram Δ of rank i is referred to as *degenerate special 8-gon of rank i* if it possesses the following properties:

(DG1) The contour $\partial\Delta$ of Δ is considered to be decomposed into the product $apbrcqds$.

(DG2) Each of the sections r and s is either smooth or geodesic.

(DG3) The section p is smooth of some rank k, $1 \le k \le i$; the section q is smooth of rank 1/2.

(DG4) $\max(|a|,\ |b|,\ |c|,\ |d|) < 2\beta^{-1}|A_k|$.

(DG5) Let a vertex o_1 be chosen on the section p and a vertex o_2 be chosen on the section either (a) r, or (b) s. Denote $p(dec,\ o_1) = p^1p^2$ and (a) $r(dec,\ o_2) = r^1r^2$ or (b) $s(dec,\ o_2) = s^1s^2$. Suppose that $x = o_1 - o_2$ is a simple path joining the vertices o_1 and o_2 with $|x| < 2\beta^{-1}|A_k|$. Then the following is true:

1) the subdiagram Δ_0 with the contour (a) $p^2br^1x^{-1}$ or (b) p^1xs^2a contains no cells;
2) the inequality $|x| \le |b|$ implies $x = b$ in case (a) or the inequality $|x| \le |a|$ implies $x = a^{-1}$ in case (b).

(DG6) Let a vertex o_3 be chosen on the section q and a vertex o_4 be chosen on the section either (a) r, or (b) s. Denote $q(dec,\ o_3) = q^1q^2$ and (a) $r(dec,\ o_4) = r^1r^2$ or (b) $s(dec,\ o_4) = s^1s^2$. Suppose that $y = o_3 - o_4$ is a simple path joining the vertices o_3 and o_4 with $|y| < 2\beta^{-1}|A_k|$. Then the following is true:

1) the subdiagram Δ_0 with the contour (a) q^1yr^2c or (b) $q^2ds^1y^{-1}$ contains no cells;
2) the inequality $|y| \le |c|$ implies $y = c^{-1}$ in case (a) or the inequality $|y| \le |d|$ implies $y = d$ in case (b).

(DG7) There are no bonds between sections r and s in Δ.

A disk tame diagram Δ of rank i is referred to as *degenerate special $8'$-gon of rank i* if it possesses the following properties:

(DG1′) The contour $\partial\Delta$ of Δ is considered to be decomposed into the product $apbrcqds$.

(DG2′) Each of the sections r and s is either smooth or geodesic.

(DG3′) The section p is smooth of some rank k, $1 \le k \le i$; the section q is smooth of rank 1/2.

(DG4′) $\max(|a|,\ |b|,\ |c|,\ |d|) < 2\beta^{-1}|A_k|$.

(DG5′) There are no bonds between $p,\ r$ and between $p,\ s$ in Δ.

(DG6′) There are no bonds between $q,\ r$ and between $q,\ s$ in Δ.

(DG7′) There are no bonds between sections r and s in Δ.

The study of degenerate special 8- and 8′-gons proceeds in the same way as the study of special 8- and 8′-gons in Sections 7, 8 in [3]. Lemma 7.1 remains valid if Δ is a degenerate special 8-gon of rank i. Dealing with a degenerate special 8-gon of rank i, we consider the same cases as are considered in [3] for special 8-gon of rank i. Note that the section q is a β-section by Lemma 2.2. So the case (L4) is impossible. The analog of Lemma 7.2 for degenerate special 8-gon of rank i says that there are 16 cases to consider: (K1)&(L1), ..., (K4)&(L3), (K1)&(L5), ..., (K4)&(L5). Analog of Lemma 7.3 for degenerate special 8-gon of rank i says that the copy of Δ with decomposition of its contour $b^{-1}p^{-1}a^{-1}s^{-1}d^{-1}q^{-1}c^{-1}r^{-1}$ is also a degenerate special 8-gon of rank i. Thus, the cases (K1)&(L5), ..., (K4)&(L5) are symmetric to the cases (K1)&(L2), ..., (K4)&(L2). Therefore it suffices to consider 12 cases (K1)&(L1), ..., (K4)&(L3) only. Note that the same cases were considered in Lemmas 7.4 - 7.15 of [3] studying special 8-gons of rank i. Analogs of Lemmas 7.4 - 7.15 with some minor changes are valid for degenerate special 8-gon of rank i.

Thus, using arguments similar to those of Sections 7, 8 from [3] and Lemmas 2.1 - 2.5 one can prove the following new versions of Lemmas 8.7 and 8.8.

Lemma 8.7. *Let Δ be a special 8-gon of rank i. Then*

$$|p| + |q| < 70\beta^{-1}M\ , \quad |r| + |s| < 102\beta^{-1}M\ , \quad |\partial\Delta| < 180\beta^{-1}M\ ,$$

where $M = \max(|A_k|,\ |A_l|)$.

If Δ is a degenerate special 8-gon of rank i, then the same estimates hold with $M = |A_k|$.

Lemma 8.8. *Let Δ be a special 8′-gon of rank i. Then*

$$|apb| + |cqd| < 100\beta^{-1}M\ , \quad |r| + |s| < 130\beta^{-1}M\ , \quad |\partial\Delta| < 230\beta^{-1}M\ ,$$

where $M = \max(|A_k|,\ |A_l|)$.

For a degenerate special 8′-gon Δ of rank i the same estimates hold with $M = |A_k|$.

Here is the new version of Lemma 9.1.

Lemma 9.1. *Let Δ be a tame diagram of rank i with the contour $\partial\Delta = bpcq$, where each of the sections p and q is either smooth or geodesic. Suppose also that Δ itself is a contiguity diagram between p and q. Then there exists a system $\mathcal{C}(\Delta)$ of bonds $E_1,\ E_2,\ \dots,\ E_k$ between p and q in Δ whose standard contours are $\partial E_t = x_t p_t y_t q_t$, where $p_t = E_t \wedge p$, $q_t = E_t \wedge q$, $t = 1,\ 2,\ \dots,\ k$, such that $E_1,\ E_2,\ \dots,\ E_k$ pairwise have no cells in common and $\mathcal{C}(\Delta)$ has the following properties:*

(a) $x_1 = b,\ y_k = c$ *and* $p = p_1 r_1 \dots p_{k-1} r_{k-1} p_k,\ q = q_k s_{k-1} q_{k-1} \dots s_1 q_1$ *for some paths* $r_1,\ s_1,\ \dots,\ r_{k-1},\ s_{k-1}$.

(b) Denote the principal cell of the bond E_t by π_t, $t = 1,\ 2,\ \dots,\ k$, (we set $|\partial\pi_t| = 0$ if E_t is a 0-bond) and let the subdiagram Δ_t be given by $\partial\Delta_t = y_t^{-1} r_t x_{t+1}^{-1} s_t$, where $t = 1,\ 2,\ \dots,\ k-1$. Then, in this notation, there are no bonds between the sections r_t and s_t in Δ_t and the following inequalities hold for every t, $t = 1,\ 2,\ \dots,\ k-1$:

$$|y_t| + |x_{t+1}| \le 100\beta^{-1} n^{-1} \max(|\partial\pi_t|,\ |\partial\pi_{t+1}|)\ ,$$

$$|s_t| + |r_t| \le 130\beta^{-1} n^{-1} \max(|\partial\pi_t|,\ |\partial\pi_{t+1}|)\ ,$$

$$|\partial\Delta_t| \le 230\beta^{-1} n^{-1} \max(|\partial\pi_t|,\ |\partial\pi_{t+1}|)\ ,$$

provided both $\pi_t,\ \pi_{t+1}$ are $\mathcal{R}$-cells, or

$$|y_t| + |x_{t+1}| \le 100\beta^{-1} n^{-1} |\partial\pi|\ ,$$

$$|s_t| + |r_t| \le 130\beta^{-1} n^{-1} |\partial\pi|\ ,$$

$$|\partial\Delta_t| \le 230\beta^{-1} n^{-1} |\partial\pi|\ ,$$

where π is the only $\mathcal{R}$-cell among $\pi_t,\ \pi_{t+1}$, or

$$|y_t| < \frac{\xi}{10} \min(|\partial\pi_t|,\ |\partial\pi_{t+1}|)$$

if there are no $\mathcal{R}$-cells among $\pi_t,\ \pi_{t+1}$.

In the latter case the subdiagram Δ_t does not have cells, $|s_t| = |r_t| = 0$ and $y_t = x_{t+1}^{-1}$.

(c) If $r(\Delta) = j > 0$ and Π is a cell of rank j in Δ, then Π is the principal cell π_t of E_t for some t, $t = 1, 2, \ldots, k$. In particular,

$$r(\Delta) = \max_{1 \le t \le k} (r(\pi_t)) .$$

Proof. The system of bonds $\mathcal{C}(\Delta)$ is constructed in the same way as in [3].

Assume that for some t, $1 \le t < k$ the bond E_{t+1} (or E_t) consists of a $\mathcal{T}$-cell, and the bond E_t (resp. E_{t+1}) is either a 0-bond or consists of a $\mathcal{T}$-cell. We will show that the subdiagram Δ_t has no cells.

Assuming that Δ_t contains at least one cell, by Lemma 5.7 there is a θ-cell Π in Δ_t. For the sum of the contiguity degrees of Π to the sections r_t, s_t, we have that $(\Pi, \Gamma_r, r_t) + (\Pi, \Gamma_s, s_t) < \alpha + \beta$, since each of the degrees does not exceed α (following from the fact that p, q are smooth or geodesic) and if both the degrees of contiguity are at least β we would have a bond between r_t and s_t.

The sum of the degrees of contiguity of Π to the sections y_t and x_{t+1} does not exceed $(\Pi, \Gamma_y, y_t) + (\Pi, \Gamma_x, x_{t+1}) < 2\beta$ since x_{t+1} is smooth of rank 1/2, y_t is smooth of rank 1/2 provided E_t consists of a $\mathcal{T}$-cell or $|y_t| = 0$ provided E_t is a 0-bond (in the latter case the contiguity subdiagram Γ_y does not exist). Thus

$$\theta < (\Pi, \Gamma_r, r_t) + (\Pi, \Gamma_s, s_t) + (\Pi, \Gamma_y, y_t) + (\Pi, \Gamma_x, x_{t+1}) < \alpha + 3\beta,$$

which is a contradiction. Thus, $r(\Delta_t) = 0$. The equalities $|s_t| = |r_t| = 0$ and $y_t = x_{t+1}^{-1}$ follow now from the fact that there are no bonds between sections s_t abd r_t. In the case when both E_t and E_{t+1} are 1/2-bonds the inequality

$$|y_t| < \frac{\xi}{10} \min(|\partial\pi_t|, |\partial\pi_{t+1}|)$$

follows from Lemma 0.1(b) and the fact that Δ is reduced.

In view of the above argument, if the subdiagram Δ_t contains cells and if one of the bonds E_t, E_{t+1} consists of a $\mathcal{T}$-cell, then the other bond is a k-bond for some $k \ge 1$. Let E_t be a 1/2-bond and E_{t+1} be a k-bond, $k \ge 1$ (the other case is similar). In the notations from [3], $|b_t| = |c_t| = 0$ since $r(\Gamma_p^t) = r(\Gamma_q^t) = 0$ by Lemmas 2.3, 2.5. The diagram Δ_t with the decomposition of its contour

$$\partial\Delta_t = a_{t+1}^{-1} (v_1^{t+1})^{-1} d_{t+1}^{-1} s_t v_2^t r_t$$

has properties (DG1′)-(DG7′) of a degenerate special 8′-gon of rank i (if we set $a = a_{t+1}^{-1}$, $p = (v_1^{t+1})^{-1}$, $b = d_{t+1}^{-1}$, $r = s_t$, $|c| = 0$, $q = v_2^t$, $|d| = 0$,

$s = r_t$). The remaining part of the proof proceeds along the lines of the proof of Lemma 9.1 given in [3] using the estimates from the new versions of Lemmas 8.7 and 8.8. □

The formulation of Lemma 9.3 is preserved. Proving part (a) in the case when E_t is a 1/2-bond, the estimate $|q_t| < (1+\varepsilon)|B|$ is immediate from Lemmas 2.5 and 0.2 as $\xi < \varepsilon$. If in addition E_{t+1} is a 1/2-bond, then by Lemmas 9.1(b) and 0.2 $|r_t| = |s_t| = 0$ and

$$|\partial\Delta_t| = |y_t| + |x_{t+1}| < \frac{\xi}{10}\min(|\partial\pi_t|,\ |\partial\pi_{t+1}|) <$$

$$< \frac{\xi}{10}(1+\xi)\beta^{-1}|B| < 0.001|B|\ .$$

The remaining part of the proof of Lemma 9.3 goes in the same way as in [3].

The formulation of Lemma 9.4 is preserved. The proof proceeds as in [3] with some minor changes. The formulation of Lemma 9.5 does not change. In the case when Π is an $\mathcal{R}$-cell there are no changes in the proof of 9.5(a). Assume that Π is a $\mathcal{T}$-cell.

First, let Π belong to a subdiagram Δ_s, $m_1 \le s \le m_2 - 1$, of $\Delta(m_1,\ m_2)$. By Lemmas 6.2, 9.1(b) and Lemma 9.5(a) for an $\mathcal{R}$-cell in $\Delta(m_1,\ m_2)$

$$|\partial\Pi| < 230\rho^{-1}\beta^{-1}n^{-1} \times 2.22|B| < 2.22|B|\ .$$

Let now Π be a principal cell of a bond E_t, $m_1 \le t \le m_2$. The bond E_t coincides with the cell Π, and therefore $\partial\Pi = x_t p_t y_t q_t$. Applying Lemma 2.3 (if p is geodesic) or Lemma 2.5 (if p is smooth), one gets $|p_t| < \alpha|\partial\Pi|$. Therefore, in view of Lemma 9.3(a),

$$|\partial\Pi| < (1-\alpha)^{-1}(|q_t|+|x_t|+|y_t|) < (1-\alpha)^{-1}(1+\varepsilon+0.002)|B| < 2.22|B|,$$

as required.

Lemmas 10.1 - 10.9 are formulated and proved in the same way as in [3].

The cells Π_1, Π_2 from the definitions of R- and S-diagrams are now required to be $\mathcal{R}$-cells. We define T-diagrams of rank i for $i \ge 1$ only. The formulation of Lemma 12.1 is preserved. The proof is decomposed into cases in the same way as in [3]. Considering the case 1.1.1 in the proof of Lemma 12.1, the diagram Δ_2 is a degenerate special $8'$-gon provided the bond E between r_1 and r_2 is a 1/2-bond. Therefore the required estimate can be obtained from Lemma 8.8.

The investigation of Cases 2.3 - 2.14 of the proof of Lemma 12.1 differs from the one in [3] if the θ-cell π is a $\mathcal{T}$-cell.

2.3. By Lemma 2.1 the bonds defining Γ_a do not contain cells. Denote $\partial\pi = uc_{12}vc_{22}$, where $c_1 = c_{11}c_{12}c_{13}$, $c_2 = c_{21}c_{22}c_{23}$ and the 0-bonds E_1 and E_2 (between π and the sections c_1 and c_2 respectively), given by $\partial E_1 = c_{12}c_{12}^{-1}$ and $\partial E_2 = c_{22}c_{22}^{-1}$, are chosen such that the sum $|c_{13}|+|c_{21}|$ is minimal.

If there are no bonds between the sections v and q, then, by Lemma 6.5, $|v| + |q| \leq \mu(|c_{13}| + |c_{21}|)$. Therefore

$$|q| \leq \mu(|c_{13}| + |c_{21}|) < 4\mu|A_j| < 0.04n|A_j|.$$

Now assume that there is a bond between v and q. Consider the contiguity subdiagram Γ between sections v and q, maximal with respect to the sum $|v_2| + |q_2|$, where $q = q_1q_2q_3$, $v = v_1v_2v_3$. By Lemma 2.5(1) $r(\Gamma) = 0$ and therefore $q_2 = v_2$. By the choice of Γ, we can apply Lemma 6.5 to the diagrams Δ_1 and Δ_2 given by $\partial\Delta_1 = c_{13}q_1v_1^{-1}$ and $\partial\Delta_2 = c_{21}v_3^{-1}q_3$. Hence $|q_1| + |q_3| \leq \mu(|c_{13}| + |c_{21}|) < 4\mu|A_j|$. In view of Lemma 0.1(a), $|q_2| < 11|A_j|$. Thus,

$$|q| = |q_1| + |q_2| + |q_3| < (11 + 4\mu)|A_j| < 0.04n|A_j| \ ,$$

as required.

Case 2.3 is complete.

2.4. By Lemmas 2.1 and 2.5(1) the bonds between $\partial\pi$ and each of the sections r, q are 0-bonds, and the contiguity subdiagrams of π to the sections r and q are of rank 0. Denote $\partial\pi = r_{12}uq_{21}v$, $r = r_{11}r_{12}r_2$, $q = q_1q_{21}q_{22}$, where r_{12} (resp. q_{21}) is the maximal common subpath of $\partial\pi$ and r (resp. q).

Assume that $|q_{21}| > \beta|\partial\pi|$. Then by Lemma 0.2 $|\partial\pi| < (1+\xi)\beta^{-1}|A_j|$. Thus, $|u| < |\partial\pi| < (1+\xi)\beta^{-1}|A_j| < 2\beta^{-1}|A_j|$, giving a contradiction to the condition (T6).

Let now $|q_{21}| \leq \beta|\partial\pi|$. Then, by Lemma 2.5(2),

$$|v| > (\theta - (\alpha + 2\beta))|\partial\pi| > \frac{1}{3}|\partial\pi|.$$

Recall also that $u < (1 - (\theta - \beta))|\partial\pi|$. Consider the diagram Δ_1 given by $\partial\Delta = c_2r_{11}v^{-1}q_{22}$. Assuming that Δ_1 has cells, by Lemma 5.7 there is a θ-cell Π in Δ_1. The contiguity degree of Π to v is less than β by Lemma 2.1 applied to Δ. One of the contiguity degrees of Π to r_{11} and q_{22} is less than β by Lemma 12.1.1. In view of (T6), the arguments from the proof of Lemma 7.1(c) can be used to show that the sum of contiguity degrees of Π to r_{11} andc_2, as well as the sum of contiguity degrees of Π

to q_{22} andc_2, is less than 0.8. Therefore the sum of the contiguity degrees of Π to all four sections of $\partial\Delta_1$ is less than $0.8 + 2\beta$, contrary to the definition of a θ-cell, as $\theta > 0.8 + 2\beta$. Therefore Δ_1 does not have cells, and $|v| \leq |c_2|$ by the choice of r_{12} and q_{21}. It follows that $|v| \leq 2\beta^{-1}|A_j|$. Thus,

$$\frac{1}{3}|\partial\pi| < |v| \leq 2\beta^{-1}|A_j|.$$

Consequently, $|\partial\pi| < 6\beta^{-1}|A_j|$ and

$$|u| < (1-(\theta-\beta))|\partial\pi| < 6\beta^{-1}(1-(\theta-\beta))|A_j| < 2\beta^{-1}|A_j|\ ,$$

contrary to (T6).

Case 2.4 is complete.

2.5. The bonds defining Γ_a are 0-bonds. Denote $\partial\pi = c_{22}uc_{12}v$, where $c_{22}uc_{12} = \Gamma_a \wedge \partial\pi$, $c_1 = c_{11}c_{12}c_{13}$, $c_2 = c_{21}c_{22}c_{23}$. By (T6) the paths $c_{11}c_{12}$ and $c_{22}c_{23}$ are geodesic in the disk subdiagram of Δ consisting of π and Γ_a. Therefore, in view of Lemma 2.3(1), we may assume that c_{12} (resp. c_{22}) is the maximal common subpath of $\partial\pi$ and c_1 (resp. c_2). Consider the disk subdiagram Δ_1 of Δ given by $\partial\Delta_1 = c_{23}rc_{11}u$.

First assume that there is a bond between u and r in Δ_1. Let Γ be a contiguity subdiagram between u and r maximal with respect to the sum $|u_2| + |r_2|$, where $u = u_1u_2u_3$, $r = r_1r_2r_3$. The diagram Γ does not have cells. It follows from Lemmas 2.2, 2.3, 3.3 and 5.7 provided $\phi(r)$ is a reduced in rank i word, or from Lemmas 2.2, 2.5, 3.4 and 5.7 provided $\phi(r)$ is A-periodic with a simple in rank i word A. Let Δ_2 be the diagram given by $\partial\Delta_2 = r_3c_{11}u_3^{-1}$. In view of the property (T6) of Δ one can estimate the sum of contiguity degrees of a cell from Δ_2 to the sections r_3 and c_{11}. Arguing as in the proof of Lemma 7.1(c), and using Lemma 5.7 and the fact that u_3 is a β-section of $\partial\Delta_2$, one can show that Δ_2 does not have cells. Similarly, one gets that Δ_3 given by $\partial\Delta_2 = u_1^{-1}c_{23}r_1$ does not have cells. Therefore $\partial\pi = c_{22}c_{23}rc_{11}c_{12}v$. Lemmas 2.3 and 2.5 imply that $|r| < \alpha|\partial\pi|$. Therefore $|c_{22}| + |c_{23}| + |c_{11}| + |c_{12}| > (\theta-\beta-\alpha)|\partial\pi|$. Thus,

$$|v| < (1-(\theta-\beta))|\partial\pi| < (1-(\theta-\beta))(\theta-\beta-\alpha)^{-1}\times 4\beta^{-1}|A_j| < 4\beta^{-1}|A_j|.$$

If there are no bonds between u and r in Δ_1, then, by Lemma 6.5,

$$|u| < \mu(|c_{11}| + |c_{23}|) < 6\beta^{-1}|A_j|\ .$$

Consequently,

$$|v| < (\theta-\beta)^{-1}(1-(\theta-\beta))(|c_{22}| + |u| + |c_{12}|) <$$

$$< (\theta-\beta)^{-1}(1-(\theta-\beta)) \times 10\beta^{-1}|A_j| < 4\beta^{-1}|A_j|.$$

Thus, regardless of whether there is a bond between u and r or there are no such bonds, the length of the path $f = c_{21}vc_{11}$ in Δ is estimated as follows:

$$|f| = |c_{21}| + |v| + |c_{11}| < 8\beta^{-1}|A_j|.$$

Repeating the arguments from [3] following the inequality (12.12), we obtain the estimate $|q| < 0.02n|A_j|$.

Case 2.5 is complete. The Case 2.6 is analogous to the Case 2.4, so we pass to the Case 2.7.

2.7. Consider the following subcases:

(2.7.1.) There is a bond between π, q and there is a bond between π, c_2 in Δ.

(2.7.2.) There are no bonds between π, q and there is a bond between π, c_2 in Δ.

(2.7.3.) There is a bond between π, q and there are no bonds between π, c_2 in Δ.

(2.7.4.) There are no bonds between π, q and between π, c_2 in Δ.

2.7.1. The property (T6) of Δ makes it possible to use arguments from the proof of Lemma 7.1(c) to show that Γ_a does not have cells in this case. Therefore $|q| < 11|A_j|$ by Lemma 0.1(a).

2.7.2. Again referring to the proof of Lemma 7.1(c), we denote $\partial\pi = uc_{12}vc_{22}r_1$, where $c_1 = c_{11}c_{12}c_{13}$, $c_2 = c_{21}c_{22}$, $r = r_1r_2$. Applying Lemma 6.5 to the disk subdiagram Δ_1 given by $\partial\Delta_1 = c_{13}qc_{21}v^{-1}$, one gets that

$$|q| < \mu(|c_{13}| + |c_{21}|) < 6\beta^{-1}|A_j| \ .$$

2.7.3. Making use of the arguments from the proof of Lemma 7.1(c), we denote $\partial\pi = uc_{12}q_1vr_2$, where $c_1 = c_{11}c_{12}$, $q = q_1q_2$, $r = r_1r_2r_3$ and there are no bonds between v, q_2, between v, r_1 and between v, c_2. By Lemma 0.1(a) $|q_1| < 11|A_j|$. By Lemma 12.1.1 there are no bonds between r_1, q_2. Consider the disk subdiagram Δ_1 of Δ given by $\partial\Delta_1 = r_1v^{-1}q_2c_2$. Assume that Δ_1 has cells. By Lemma 5.7 there is a θ-cell Π in Δ_1. The degree of contiguity of Π to one of the sections r_1, q_2 is less than β since there are no bonds between those sections. Let this section be r_1 (the case when it is q_2 is analogous). Arguing as in Lemma 7.1(c), we get that the sum of contiguity degrees of Π to the sections c_2 and q_2 is less than 0.8. The section v^{-1} is a β-section of $\partial\Delta_1$. So, the sum of contiguity degrees of Π to all four sections of $\partial\Delta_1$ is less than $0.8 + 2\beta$. Thus, the inequality $\theta > 0.8 + 2\beta$ means that Δ_1 does not have cells. It follows from the condition (T2) and the fact that there are no bonds between r_1, q_2 and between v, q_2 that $|q_2| < 2|A_j| + 2\beta^{-1}|A_j|$. Finally, $|q| = |q_1| + |q_2| < (13 + 2\beta^{-1})|A_j|$.

2.7.4. Denote $\partial\pi = uc_{12}vr_2$, where $r = r_1r_2r_3$, $c_1 = c_{11}c_{12}c_{13}$ and there are no bonds between v, r_1 and between v, c_{13}. The disk subdiagram Δ_1 of Δ given by $\partial\Delta_1 = c_{13}qc_2r_1v^{-1}$ has properties (DG1′)-(DG7′) of a degenerate special 8′-gon of rank i (if we set $a = c_{13}$, $p = q$, $b = c_2$, $r = r_1$, $|c| = 0$, $q = v^{-1}$, $|d| = |s| = 0$). Referring to Lemma 8.8 we get that $|q| < 100\beta^{-1}|A_j|$.

Case 2.7. is complete.

The remaining Cases 2.8 - 2.14 of the proof of Lemma 12.1 with the θ-cell π of rank 1/2 can be considered in a similar manner.

In the definition of a U-diagram Δ we require in addition $r(\Delta) = j \geq 1$. In the formulation of Lemma 13.1 we add an assumption $j = r(\Delta) > 1/2$. Similarly, in Lemmas 13.2, 13.4 we require that $j = r(\Delta(1)) > 1/2$ and in Lemma 13.3 - $j = r(\Delta) > 1/2$.

The formulation and the proof of Lemma 14.1 do not change. Proving Lemma 14.2 we need to consider the case $r(\Pi) = 1/2$. If p_Γ is a subpath of q, then the needed inequalities are immediate from Lemma 0.2. Since the bonds defining Γ are 0-bonds, it is possible to obtain the same inequalities (14.1)-(14.3) modulo substitution 0 for γ. Arguing in the same way as in [3], instead of (14.4) we get $|u| > \beta|\partial\Pi|$.

If there are no bonds between sections y and u in Δ_5, then by Lemma 6.5

$$|y| + |u| \leq \mu(|d_1| + |d_3|) < \mu(1+\delta)|A_j|,$$

as $|d_1| = 0,\ |d_3| < (1+\delta)|A_j|$. Consequently,

$$|\partial\Pi| < \beta^{-1}|u| < \beta^{-1}\mu(1+\delta)|A_j| < 4\beta^{-1}|A_j|$$

and

$$|p_\Gamma| = |y| + |q_{12}| + |t| < (1+\mu)(1+\delta)|A_j| < 5|A_j| \ .$$

If there are bonds between y and u in Δ_5, then all these bonds are 0-bonds. Since a diagram with contour $y'u'$ with y' being smooth and u' being smooth of rank 1/2 does not have cells, there are decompositions $y = y_1y_2$, $u = u_2u_1$, where $y_1 = u_1^{-1}$ is the longest common subpath of y and u^{-1}. By Lemma 6.5 $|y_2| + |u_2| \leq \mu|d_3| < \mu(1+\delta)|A_j|$. The fact that $y_1 = u_1^{-1}$ is a common subpath of a $\mathcal{T}$-section and an A_j-periodic section means that either $|y_1| = |u_1| < (1+\xi)|A_j|$, or $(1+\xi)|A_j| < |y_1| = |u_1| < 11|A_j|$ and $|A_j| < \xi^{-3/2}$. The latter case is impossible since otherwise

$$|\partial\Pi| < \beta^{-1}|u| = \beta^{-1}(|u_1| + |u_2|) < \beta^{-1}(\mu(1+\delta) + 11)\xi^{-3/2} < n^2 \ ,$$

contrary to Lemma 0.1(3). Therefore

$$|y|,\ |u| < (1 + \xi + \mu(1+\delta))|A_j|$$

and the inequalities sought follow as before.

The formulations and proofs of Lemmas 14.3 - 14.6 do not change. Proving Lemma 14.7, notice that the inequality 14.19 holds also in the case when Π is a 1/2-cell. Indeed, in this case when $\Gamma \wedge p_1$ is a subpath of q_1, $r(\Gamma) = 0$ by Lemma 2.5 and, in view of the inequality $|p^\Gamma| = |q_\Gamma| > \beta|\partial\Pi|$, Lemma 0.2 imply the estimate $|p^\Gamma| < (1+\xi)|A_{j_1}|$. If $\Gamma \wedge p_1$ is a subpath of t_1, then using Lemmas 14.1 and 2.3 we again obtain that $r(\Gamma) = 0$. Therefore $|p_\Gamma| \le |t_1| < \delta|A_{j_1}|$. It remains to note that $1+\xi < \delta$ and the rest of the proof proceeds as in [3]. The formulations and proofs of Lemmas 14.8 and 14.9 do not change.

Proving Lemma 14.10 we need to consider the case when Π_1 is a cell of rank 1/2. In the notations of [3], if p^Γ is a subpath of either q_1 or t_1, then $r(\Gamma) = 0$ by Lemmas 2.5, 14.1 and 2.3. Therefore, by Lemma 0.1(a) provided p^Γ is a subpath of q_1 or by Lemma 18.5(c) provided p^Γ is a subpath of t_1, $|p^\Gamma| < 11|A_{j_1}|$. The obtained estimate implies the inequality (14.25) from [3].

Now suppose that p^Γ is a subpath of p_1 but not a subpath of either q_1 or t_1. By Lemma 2.1 the bonds defining Γ are 0-bonds. By Lemma 6.1 applied to Γ, $\rho|q_\Gamma| \le |p^\Gamma|$. As in the proof of Lemma 14.2 one can obtain a reduced diagram Δ_1 with $\partial\Delta_1 = q_\Gamma yz$, where y is a strictly smooth A_{j_1}-periodic section, $|y| > |p^\Gamma| - (1+\delta)|A_{j_1}|$, and $|z| < (1+\delta)|A_{j_1}|$. Arguing as in the proof of Lemma 14.2, $|y| + |q_\Gamma| \le \mu|z| < \mu(1+\delta)|A_{j_1}|$ by Lemma 6.5 provided there are no bonds between sections y and q_Γ in Δ_1. It follows that $|p^\Gamma| < |y| + (1+\delta)|A_{j_1}| < (\mu+1)(1+\delta)|A_{j_1}|$. If there are bonds between y and q_Γ in Δ_1, then, as in the proof of Lemma 14.2, we obtain the estimate $|y| < (11+\mu(1+\delta))|A_{j_1}|$. Therefore $|p^\Gamma| < (11 + (\mu+1)(1+\delta))|A_{j_1}|$. In both cases $|p^\Gamma| < \beta^{-1}|A_{j_1}|$, which forces the inequality (14.25).

If now p^Γ is a subpath of a contour of a cell Π_2 in Δ, then $r(\Gamma) = 0$ by Lemma 2.1. Therefore $|p^\Gamma| = |q_\Gamma|$ and the inequality (14.28) can be obtained as in [3].

There are no changes in the formulation and in the proof of Lemma 14.11. Note that in the proof the weight of a cell π of rank 1/2 is given by the formula $\nu(\pi) = |\partial\pi|^{2/3}$. Lemmas 14.12 - 16.6 are formulated and proved without changes.

In the formulations of Lemmas 17.1, 17.2 we add the condition $r(\Delta) > 1/2$. The conclusion of Lemma 17.2 now reads:

Then Δ_0 contains either a $16\beta^{-1}n^{-1}$-contractile cell Π of rank $r(\Pi) > 1/2$ or a cell π of rank $j \le i$, $j > 1/2$ such that if $o \in \partial\pi$ is a phase vertex and $t = o - o$ is a simple path in Δ_0 homotopic as a cyclic path in Δ_0 to a contour of Δ_0, then $\phi(t) \overset{i}{=} T$, where T is an $\mathcal{F}(A_j)$-involution.

Proof of Lemma 17.2 proceeds in the same way as in [3]. Remark only that in the sequence of diagrams $\Delta_0 = \Delta_0^{(1)}, \Delta_0^{(2)}, \ldots$ the diagram $\Delta_0^{(k)}$ (for every $k > 1$) is obtained from $\Delta_0^{(k-1)}$ by removal of a reducible j-pair (for some j, $1 \le j \le i$) and all reducible 1/2-pairs.

The formulation of Lemma 17.3 and the scheme of proof of this lemma in the case $r(\Delta) > 1/2$ coincide with those in [3]. The details of the proof that require additional consideration caused by presence of 1/2-cells will be considered below. The equality $r(\Delta) = 1/2$ means that Δ consists of 1/2-cells only. Therefore Δ can be considered as a diagram over presentation considered in Section 4.2 [8] with additional assumption that the subgroup N of H is the whole H. It means that Lemma 4.12 from [8] (up to change of notations) may be applied to Δ (indeed, the estimates on the lengths of the sections of $\partial\Delta$ used in the formulation of Lemma 4.12 in [8] follow from the analogous estimates in the formulation of Lemma 17.3 and the fact that Δ itself is a contiguity subdiagram between the sections p and q; simplicity of the word A mentioned in the formulation of Lemma 17.3 implies that in context of [8] the word A is simple in some positive rank and can be considered as a period of some rank; in the arguments from [8] (and [7]) that use the fact that the exponent is odd we refer to Lemma 0.3 instead). Thus, the formulation of Lemma 17.3 admits the following addition:

If $r(\Delta) = 1/2$, then there are two phase vertices $o_p \in p$, $o_q \in q$ that can be joined in Δ by a path whose label is $T \overset{1/2}{=} 1$, and the subgroup

$$\langle A^{-l} T A^{l} \mid l = 0, 1, 2, 3 \rangle$$

of $G(i)$ is trivial.

Proving Lemma 17.3 in the case when $r(\Delta) = j > 1/2$, as in [3], by Lemmas 9.4 and 9.5 we find a rigid subdiagram $\Delta(m_1, m_2)$ in Δ. Note that there might be $\mathcal{T}$-cells in $\Delta(m_1, m_2)$ with boundaries longer than $n|A_j|$, however, by Lemma 9.5(a) $|\partial\Pi| < 2.22|A|$ for any cell $\Pi \in \Delta(m_1, m_2)$. So, instead of the inequality (17.30) we use two inequalities:

$$|\partial\Pi| \le n|A_j| < 2.22|A|, \quad \text{if } r(\Pi) > 1/2,$$

$$|\partial\Pi| < 2.22|A|, \quad \text{if } r(\Pi) = 1/2.$$

Proving lemma 17.3.1, notice that the cell Π is an $\mathcal{R}$-cell by Lemma 17.2. In the case 4 considered in the proof of Lemma 17.3.1 the subdiagram Γ_q^t contains Π and therefore $j_q^t = r(\Gamma_q^t) > 1/2$. The inequality (17.39) is true for the length of contours of all subdiagrams $\Delta_s(\Gamma_q^t)$, and those of the subdiagrams $E_s(\Gamma_q^t)$ that have an $\mathcal{R}$-cell as its principal cell.

In the formulation of Lemma 17.3.2 we remark that $r(\pi_{t'}) = j > 1/2$, and similarly, $r(\pi_{m_3}) = j > 1/2$ in the definition of the diagram $\Delta(m_3,\ m_4)$ and in the formulation of Lemma 17.3.3. The proofs of these lemmas are preserved.

The formulation of Lemma 17.3.4 does not change. Proving part (a) of this lemma, we need to consider the situation when the initial vertex $(p_{m_3})_-$ of the path p_{m_3} belongs to a subpath $\tilde{p}_t = \tilde{E}_t \wedge \tilde{p}(m_3,\ m_4)$ for some t, $m_3 \le t \le m_4$, with the bond $\tilde{E}_t$ consisting of a 1/2-cell $\tilde{\pi}_t$. As in [3], denote $\tilde{p}_{t2}$ to be the subpath of $\tilde{p}_t$ that connects vertex $(p_{m_3})_-$ to the terminal vertex $(\tilde{p})_+$ of the path $\tilde{p}_t$.

Assume that $|\tilde{p}_{t2}| < \max(\beta|\partial\tilde{\pi}_t|,\ \beta n|A_j|)$. Then, using Lemma 9.1(b) and the inequalities (17.30), the length of the path $f_1 = \tilde{p}_{t2}\tilde{y}_t$ is estimated as follows

$$|f_1| < \beta \max(|\partial\tilde{\pi}_t|,\ n|A_j|) + \max\left(\frac{\xi}{10}|\partial\tilde{\pi}_t|,\ 100\beta^{-1}|A_j|\right) < 2.5\beta|A|\ .$$

Now assume that $|\tilde{p}_{t2}| > \max(\beta|\partial\tilde{\pi}_t|,\ \beta n|A_j|)$ and, as in [3], consider two cases:

1. $(\tilde{p})_+ \in p_{m_3}$.
2. $(\tilde{p})_+ \notin p_{m_3}$.

1. Consider the diagram Γ_1 given by

$$\partial\Gamma_1 = a_{m_3}\tilde{p}_{t2}h_2v.$$

Here $|a_{m_3}| < 2\beta^{-1}|A_j|$, $|h_2| < \beta^{-1}|A_j|$ by Lemmas 3.1 and 9.3(b), and v is an arc of $\partial\pi_{m_3}$. By Lemma 6.1 applied to Γ_1

$$|v| \ge \rho|\tilde{p}_{t2}| - |a_{m_3}| - |h_2| > 0.94\beta n|A_j|\ .$$

In view of $|\tilde{p}_{t2}| > \beta|\partial\tilde{\pi}_t|$ and Lemma 0.1(b) we can consider $\tilde{p}_{t2}$ as a $\mathcal{T}$-section of $\partial\Gamma_1$ and remove (if necessary) 1/2-cells in Γ_1 compatible with $\tilde{p}_{t2}$. Thus we may assume that $\tilde{p}_{t2}$ is a smooth section of rank 1/2 of $\partial\Gamma_1$.

Let Γ_2 be a contiguity subdiagram between v and $\tilde{p}_{t2}$ in Γ_1 maximal with respect to the sum $|v^2| + |\tilde{p}_{t2}^2|$, where $v^2 = \Gamma_2 \wedge v$, $\tilde{p}_{t2}^2 = \Gamma_2 \wedge \tilde{p}_{t2}$, $v = v^1v^2v^3$, $\tilde{p}_{t2} = \tilde{p}_{t2}^1\tilde{p}_{t2}^2\tilde{p}_{t2}^3$. By Lemma 2.2 (applied to the diagram consisting of Γ_1 and π_{m_3}) $r(\Gamma_2) = 0$. It follows now from Lemma 6.5 and the choice of Γ_2 that

$$|v^1| + |v^3| \le \mu(|a_{m_3}| + |h_2|) < 5\beta^{-1}|A_j|.$$

Then the length of $A_j^{\pm 1}$-periodic subword $\phi(\tilde{p}_{t2}^2)$ of $\phi(\tilde{p}_{t2})$ can be estimated as follows:

$$|\phi(\tilde{p}_{t2}^2)| = |\phi(v^2)| > (0.94\beta n - 5\beta^{-1})|A_j| > 0.9\beta n|A_j|.$$

But this is impossible by Lemma 0.1(a).

2. Consider the subdiagram Γ of $\Delta(l)$ consisting of π_{m_3}, $\tilde{\pi}_t$ and $\Gamma_p^{m_3}$. Assume that Γ is reduced. Then, following from rigidity of $\Delta(m_3,\ m_4)$ and Lemma 3.4, the contiguity degree of π_{m_3} to $\tilde{\pi}_t$ is greater than $\chi-\alpha > \beta$. But this contradicts Lemma 2.1. If Γ is not reduced, then the cell $\tilde{\pi}_t$ with some cell from $\Gamma_p^{m_3}$ form a 1/2-pair. Removing this 1/2-pair and taking out the cell π_{m_3} from Γ, we obtain a reduced diagram Γ_1 with $\partial\Gamma_1 = va_{m_3}p'b_{m_3}$. Removing (if necessary) 1/2-cells in Γ_1 compatible with the section p', we may assume that p' is a smooth section of rank 1/2 of $\partial\Gamma_1$. The estimate $|v| > 0.94\beta n|A_j|$ follows now from rigidity of $\Delta(m_3,\ m_4)$ and Lemma 3.4. Therefore one can obtain a contradiction to Lemma 0.1(a) in the same way as it was done in case 1.

To prove part (b) of Lemma 17.3.4 one has to consider the case when $(\tilde{y}_{m_4})_- \in p_t = E_t \wedge p(m_3,\ m_4)$ with the bond E_t consisting of a $\mathcal{T}$-cell Π. By Lemma 9.3(a) $|p_t| < (1+\xi)|A|$. The remaining part of the proof of (b) as well as the proof of (c) proceed as in [3]. Lemma 17.3.4 is proved.

The formulation of Lemma 17.3.5 remains the same. The estimate in part (a) is valid if the bond $E^0_{k^0+1}$ consists of a 1/2-cell:

Lemma 6.5 applied to $\Delta^0_{k^0+1}$ and the inequalities (17.30), (17.63) imply that

$$|r^0_{k^0+1}| + |s^0_{k^0+1}| \le \mu(|y^0_{k^0+1}| + |c^0(l)|) < \mu(2.22|A| + 0.663|A|) < 4|A|.$$

Considering case 1 of the proof of Lemma 17.3.5.1 with the additional condition that the bond E^0_1 consists of a 1/2-cell, notice that the subdiagram Δ^0_0 is a degenerate special 8-gon of rank i (if we set $a = d,\ p = v,\ b = e, r = r^0_0,\ q = (x^0_1)^{-1},\ s = s^0_0$). Now, by Lemma 8.7, $|\partial\Delta^0_0| < 180\beta^{-1}|A_j|$, and therefore $r(\Delta^0_0) < j$ by Lemma 6.2.

Dealing with case 2 of the proof of Lemma 17.3.5.1 in the situation when $r(E^0_{k^0+1}) = 1/2$, one needs to substitute the summand $(1+2\gamma)n|A_j|$ by $2.22|A|$ in the estimate of the length of $\partial\Delta^0_{k^0+1}$. The inequality $|\partial\Delta^0_{k^0+1}| < 7|A|$ is still valid and the proof of Lemma 17.3.5.1, as well as of Lemma 17.3.5, can be completed in the same way as in [3].

The formulations and proofs of Lemmas 17.3.6 - 17.3.8 do not need any changes. In the formulation of Lemma 17.4 we require in addition that the rank m of the section $\phi(q)$ is greater than 1/2. If the diagram Δ from the condition of Lemma 17.4 satisfies $r(\Delta) = 1/2$, then the subdiagram $\Delta(m_1,\ m_2)$ can be obtained from $\Delta = \Delta(1,\ k)$ by removing those of the bonds $E_1,\ E_k$ defining Δ, that are not 0-bonds. The inequalities (17.92)-(17.94) follow from Lemmas 9.3(a) and 9.5.

Proving the existence of a short path connecting the vertex $o^1_p \in p_t$ with some vertex $o^1_q \in q(m_2,\ m_1)$ in the case when $E_t = \pi_t$ is a cell of rank 1/2, notice that $|p_t| < \alpha|\partial\pi_t| < 2.22\alpha|A_m|$ by Lemma 2.5 and the inequality (17.94). Using Lemma 9.3(a) and the inequality (17.93), one can obtain a path connecting o_p^1 with some vertex $o_q^1 \in q(m_2,\ m_1)$ of length at most

$$\Big(\frac{2.22\alpha}{2} + 0.003\Big)|A_m| < 0.6|A_m|,$$

as required. The rest of the proof proceeds as in [3].

The formulation and the proof of Lemma 18.1 need not be changed.

The formulation of Lemma 18.2 is preserved. Instead of the word B considered in [3] we take the word $\bar{B}$ obtained from B substituting every occurrence of the letter a_1 by a_1^{-1}. It follows from the way B is constructed that the word $\bar{B}$ is cyclically reduced, $|B| = |\bar{B}|$, and $\bar{B}$ does not contain subwords of the form D^3, $|D| > 0$. Moreover, no cyclic shift of $\bar{B}^{\pm 1}$ contains a positive (negative) subword of length greater than 4. Therefore the maximal length of a common subword of a $\bar{B}^{\pm 1}$-periodic word and an element from $\mathcal{T}$ is less than $4 < \beta n^2$. The remaining part of the proof proceeds as in [3].

In the formulation of Lemma 18.3 we need to consider only diagrams Δ_k of strict rank greater than 1/2. Moreover, we consider only those of diagrams Δ_k, for which the diagrams Γ_k^l and $\Gamma_k^l(m_1^k,\ m_2^k)$ constructed in the beginning of the proof of Lemma 18.3 and in Lemma 18.3.1 are of strict rank greater than 1/2. Indeed, by Lemma 17.3 applied to $\Gamma_k^l(m_1^k,\ m_2^k)$ in the case $r(\Gamma_k^l(m_1^k,\ m_2^k)) = 1/2$ and by construction of $\Gamma_k^l(m_1^k,\ m_2^k)$ and Δ_k^l, there are phase vertices $\bar{o}_1^k \in p_k$ and $\bar{o}_2^k \in q_k$ and a path $r^k = \bar{o}_1^k - \bar{o}_2^k$ in Δ_k such that $\phi(r^k) \overset{1/2}{=} 1$ and the subgroup

$$\langle A_{i+1}^l \phi(r^k) A_{i+1}^{-l} \mid l = 0,\ 1,\ 2,\ 3\rangle$$

of $G(i)$ is trivial.

In the estimate for l in the formulation of Lemma 18.3.1 we substitute i by $i+1$. The length of the chain (18.13) is now bounded from above by $i+1$. The remarks about ranks of diagrams $\Gamma_k^l(m_1^k,\ m_2^k)$ mean that $j^k = r(\pi_{m_0^k}) > 1/2$ for the cell $\pi_{m_0^k} \in \Delta_k(m_1^k,\ m_2^k)$ from the condition of Lemma 18.3.2. The proof of Lemma 18.3.1 proceeds as in [3] with obvious changes caused by new estimate for l. The proof of Lemma 18.3.2 is preserved.

In the condition of Lemma 18.3.3 $j^{k'} = r(\pi_{m_0^{k'}}) > 1/2$ by construction of $\Delta_{k'}(m_1^{k'},\ m_2^{k'})$; the proof remains unchanged. Notice that $r(\Delta_k(g)) > 1/2$ provided claim (*2*) of Lemma 18.3.3 holds.

The formulations and proofs of Lemmas 18.3.4 - 18.3.6 and the argument completing the proof of Lemma 18.3 are preserved.

The formulations of Lemmas 18.4, 18.4.1 and 18.4.1.1 are preserved. The diagrams constructed in the proof of part (a) of Lemma 18.4 and in the beginning of the proof of part (b) are of strict rank greater than 1/2 since otherwise, by Lemma 17.3, the word A_{i+1} would be of finite order in $G(i)$ provided S_k is nontrivial in $G(i)$.

In the condition of Lemma 18.4.3 we also allow the section q to be smooth $\mathcal{T}$-section of length $|q| > \beta n^2$. The proofs of Lemmas 18.4.2 and 18.4.3 do not change. In the case of a 1/2-cell Π considered in the proof of Lemma 18.4.2 every letter that occurs in $\phi(\partial\Pi)$ occurs also in $\phi(u_t)$, since $|u_t| > \beta|\partial\Pi| > \beta n^2$, the alphabet consists of two letters and the word $\phi(\partial\Pi)$ does not contain long periodic subwords.

The formulation of Lemma 18.5 does not change. In the proof of part (b) in the case when $r(\Delta) = 1/2$ Lemma 17.3 implies that some phase vertices $o_p \in p$ and $o_q \in q$ can be connected by a path of zero length.

The formulation of Lemma 19.1 is preserved. Notice that the possibility of the diagram Δ from the condition of Lemma 19.1 to be of strict rank 1/2 can be eliminated using similar arguments as in [7] (Lemma 18.9) as the group $G(1/2)$ is torsion free. The same argument can be also applied to the rigid subdiagram $\Delta(m_1,\ m_2)$ of Δ. Thus we may assume that $r(\Delta(m_1,\ m_2)) > 1/2$. The formulation and proof of Lemma 19.1.1 need not be changed in view of the above remarks. If now the contiguity subdiagram Δ_0 happened to be of strict rank 1/2, then Lemma 17.3 imply the equality (19.16) with $F_0 \overset{i}{=} 1$. The proof of Lemma 19.1 can be completed as in [3]. Lemmas 19.2 - 19.6 are formulated and proved without any changes. The formulations of Lemmas 20.1-20.3 are preserved. Proving Lemma 20.1, the diagram $\Delta(m_1,\ m_2)$ is constructed in the same way as in the proof of Lemma 17.4. The remaining part of the proof of Lemma 20.1, as well as proofs of Lemmas 20.2 and 20.3, proceed as in [3].

The induction is now complete. By $G_{\mathcal{T}}$ we denote the inductive limit of groups $G(i)$, $i = 0,\ 1/2,\ 1,\ 2,\ \ldots$:

$$G_{\mathcal{T}} = \langle a_1,\ a_2 \mid \{B = 1,\ B \in \mathcal{T}\} \cup \{A_i^n = 1\}_{i=1}^{\infty}\rangle\ .$$

3. Proofs of theorems

Proof of Theorem A. The claims that $G_{\mathcal{T}}$ is infinite and satisfies the identity $x^n = 1$ are deduced from Lemmas 10.4(a) and 18.2 in the same way as in [3]. The claim about the structure of finite subgroups of $G_{\mathcal{T}}$ follows, as in [3], from Lemma 15.9.

Assuming that the center of $G_{\mathcal{T}}$ is nontrivial, instead of the equality

(21.6) we consider the equality

$$Z\bar{D}Z^{-1}\bar{D}^{-1} \overset{i}{=} 1 \ ,$$

where $\bar{D}$ is obtained from the word D used in [3] by substituting every occurrence of a_1 by a_1^{-1}. Using the remarks made in the proof of Lemma 18.2 one can prove Lemma 21.2 and complete the argument as in [3].

Passing to a subset $\mathcal{T}'$ of $\mathcal{T}$ we may assume that presentation of the group $G_{\mathcal{T}'}$ is constructed. Let us show that the kernels of presentations of groups $G_{\mathcal{T}_1}$ and $G_{\mathcal{T}_2}$ are different for any two different subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$. Without loss of generality, there is a word $w \in \mathcal{T}_1 \setminus \mathcal{T}_2$. Thus, $w = 1$ in $G_{\mathcal{T}_1}$. Assuming that $w = 1$ in $G_{\mathcal{T}_2}$ we obtain a disk reduced diagram Δ of some rank i over the presentation of $G_{\mathcal{T}_2}$ with the boundary label w. The diagram Δ has cells since w is not equal to the identity in a free group. The word w is cyclically reduced. By Lemma 18.1 there is a cell π in Δ such that the length of a common subpath of $\partial\pi$ and $\partial\Delta$ is greater than $\beta|\partial\pi|$. But this is impossible by Lemma 0.1 and the choice of w. Therefore $w \neq 1$ in $G_{\mathcal{T}_2}$, and the kernels of presentations of $G_{\mathcal{T}_1}$ and $G_{\mathcal{T}_2}$ are different.

Thus, the groups $G_{\mathcal{T}_1}$ and $G_{\mathcal{T}_2}$ are quotients of a free group $F(a_1,\ a_2)$ over different normal subgroups provided $\mathcal{T}_1 \neq \mathcal{T}_2$. Note that there is only countably many different homomorphisms of a finitely generated group $F(a_1,\ a_2)$ onto a fixed countable group. So we conclude that the set of pairwise non-isomorphic groups among $\{G_{\mathcal{T}'}\}$ is of cardinality continuum, since so is the set of different subsets of $\mathcal{T}' \subseteq \mathcal{T}$ provided $\mathcal{T}$ is infinite.

To show the solvability of the word and conjugacy problems in $G_{\mathcal{T}'}$ in the case when $\mathcal{T}'$ is a recursive subset of $\mathcal{T}$, we repeat the arguments from [3]. In the proof of Lemma 21.1 the θ-cell Π may happen to be a 1/2-cell. In this situation Lemma 3.1 implies $|d_1| = |d_2| = 0$. Case 1 of the proof of Lemma 21.1 can be considered in the same way as in [3]. In Case 2 the subdiagram Γ given by $\partial\Gamma = w_1 u_1^{-1}$ contains cells since $r(\Gamma) = i + 1$. By Lemma 5.7 there is a θ-cell Π' in Γ. It follows from the choice of Γ that the contiguity degree of Π' to w_1 is less than α. The contiguity degree of Π' to the section u_1^{-1} is less than β by Lemma 2.1 applied to Δ. The inequality $\theta > \alpha + \beta$ means that Case 2 is impossible. □

Proof of Theorem B. Pick a subset $\mathcal{S}$ of $\mathcal{T}$ and obtain a new set of words $\mathcal{T}'$ in the following way. In every word from $\mathcal{S}$ we delete an arbitrary occurrence of a letter (a_1 or a_2). Denote the set thus obtained by $\bar{\mathcal{S}}$ and set $\mathcal{T}' = (\mathcal{T} \setminus \mathcal{S}) \cup \bar{\mathcal{S}}$. The set $\mathcal{T}'$ thus obtained has properties similar to the properties of the set $\mathcal{T}$ listed in Lemma 0.1.

More precisely, no cyclic shift of (an inverse of) an element of $\mathcal{T}'$ contains a B-periodic subword U of length greater than $(1+3\xi)|B|$ unless

$|B| < \xi^{-3/2}$ and $|U| < 23|B| < 23\xi^{-3/2}$; the symmetrized set obtained from the set $\mathcal{T}'$ satisfies the small cancellation condition $C'(\frac{3\xi}{10})$; and any word from $\mathcal{T}'$ is a positive word of length at least $n^2 - 1$. It follows that for any set $\mathcal{T}'$ obtained from the set $\mathcal{T}$ in the way described above one can use the scheme of Sections 1, 2 to construct the group $G_{\mathcal{T}'}$.

Now we are ready to make some alterations to the set $\mathcal{T}$. Consider the set $\mathcal{T}$ decomposed into pairs of words: $\{u_i,\ v_i\}$, $i = 1,\ 2,\ \dots$. For $i = 1,\ 2,\ \dots$ denote by u_i' (resp. v_i') the word obtained by deleting an arbitrary occurrence of a_1 (resp. a_2) from u_i (resp. v_i). For any sequence $\alpha = (\alpha_i)_{i=1}^{\infty}$ of 0's and 1's the set $\mathcal{T}_\alpha$ is constructed as follows: for every i, $i = 1,\ 2,\ \dots$, the words u_i, v_i are included in $\mathcal{T}_\alpha$ if $\alpha_i = 0$, and the words u_i', v_i' are included in $\mathcal{T}_\alpha$ otherwise. So, every set $\mathcal{T}_\alpha$ contains as a subset exactly one of the pairs $\{u_i,\ v_i\}$ or $\{u_i',\ v_i'\}$ for every i, $i = 1,\ 2,\ \dots$. The above remarks allow us to assume that the groups $G_{\mathcal{T}_\alpha}$ are constructed, and for every sequence α the group $G_{\mathcal{T}_\alpha} = F/N_\alpha$ ($F = F(a_1,\ a_2)$ is a free group) satisfies the identity $x^n = 1$. Note that $N_\alpha N_\beta = F$ for any two different sequences α and β. Indeed, for some index i the subgroup $N_\alpha N_\beta$ contains the words u_i, v_i, u_i' and v_i'. It follows that $a_1 = 1$ and $a_2 = 1$ in the quotient $F/N_\alpha N_\beta$, and therefore $N_\alpha N_\beta = F$.

Consider a quotient F/M_α of the group $G_{\mathcal{T}_\alpha}$ over its maximal proper normal subgroup. The group F/M_α is simple, and $M_\alpha \neq M_\beta$ for any two different sequences α and β. Indeed, $N_\alpha \subseteq M_\alpha$ for every sequence α. Therefore $M_\alpha M_\beta = F$ and consequently $M_\alpha \neq M_\beta$ since both are proper subgroups of F.

Thus, the set of different kernels M_α of homomorphisms of a free group F of rank 2 onto simple groups of exponent n is of cardinality continuum, and therefore so is the set of pairwise non-isomorphic groups in the collection $\{F/M_\alpha\}$. □

Acknowledgements

The author wishes to thank S.V. Ivanov for helpful discussions. The author is greatly thankful to A.Yu.Ol'shanskii for suggesting the problem, useful discussions, comments and constant encouragement.

References

[1] S. I. Adian, *The Burnside problem and identities in groups*, Nauka, Moscow, 1975; Translated in Ergebnisse der Mathematik und ihrer Grenzgebiete, **95**, Springer-Verlag, Berlin - New York, 1979.

[2] V. S. Atabekyan, *Residual properties of subgroups of free periodic groups*, deposited VINITI 22 July 1986, no. 5380-B86.

[3] S. V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Intern. J. of Algebra and Computation, **4**, 1994, no. 1,2.

[4] R. C. Lyndon, P. E. Schupp, *Combinatorial group theory*, Springer, Berlin, 1977.

[5] I. G. Lysenok, *Infinite Burnside groups of even exponent*, Izv. Ross. Akad. Nauk, Ser. Mat., **60**, 1996; Translated in Izv. Math. **60**, 1996, no. 3, pp. 453-654.

[6] A. Yu. Ol'shanskii, *On the Novikov-Adian theorem*, Matem. Sbornik, 118, 1982, no. 2, pp. 203-235; Translated in Math. USSR Sbornik, Vol. 46, 1983, pp. 203-236.

[7] A. Yu. Ol'shanskii, *The geometry of defining relations in groups*, Nauka, Moscow, 1989; Translated in Math and Its Applications (Soviet series), **70**, Kluwer Acad. Publishers, 1991.

[8] A. Yu. Ol'shanskii, M. V. Sapir, *Non-amenable finitely presented torsion-by-cyclic groups*, Publications of IHES, (96), 2002.

[9] B. H. Neumann, *Some remarks on infinite groups*, J. London Math. Soc., **12**, 1937.

[10] D. Sonkin, *CEP-subgroups of free Burnside groups of large odd exponents*, Comm. in Algebra., **31**, 2003, no. 10.

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Received by the editors: 12.06.2004
and final form in 15.12.2004.