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RESEARCH ARTICLE

On growth of the inverse semigroup of partially defined co–finite automorphisms of integers

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Abstract. The inverse semigroup of partially defined co–finite automorphisms of integers is considered. This semigroup is presented by generators and defining relations and its growth function is described.

1. Introduction

One of the main trends in the theory of semigroups is the study of objects by means of certain semigroups connected with the different kind of mathematical objects. For a graph, these special objects may be, e.g., automorphisms, endomorphisms, extensive transformations, etc. For the proposes of classifying and finding some characteristics of graphs were studied partial endomorphism semigroups of graphs (L.M.Gluskin, Yu.M.Važenin, A.Solomon, B.M.Schein, A.Vernitskii and others, see [13], [16]). If the set of endomorphisms of a graph is extended by including partial endomorphisms, we arrive at a more informative semigroup (in the papers of L.M.Popova, A.M.Kalmanovič, C.J.Maxson, V.A.Molčanov, V.H.Fernandes and others, see [3], [7], [9], [10], [11]).

Recently in modern combinatorial algebra has been observed the increasing of interest in growth functions, particulary for groups (see [5], [6], [15] and references therein), especially concerned with its relationships with various properties of algebraic objects ([1], [2], [4], [8], [14]). Also for many groups the growth function can be determined by calculated the growth function of some special monoid being an extension of given group ([2], [12]).

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The object of this paper is to introduce a semigroup of co–finite partial transformations (i.e. partial transformations with finite codomain) defined on a poset and to investigate the structure and properties of this semigroup. We consider the inverse semigroup ID_∞ of partially defined automorphisms of integers $\mathbb{Z}$ with a finite codomain, defined by the automorphism group of poset $\mathbb{Z}$. In Theorem 1 we describe semigroup ID_∞ by generators and defining relations:

Theorem 1. *The semigroup ID_∞ is an inverse monoid, and has the following generators and defining relations:*

$$ID_\infty =< a, b, f|$$

$$a^2 = 1, bb^{-1} = b^{-1}b = 1, aba = b^{-1}, f^2 = f, af = fa; \tag{1}$$

$$fb^{-r}fb^r = b^{-r}fb^rf, r \in \mathbb{N} > . \tag{2}$$

Such definition of ID_∞ makes it possible to compute its length function and therefore its growth function explicitly. We obtain that this semigroup has the exponential growth in contrary the polynomiallity of growth of the automorphism group of poset $\mathbb{Z}$. Theorem 2 is devoted the investigation of the growth function:

Theorem 2. *The inverse monoid ID_∞ has the exponential growth.*

2. Preliminaries

Let M be a semigroup generated by a finite set S. Then we may consider elements of M as words over the alphabet S and let h be a word in S. We denote the element of M corresponding to h as $\overline{h}$, and we define the length of h, $l_w(h)$, as the word length of h with respect to S. Recall that the length $l(m)$ of an element $m \in M$ (with respect to S) is the least number of factors in all representations of m as a product of elements of S, so $l(m) = \min\{l_w(h) : \overline{h} = m\}$. Obviously for any $m \in M$ the length $l(m)$ is great than 0, let's put $l(1) = 0$ when the semigroup M is a monoid. The function

$$\gamma_{M,S} : \mathbb{N} \to \mathbb{N} \quad (\mathbb{N} \text{ is the set of all natural numbers})$$

defined by

$$\gamma_{M,S}(n) = |\{w \in M : l(w) \le n\}|$$

(where |E| means the cardinality of set E) is called the growth function of M with respect to the set of generators S.

We say that a function $f : \mathbb{N} \to \mathbb{R}$ is dominated by a function $g : \mathbb{N} \to \mathbb{R}$, denote by $f \preceq g$, if there are constants $C_1, C_2, N_0 \in \mathbb{N}$ such that

$$f(n) \leq C_1 \; g(C_2 \; n)$$

for all $n \geq N_0$. Two functions $f, g : \mathbb{N} \to \mathbb{R}$ are called equivalent, denoted by $f \sim g$, if $f \preceq g$ and $g \preceq f$. It is well known that for any two finite sets of generators S_1, S_2 of semigroup M the corresponding two growth functions $\gamma_{M,S_1}(n)$ and $\gamma_{M,S_2}(n)$ are equivalent, that is to say a semigroup growth is independent of the choice of generators. For this reason let denote the growth function of M by $\gamma_M(n)$ or briefly $\gamma(n)$. Note also that if $|S| = k$, then $\gamma(n) \leq k^n$.

A growth of semigroup M is called exponential if $\gamma(n) \sim e^n$. A growth of semigroup M is called polynomial if $\gamma(n) \sim n^c$ for some $c > 0$. A growth of semigroup M is called intermediate if M hasn't neither an exponential nor a polynomial growth.

3. General construction of the inverse monoid IG

For convenience, regarding the purpose of this paper, in what follows we will always consider a monoid rather than a semigroup.

Let X be a nonempty poset. An order preserving partial automorphism of set X is a bijection $g : Dom\ g \to Ran\ g$ between subsets $Dom\ g$ and $Ran\ g$ of set X, that is to say, g is a partial permutation of set X, which preserves order on the set X. The set $Dom\ g$ is the domain of the automorphism g and $Ran\ g$ is the range. The automorphism g is defined in a point $x \in X$ if and only if $x \in Dom\ g$. The set $X \backslash Dom\ g$ is the codomain of g (let's denote its by $Codom\ g$).

The composition of two partial automorphisms g_1 and g_2 is the partial automorphism $g_1 g_2$ with the domain $Dom\ g_1 g_2 = g_1^{-1}(Dom\ g_2 \bigcap Ran\ g_1)$ and defined on the domain by the condition $g_1 g_2(x) = g_2(g_1(x))$ (here the composition acts from the left to the right). For the partial automorphism g of set X its inverse is the automorphism g^{-1} with the domain $Dom\ g^{-1} = Ran\ g$ and the range $Ran\ g^{-1} = Dom\ g$ such that the product gg^{-1} is the identical map $id|_{Ran\ g}$ of set $Ran\ g$ (or, equivalently, $g^{-1}g$ is the identical map of set $Dom\ g$).

Let $IS(X)$ be a set of all partial automorphisms of poset X. Then $IS(X)$ is closed with respect to the composition of automorphisms and $IS(X)$ is the inverse partial automorphism monoid action on the set X.

Let's now G be an infinite group of automorphisms over a poset X, acting transitively on X. For a subset $Y \subseteq X$ and an automorphism $g \in G$ let's denote by g_Y the partial automorphism with $Dom\ g_Y = X \setminus Y$ and

Codom $g_Y = Y$ such that $g_Y|_{Dom\ g_Y} = g|_{Dom\ g_Y}$ (here the symbol $f|_A$ is used to denote the restriction of f onto the set A). We put separately $g_\emptyset = g$. The set IG of all such defined partial automorphisms g_Y with a finite codomain Y forms the monoid under the composition. And it is clear that any partial automorphism of IG is also a partial automorphism of $IS(X)$, i.e. the monoid $IS(X)$ contains IG.

Definition 1. *The inverse monoid IG are called an inverse monoid of partially defined automorphisms of X with a finite codomain, defined by the group G of partial order preserving automorphisms of poset X, acting transitively on X.*

For some fixed point $x_0 \in X$ by e_{x_0} we denote a partial automorphism with *Codom* $e_{x_0} = \{x_0\}$, which acts identically on *Dom* $e_{x_0} = X\setminus \{x_0\}$. The partial automorphism e_{x_0} is an idempotent of IG and

Proposition 1. *Let S be a set of generators of G. Then the monoid IG is generated by $S\bigcup\{e_{x_0}\}$.*

Proof. Let g_Y, where $g \in G$, $Y = \{y_1, \ldots, y_k\} \subset X$, be an arbitrary element. The group G acts transitively on X, so that there exists $h_1, \ldots, h_k \in G$ such that $y_1^{h_1} = x_0$, $\ldots$, $y_k^{h_k} = x_0$. Then $h = h_1 e_{x_0} h_1^{-1} \ldots h_k e_{x_0} h_k^{-1} \in IG$, *Codom* $h = Y$, $h|_{X\setminus Y} = id|_{X\setminus Y}$ and $g_Y = hg$. Therefore, $IG \subseteq < S, e_{x_0} >$.

The inclusion $IG \supseteq < S, e_{x_0} >$ is obvious. □

From Proposition 1 it is easy to obtain

Corollary 1. *Every $g \in ID_\infty$ can be expressed in the form*

$$g = w_1 e_{x_0} w_2 e_{x_0} \ldots w_{t-1} e_{x_0} w_t,$$

where w_2, $\ldots$, $w_{t-1} \in G\setminus\{1\}$, $w_1, w_t \in G$, $t \in \mathbb{N}$.

Corollary 2. *1. An idempotent of IG has the following form $w_1 e_{x_0} \ldots e_{x_0} w_t$, where w_2, $\ldots$, $w_{t-1} \in G\setminus\{1\}$, $w_1, w_t \in G$, $t \in \mathbb{N}$, such that $(w_1 \ldots w_t)|_Y = id|_Y$ for $Y = \{y_1, \ldots, y_{t-1}\}$ and $y_1^{w_1} = x_0$, $\ldots$, $y_{t-1}^{w_1 \ldots w_{t-1}} = x_0$.*

2. All idempotents of IG form a lower semilattice.

Proof. 1. Let $g \in IG$ be an idempotent, that is $g = w_1 e_{x_0} \ldots e_{x_0} w_t$, where w_2, $\ldots$, $w_{t-1} \in G\setminus\{1\}$, $w_1, w_t \in G$, $t \in \mathbb{N}$, and $g^2 = g$. Then *Codom* $g = \{y_1, \ldots, y_{t-1}\}$, where $y_1^{w_1} = x_0$, $\ldots$, $y_{t-1}^{w_1 \ldots w_{t-1}} = x_0$, and *Codom* $g^2 = \{y_1, \ldots, y_{t-1}, \breve{y}_1, \ldots, \breve{y}_{t-1}\}$, where $\breve{y}_1^{w_1 \ldots w_t w_1} = x_0$, $\ldots$, $\breve{y}_{t-1}^{w_1 \ldots w_t w_1 \ldots w_{t-1}} = x_0$. Thus $\{\breve{y}_1, \ldots, \breve{y}_{t-1}\} \subseteq \{y_1, \ldots, y_{t-1}\}$ and $(w_1 \ldots w_t)|_{Dom\ g} = (w_1 \ldots w_t)^2|_{Dom\ g}$. □

Remark 1. *We would like to make a notice that it is possible to define such construction for any relative algebra X with transitive action of $Aut\ X$ on X.*

4. Presentation for the inverse monoid ID_∞

Let $G = D_\infty$ be the automorphism group of integers $\mathbb{Z}$, that is to say $D_\infty =< a, b | a^2 = 1, aba = b^{-1} >$. And let's denote by ID_∞ the semigroup of partially defined automorphisms of $\mathbb{Z}$ with a finite codomain defined by D_∞. Then from Proposition 1 we obtain that ID_∞ is generated by following partial automorphisms a, b, f, where for all $x \in \mathbb{Z}$ $a : x \to -x$, $b : x \to x+1$, $Codom\ f = \{0\}$ and for all $x \in \mathbb{Z}\backslash\{0\}$ $f : x \to x$. Hence we can identify elements of semigroup ID_∞ with correspond semigroup words over the alphabet $S = \{a, b, f\}$.

Proposition 2. *Every $g \in ID_\infty$ admits an unique representation as a word of the form*

$$h_1 \dots h_t h,\ \textit{where}\ h_i = b^{-\alpha_i} f b^{\alpha_i}\ \textit{for}\ i \in \{1, \dots, t\}, h = b^k a^\varepsilon, \tag{3}$$

$$\alpha_1, \dots, \alpha_t \in \mathbb{Z}\ \textit{such that}\ \alpha_1 < \dots < \alpha_t, t \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}, \varepsilon \in \{0, 1\}.$$

Proof. Since $a^2 = 1$, $aba = b^{-1}$ and also clearly that $f^2 = f$, $af = fa$, it follows that every $g \in ID_\infty$ admits a representation as a word of such form $g = b^{k_1} f b^{k_2} \dots b^{k_{s-1}} f b^{k_s} a^\varepsilon$, where $k_1, \dots, k_s \in \mathbb{Z}$, $\varepsilon \in \{0, 1\}$. Then $g = b^{k_1} f b^{-k_1}\ b^{k_1+k_2} f\ \dots\ f b^{-k_1-\dots-k_{s-1}}\ b^{k_1+\dots+k_s} a^\varepsilon$. But elements $b^k f b^{-k}$ and $b^l f b^{-l}$ are commuted because of Corollary 2 they are idempotents. Moreover, $b^k f b^{-k} b^k f b^{-k} = b^k f b^{-k}$, and this proves the existence of the representation (3). Now let's prove an unique of (3). Let $g = h_1 \dots h_t h = \tilde{h}_1 \dots \tilde{h}_s \tilde{h}$ be two representations of the form (3), where $h_i = b^{-\alpha_i} f b^{\alpha_i}$, $i \in \{1, \dots, t\}$, $h = b^{k_1} a^{\varepsilon_1}$, $\tilde{h}_j = b^{-\beta_j} f b^{\beta_j}$, $j \in \{1, \dots, s\}$, $\tilde{h} = b^{k_2} a^{\varepsilon_2}$. Then $Codom\ g = Codom\ (h_1 \dots h_t) = Codom\ (\tilde{h}_1 \dots \tilde{h}_s)$ and by virtue of Corollary 2 $(h_1 \dots h_t)|_{Dom\ g} = (\tilde{h}_1 \dots \tilde{h}_s)|_{Dom\ g} = id|_{Dom\ g}$ we conclude that $h|_{Dom\ g} = \tilde{h}|_{Dom\ g}$ and hence $h = \tilde{h}$. Moreover,

$$Codom\ (h_1 \dots h_t) = \{\alpha_1, \dots, \alpha_t\},\ Codom\ (\tilde{h}_1 \dots \tilde{h}_s) = \{\beta_1, \dots, \beta_s\}$$

and $\alpha_1 < \dots < \alpha_t$, $\beta_1 < \dots < \beta_s$. Therefore we can see easily that $t = s$ and $\alpha_1 = \beta_1, \dots, \alpha_t = \beta_t$. □

Definition 2. *Let $g \in ID_\infty$ be an arbitrary element. The canonical form of this element is a representation as a word of the form* (3) *(let's denote the representation* (3) *of an element g by $[g]$). Also, the number $r \in \{1, \dots, t\}$, such that $\alpha_1 < \dots < \alpha_r < 0 \leq \alpha_{r+1} < \dots < \alpha_t$, is called the crossing index and is denoted by $i([g])$.*

5. Proof of Theorem 1

Proof. It is easy to check that the relations (1) and (2) hold. Moreover, from the proof of Proposition 2 we also can obtain the following commuting of idempotents from ID_∞ :

$$b^{-l}fb^lb^{-k}fb^k = b^{-k}fb^kb^{-l}fb^l,\ l,k \in \mathbb{Z},\ l \neq k, \tag{4}$$

where $k = r + l$. And by using relations (1) and (4) each elements can be ambiguously reduced to the canonical form (3). If semigroup elements $g_1 = b^{-\alpha_1}fb^{\alpha_1}\dots b^{-\alpha_t}fb^{\alpha_t}b^ka^{\varepsilon_1}$ and $g_2 = b^{-\beta_1}fb^{\beta_1}\dots b^{-\beta_s}fb^{\beta_s}b^la^{\varepsilon_2}$ define the same partial automorphism over $\mathbb{Z}$, then we can easily conclude that $t = s$, $\alpha_1 = \beta_1$, $\dots$, $\alpha_t = \beta_t$, $\varepsilon_1 = \varepsilon_2$ and $k = l$ in a way analogous to that used in the proof of Proposition 2. Therefore, semigroup elements, which are written in different canonical forms, define different partial automorphisms. The theorem is completely proved. □

Remark 2. *The infinite system of relations* (1) *and* (2) *is independent, i.e., none of these relations can be derived from the rest.*

6. Length function

We have already been discussed that every element g of semigroup ID_∞ is identified with a semigroup word over the alphabet $S = \{a, b, f\}$ in the canonical form (3). Then the length $l_w([g])$ of semigroup word $[g]$ is equal to $l_w([g]) = |\alpha_1| + 1 + |\alpha_1 - \alpha_2| + 1 + \dots + |\alpha_{t-1} - \alpha_t| + 1 + |\alpha_t + k| + \varepsilon = |\alpha_1| + \alpha_t - \alpha_1 + |\alpha_t + k| + t + \varepsilon$, as $\alpha_1 < \dots < \alpha_t$. Denote by g_π the word $h_{\pi(1)}\dots h_{\pi(t)}h$ for some permutation $\pi \in S_t$. Likewise, the length of g_π is equal to $l_w(g_\pi) =$

$$|\alpha_{\pi(1)}|+1+|\alpha_{\pi(1)}-\alpha_{\pi(2)}|+1+\dots+|\alpha_{\pi(t-1)}-\alpha_{\pi(t)}|+1+|\alpha_{\pi(t)}+k|+\varepsilon, \tag{5}$$

and

Proposition 3. *For any arbitrary element $g \in ID_\infty$*

$$l(g) = \min_{\pi\in S_t}\{l_w(g_\pi)\}, \tag{6}$$

where $g = h_1 \dots h_th$ is the canonical form of element g.

Proof. Let π be a permutation in S_t such that $l_w(g_\pi) = \min_{\pi\in S_t}\{l_w(g_\pi)\}$. The inequality $l(g) \leq l_w(g_\pi)$ is obviously. Let $d = d_1d_2\dots d_j$ be a semigroup word such that $l(g) = l_w(d) = j < l_w(g_\pi)$. Suppose that for some $i \in \{1, \dots, j-1\}$ $d_i = a$. Then $d_{i+1} \neq a$. If $d_{i+1} = f$ it follows from (1) that $d = d_1\dots d_{i-1}d_{i+1}d_id_{i+2}\dots d_j$, and hence $d_{i+2} \neq a$,

$d_{i+2} \neq f$ because in another way $l_w(d_1 \dots d_{i-1}d_{i+1}d_i d_{i+2} \dots d_j) = j-1$ and $l(g) < j$. If $d_{i+1} = b$, then similarly $d = d_1 \dots d_{i-1}d_{i+1}d_i d_{i+2} \dots d_j$ and hence $d_{i+2} \neq a$. So, we may further rewrite d so that it has the form $d = b^{\delta_1} f b^{\delta_2} f \dots f b^{\delta_q} a^{\mu}$, where $\delta_2 \neq 0, \dots, \delta_{q-1} \neq 0$, $\mu \in \{0,1\}$ and $|\delta_1| + \ldots + |\delta_q| + (q-1) + \mu = j$. Our aim now to show that $l_w(d) = l_w(g_\pi)$ for some permutation $\pi \in S_t$. Since $d = g_\pi$ by Proposition 2 we have that $h = b^{\delta_1 + \ldots + \delta_q} a^{\mu}$ and

$$b^{\delta_1} f b^{-\delta_1} b^{\delta_1+\delta_2} f b^{-\delta_1-\delta_2} \dots b^{\delta_1+\ldots+\delta_{q-1}} f b^{-\delta_1-\ldots-\delta_{q-1}} = h_1 \dots h_t$$

are idempotents. From $l(g) = l_w(d)$ it follows that all $\delta_1, \delta_1+\delta_2, \dots, \delta_1 + \ldots + \delta_{q-1}$ are different. Now the required equality follows from the comparison of codomains $\{\alpha_1, \dots, \alpha_t\}$ and $\{\delta_1, \delta_1+\delta_2, \dots, \delta_1+\ldots+\delta_{q-1}\}$ of idempotents under the consideration and Corollary 2. $\square$

We can compute the length function $l(g)$ explicitly.

Proposition 4. *Let g be an arbitrary element of ID_∞, $[g]$ is its canonical form* (3) *and $r = i([g])$ is its crossing index. Then for $k > 0$*

$$l(g) = \begin{cases} 2\alpha_t - \alpha_1 + |\alpha_1 + k| + t + \varepsilon, & \text{if } r \neq 0, t, \\ 2\alpha_t + k + t + \varepsilon, & \text{if } r = 0, \\ -\alpha_1 + |\alpha_1 + k| + t + \varepsilon, & \text{if } r = t; \end{cases}$$

and for $k \leq 0$

$$l(g) = \begin{cases} -2\alpha_1 + \alpha_t + |\alpha_t + k| + t + \varepsilon, & \text{if } r \neq 0, t, \\ \alpha_t + |\alpha_t + k| + t + \varepsilon, & \text{if } r = 0, \\ -2\alpha_1 - k + t + \varepsilon, & \text{if } r = t. \end{cases}$$

Proof. It follows from (5) and (6) that for $k > 0$ the length of word g_π will be the least when

$$\pi(1) = r+1, \ \dots, \ \pi(t-r) = t, \ \pi(t-r+1) = r, \ \dots, \ \pi(t) = 1, \quad (7)$$

that is $l(g) = l_w(g_\pi) = l_w(h_{r+1} \dots h_t h_r \dots h_1 h)$. For $k \leq 0$ the length will be the least when

$$\pi(1) = 1, \ \dots, \ \pi(t) = t, \quad (8)$$

and we have $l(g) = l_w(h_1 \dots h_r h_{r+1} \dots h_t h)$. It remains to count by using (5) the length $l_w(g_\pi)$ in all cases which have been indicated above. $\square$

Definition 3. *Let g be an arbitrary element of ID_∞, $[g]$ is its canonical form* (3) *and $r = i([g])$ is its crossing index. An amplification canonical form of element g (let's denote it by $[g]_{amp}$) is the representation g as a word g_π, were $\pi \in S_t$ is a permutation of the form* (7) *for $k > 0$ and the form* (8) *for $k \leq 0$.*

7. Growth function for ID_∞

From now let $\gamma(n)$ be the growth function of ID_∞, that is $\gamma(n)$ is the number of all irreducible elements in the semigroup ID_∞ of a length not more that n. We can to compute the growth function $\gamma(n)$ explicitly in the following way.

Let $g \in ID_\infty$ and $h_{\pi(1)} \dots h_{\pi(t)} h$, where $h = b^k a^\varepsilon$, be the amplification canonical form $[g]_{amp}$ of element g. Denote by $k(g)$ the index k, by $\varepsilon(g)$ the index ε in h, and will be denote by $t(g)$ the number t in $[g]_{amp}$.

Now let's fix some $m \in \mathbb{N}$ and find the number of all g from ID_∞ such that $l(g) = m$, $k(g) = 0$, $i([g]) = 0$, $\varepsilon(g) = 0$ (denote this number by $Q_1(m)$). By Proposition 4 and relation (5) for all such elements we have $l(g) = m = 2\alpha_t + t = \alpha_1 + (\alpha_2 - \alpha_1) + \dots + (\alpha_t - \alpha_{t-1}) + \alpha_t + t$. Count the number $p_{1,m,t}$ of decompositions of $\alpha_t = \frac{m-t}{2}$ into t terms $k_1 = \alpha_1 \geq 0$, $k_2 = \alpha_2 - \alpha_1 > 0, \dots, k_t = \alpha_t - \alpha_{t-1} > 0$. So, $p_{1,m,t}$ is equal to $\binom{(m-t)/2}{t-1}$ when $m - t$ is even, and is equal to 0 when $m - t$ is odd. The parameter t varies from 1 to $[\frac{m+2}{3}]$ (here $[x]$ is the integer part of number x), because of $\alpha_t = \frac{m-t}{2} \geq t - 1$. Hence $Q_1(m) = \sum_{t=1}^{[(m+2)/3]} p_{1,m,t}$.

Find the number of all g from ID_∞ such that $l(g) = m$, $k(g) = 0$, $i([g]) \neq 0, t$, $\varepsilon(g) = 0$ (denote this number by $Q_2(m)$). Then we have $l(g) = m = 2(\alpha_t - \alpha_1) + t = -\alpha_1 + (\alpha_2 - \alpha_1) + \dots + (\alpha_t - \alpha_{t-1}) + \alpha_t + t$. Count the number $p_{2,m,t}$ of decompositions of $\alpha_t - \alpha_1 = \frac{m-t}{2}$ into $t - 1$ terms $k_1 = \alpha_2 - \alpha_1 > 0, \dots, k_{t-1} = \alpha_t - \alpha_{t-1} > 0$. So, $p_{2,m,t}$ is equal to $\binom{(m-t)/2-1}{t-2}$ when $m - t$ is even, and is equal to 0 when $m - t$ is odd. The number $q_{2,m,t}$ of decompositions of $\frac{m-t}{2}$ into terms $s_1 = \alpha_t \geq 0$ and $s_2 = -\alpha_1 > 0$ is equal to $\frac{m-t}{2}$ when $m - t$ is even, and is equal to 0 when $m - t$ is odd. The parameter t varies from 1 to $[\frac{m+2}{3}]$, because of $\alpha_t - \alpha_1 = \frac{m-t}{2} \geq t - 1$. Hence $Q_2(m) = \sum_{t=1}^{[(m+2)/3]} p_{2,m,t} q_{2,m,t}$.

Next find the number of all g from ID_∞ such that $l(g) = m$, $k(g) > 0$, $i([g]) = 0$, $\varepsilon(g) = 0$ (denote this number by $Q_3(m)$). Then we have $l(g) = m = 2\alpha_t + t + k = \alpha_1 + (\alpha_2 - \alpha_1) + \dots + (\alpha_t - \alpha_{t-1}) + \alpha_t + t + k$. Count the number $p_{3,m,t}$ of decompositions of $i = \alpha_t$ into t terms $k_1 = \alpha_1 \geq 0$, $k_2 = \alpha_2 - \alpha_1 > 0, \dots, k_t = \alpha_t - \alpha_{t-1} > 0$. So, $p_{3,m,t}$ is equal to $\binom{i}{t-1}$. The parameter t varies from 1 to $[\frac{m+1}{3}]$ as before, and the parameter i varies from $t - 1$ to $[\frac{m-t-1}{2}]$ because of $i = \alpha_t$. Hence $Q_3(m) = \sum_{t=1}^{[(m+1)/3]} \sum_{i=t-1}^{[(m-t-1)/2]} p_{3,m,t}$.

Now let's find the number of all g from ID_∞ such that $l(g) = m$, $k(g) > 0$, $r := i([g]) \neq 0, t$, $\varepsilon(g) = 0$ (denote this number by $Q_4(m)$). Then we have $l(g) = m = 2\alpha_t - \alpha_1 + |\alpha_1 + k| + t = \alpha_{r+1} + (\alpha_{r+2} - \alpha_{r+1}) + \dots + (\alpha_t - \alpha_{t-1}) + (\alpha_t - \alpha_r) + (\alpha_r - \alpha_{r-1}) + \dots + (\alpha_2 - \alpha_1) + |\alpha_1 + k| + t$. Count the number $p_{4,m,t}$ of decompositions of $i = \alpha_t - \alpha_1$ into t terms

$k_1 = \alpha_{r+1} \geq 0$, $k_2 = \alpha_{r+2} - \alpha_{r+1} > 0, \ldots, k_{t-r} = \alpha_t - \alpha_{t-1} > 0$, $k_{t-r+1} = -\alpha_r > 0$, $k_{t-r+2} = \alpha_r - \alpha_{r-1} > 0, \ldots, k_t = \alpha_2 - \alpha_1 > 0$. So, $p_{4,m,t}$ is equal to $\binom{i}{t-1}$. The parameter t varies from 1 to $[\frac{m}{2}]$ and the parameter i varies from t to $m-t$ as before. Hence $Q_4(m) = \sum_{t=1}^{[m/2]} \sum_{i=t}^{m-t} p_{4,m,t}$.

Finitely, find the number of all g from ID_∞ such that $l(g) = m$, $k(g) < 0$, $i([g]) \neq 0, t$, $\varepsilon(g) = 0$ (denote this number by $Q_5(m)$). Then we have $l(g) = m = -2\alpha_1 + \alpha_t + |\alpha_t + k| + t = -\alpha_1 + (\alpha_2 - \alpha_1) + \ldots + (\alpha_t - \alpha_{t-1}) + |\alpha_t + k| + t$. Count the number $p_{5,m,t}$ of decompositions of $i = \alpha_t - \alpha_1$ into $t-1$ terms $k_1 = \alpha_2 - \alpha_1 > 0, \ldots, k_{t-1} = \alpha_t - \alpha_{t-1} > 0$. So, $p_{5,m,t}$ is equal to $\binom{i-1}{t-2}$. The number $q_{5,m,t}$ of decompositions of $m-2t$ into terms $s_1 = -\alpha_1 > 0$, $s_2 = \alpha_t - \alpha_1 - t \geq 0$ and $s_3 = |\alpha_t + k| \geq 0$ is equal to $\frac{(m-2t)(m-2t+1)}{2}$. The parameter t varies from 1 to $[\frac{m-1}{2}]$ and the parameter i varies from t to $m-t-1$ as before. Hence $Q_5(m) = \sum_{t=1}^{[(m-1)/2]} \sum_{i=t}^{m-t-1} p_{5,m,t} q_{5,m,t}$.

In the same way we can consider the another cases of Proposition 4 and count $Q_6(m), \ldots, Q_{18}(m)$, but its will be omitted.

Proposition 5. *The growth function $\gamma(n)$ of ID_∞ is equal to the following sum*

$$\sum_{m=1}^{n} (Q_1(m) + \ldots + Q_{18}(m)) \; .$$

8. Proof of Theorem 2

Now let's prove Theorem 2.

Proof. Let us separately consider the sum

$$\Sigma(n) = \sum_{m=1}^{n} Q_1(m).$$

Denote by $q(s,t)$ the binomial coefficient $\binom{\frac{s-t}{2}}{t-1}$. Then we have

$$Q_1(m) = \sum_{\substack{t=1 \\ m-t \text{ is even}}}^{[\frac{m+2}{3}]} q(m,t).$$

Let $N = [\frac{n}{3}]$. Then $q(2N+1),\ q(2N+2,2),\ \ldots,\ q(3N,N) \in \{q(s,t), t \in \{1, \ldots, m-t\}, m \in \{1, \ldots, n\}, m-t \text{ is even}\}$, that is such binomial coefficients are different components of sum $\Sigma(n)$ because of $3N+1 = 3[\frac{n}{3}] \leq n$. So, $\Sigma(n) > 2^{[\frac{n}{3}]}$ and hence ID_∞ has the exponential growth.

The theorem is completely proved. $\square$

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