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On groups whose subgroups of infinite special rank are transitively normal

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Abstract. This paper sheds a light on periodic soluble groups whose subgroups of infinite special rank are transitively normal.

Introduction

Groups with certain prescribed properties of subgroups form one of the central subjects of research in group theory. Their investigation introduced many important notions such as finiteness conditions, local nilpotence, local solubility, group rank, and others. Choosing specific prescribed properties and concrete families of subgroups, which possess these properties, we come to distinct classes of groups. There is an enormous array of papers devoted to these topics. This particular article discloses the influence of a family of subgroups of finite special rank and the one of transitively normal groups on a group structure.

A group G is said to have a finite special rank r if every finitely generated subgroup of G can be generated by at most r elements and there exists a finitely generated subgroup H, which has exactly r generators [1]. The theory of groups of finite special rank is one of the most profoundly developed parts of the group theory (for instance, surveys [2, 3, 4]). In a paper [5] M.R. Dixon, M.J. Evans and H. Smith have considered groups

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whose subgroups of infinite special rank have some fixed property P. A bunch of authors expanded the research area taking into account distinct natural properties P (for example, survey [4]). This paper focuses on groups whose subgroups of infinite special rank are transitively normal.

A subgroup H of a group G is transitively normal if H is normal in every subgroup $K \geqslant H$, in which H is subnormal [6]. In [7] these subgroups have been introduced under a different denomination. Namely, a subgroup H of a group G is said to satisfy the subnormalizer condition in G if for every subgroup K, such that H is normal in K, we have $N_G(K) \leqslant N_G(H)$. There are many natural types of subgroups, which are transitively normal, for instance, pronormal subgroups and their generalizations (see [8]).

A relation "to be a normal subgroup" is not transitive. A group G is said to be a T-group if this relation is transitive in G. A group G is said to be a $\bar{T}$-group if every subgroup of G is a T-group. It is obvious that every subgroup of G is transitively normal if and only if G is a $\bar{T}$-group.

A locally nilpotent residual G^{LN} of a group G is the intersection of all normal subgroups H such that G/H is locally nilpotent. Note, that if G is locally finite, then G/G^{LN} is locally nilpotent.

Theorem 1. *Let G be a locally finite group whose subgroups are transitively normal, and let L be a locally nilpotent residual of G. Then G satisfies the following conditions:*

(i) *L is an abelian and G is a metabelian group;*
(ii) *Every subgroup of L is a G-invariant;*
(iii) *$2 \notin \Pi(L)$;*
(iv) *$\Pi(L) \bigcap \Pi(G/L) = \varnothing$;*
(v) *G/L is a Dedekind group and $G/C_G(L)$ is an abelian group.*

Conversely, if G satisfies conditions (i)–(v)*, then every subgroup of G is a transitively normal one.*

Indeed, a finite group whose subgroups are transitively normal is a metabelian one [9], so that G is a metabelian group. Now, we can apply the results of [10].

This paper aims at describing periodic soluble groups whose subgroups of infinite special rank are transitively normal. The main results are summarized as follows.

Theorem 2. *Let G be a periodic soluble group of infinite special rank whose subgroups of infinite special rank are transitively normal. Then every subgroup of G is a transitively normal one.*

1. On the structure of locally nilpotent subgroups

Lemma 1. *Let G be a group whose subgroups of infinite special rank are transitively normal.*

If H is a subgroup of G, then every subgroup of H with infinite special rank is transitively normal in H.

If L is a normal subgroup of G, such that G/H has infinite special rank, then every subgroup of G/L with infinite special rank is transitively normal in G/L.

If U, V are subgroups of G, such that U is normal in V and V/U has infinite special rank, then every subgroup of V/U with infinite special rank is transitively normal in V/U.

The proof of this assertion is evident.

Lemma 2. *Let G be a group and F be a finite subgroup of G. Suppose that A is an infinite elementary abelian p-subgroup of G, where p is a prime. If A is an F-invariant, then A includes a subgroup $B = Dr_{n\in N} B_n$ where B_n is a finite F-invariant subgroup for each $n \in N$ and $F \bigcap B = \langle 1\rangle$.*

Proof. We have $A = Dr_{\lambda\in\Lambda}\langle a_\lambda\rangle$, where the set of indices Λ is infinite and $|a_\lambda| = p$ for each $\lambda \in \Lambda$. Since $F \bigcap A$ is cyclic, there exists a finite subset $M \subseteq \Lambda$, such that $Supp(F \bigcap A) \subseteq Dr_{\lambda\in M}\langle a_\lambda\rangle$. Then a subset $\sum = \Lambda M$, is infinite and $F \bigcap Dr_{\lambda\in\Sigma}\langle a_\lambda\rangle = \langle 1\rangle$. Put $Dr_{\lambda\in\Sigma}\langle a_\lambda\rangle = A_0$. Then an index $|A : A_0|$ is finite. The subgroup A_0^x has finite index in A for each element $x \in F$ and family $\{A_0^x | x \in F\}$ is finite, because a subgroup F is finite. Then, a subgroup $\bigcap_{x\in F} A_0^x = D$ has finite index in A. In particular, D is infinite. Since $D \leqslant A_0$, $F \bigcap D = \langle 1\rangle$. By the choice, a subgroup D is an F-invariant. Let $1 \neq b_1 \in D$ and $B_1 = \langle b_1\rangle^F$. Since D is an elementary abelian subgroup, then there exists a subgroup A_1 of D, such that $D = B_1 \times A_1$. Finiteness of B_1 implies that a subgroup A_1 has finite index in D. Using the above-mentioned arguments we obtain that a subgroup $\bigcap_{x\in F} A_1^x = D_1$ has finite index in D. An inclusion $D_1 \leqslant A_1$ shows that $B_1 \bigcap D_1 = \langle 1\rangle$. By the choice, a subgroup D_1 is an F-invariant. Let $1 \neq b_2 \in D_1$ and $B_2 = \langle b_2\rangle^F$. A subgroup B_2 is finite and $B_1 \bigcap B_2 = \langle 1\rangle$. Since D is an elementary abelian subgroup, then there exists a subgroup A_2 of D, such that $D = (B_1B_2) \times A_2$. Repeating the above-mentioned arguments, we construct a family $\{B_n | n \in N\}$ of finite F-invariant subgroups, such that $B = \langle B_n | n \in N\rangle = Dr_{n\in N} B_n$ and $B \bigcap F = \langle 1\rangle$. □

Lemma 3. *Let G be a group whose subgroups of infinite special rank are transitively normal. If L is a periodic locally nilpotent subgroup of G with infinite special rank, then L is a Dedekind subgroup.*

Proof. Firstly, suppose that there exists a prime p, such that the Sylow p-subgroup P of L has infinite special rank. Let $1 \neq g \in P$, $1 \neq x \in P$ and $F = \langle g, x \rangle$. Then F is a finite subgroup. Since P has infinite special rank, it includes an infinite elementary abelian subgroup A, which is an F-invariant [11]. By Lemma 2 A includes an infinite subgroup $B = Dr_{n \in N} B_n$ such that $B_n = \langle b_n \rangle^F$ and $B \bigcap F = \langle 1 \rangle$. In particular, $|B_n| \leqslant p^k$, where $k = |F|$. Since FB_n is a p-subgroup, $B_n \leqslant \zeta_k(FB_n)$. It is true for each $n \in N$, therefore $B \leqslant \zeta_k(FB)$. It follows that a subgroup FB is a nilpotent one. Then every subgroup of FB is a subnormal one.

Choose two infinite subsets Γ, $\Delta \subseteq N$, such that $\Gamma \bigcap \Delta = \varnothing$ and $\Gamma \bigcup \Delta = N$. Then each subgroup $C = Dr_{n \in \Gamma} B_n$ and $D = Dr_{n \in \Delta} B_n$ is an infinite and F-invariant one. Then the both subgroups $\langle g \rangle C$ and $\langle g \rangle D$ have infinite special rank, and, hence, are transitively normal in G. Being transitively normal and subnormal in FB, $\langle g \rangle C$ and $\langle g \rangle D$ are normal in FB. The choice of subgroups C, D implies that $\langle g \rangle C \bigcap \langle g \rangle D = \langle g \rangle$. It follows that a subgroup $\langle g \rangle$ is normal in FB. In particular, $\langle g \rangle$ is $\langle x \rangle$-invariant. Since it is true for each element $x \in P$, $\langle g \rangle$ is P-invariant, and hence $\langle g \rangle$ is L-invariant. It follows that every subgroup of P is L-invariant, in particular, P is a Dedekind group.

Let $q \in \Pi(L)$ and $q \neq p$. Let Q be a Sylow q-subgroup of L. Choose again the arbitrary elements $y, z \in Q$. A subgroup $K = \langle y, z \rangle$ is a finite one, in particular, it is nilpotent. Choose an infinite elementary abelian p-subgroup V in P. Then VK is nilpotent. There are infinite subgroups U, W of V, such that $V = U \times W$. Then both subgroups $\langle y \rangle U$ and $\langle y \rangle W$ have infinite special rank, and, hence. are transitively normal in G. Being transitively normal and subnormal in KV, $\langle y \rangle U$ and $\langle y \rangle W$ are normal in KV. The choice of subgroups U, W implies that $\langle y \rangle = \langle y \rangle U \bigcap \langle y \rangle W$. It follows that a subgroup $\langle y \rangle$ is normal in KV. In particular, $\langle y \rangle$ is $\langle z \rangle$-invariant. Since it is true for each element $z \in Q$, $\langle y \rangle$ is Q-invariant, and hence $\langle y \rangle$ is L-invariant. It follows that every subgroup of Q is L-invariant. It is true for each prime q, which follows that L is a Dedekind group.

Suppose now that the Sylow p-subgroup has finite special rank for each prime p. We have $L = Dr_{p \in \Pi(L)} L_p$ where L_p is a Sylow p-subgroup of L. Since L has infinite special rank, then the set $\Pi(L)$ is infinite and the set $\{r(L_p) | p \in \Pi(L)\}$ is not bounded. The fact that L_p has finite special rank implies that it is Chernikov subgroup [12]. Being a Chernikov p-group, L_p

is hypercentral. It is true for all primes p, thus L is hypercentral. It follows that every subgroup of L is ascendant. Let d be an arbitrary element of L. There exists a finite subset there exists a finite subset $M \subseteq \Pi(L)$ such that $\langle g\rangle \bigcap Dr_{p\in M} L_p = \langle 1\rangle$. Then the subset $\Sigma = \Pi(L)M$ is infinite and a subgroup $Dr_{p\in\Sigma} L_p$ has infinite special rank. Since Σ is infinite, we can choose in Σ two infinite subsets Λ, Θ such that $\Lambda \bigcap \Theta = \varnothing$, $\Lambda \bigcup \Theta = \Sigma$ and the both subgroups $X = Dr_{p\in\Lambda} L_p$ and $Y = Dr_{p\in\Theta} L_p$ have infinite special rank. Then the both subgroups $\langle d\rangle X$ and $\langle d\rangle Y$ have infinite special rank, and, hence. are transitively normal in G. Being transitively normal and ascendant in L, the subgroup $\langle d\rangle X$ and $\langle d\rangle Y$ are normal in L [6]. The choice of subgroups X, Y implies that $\langle d\rangle X \bigcap \langle d\rangle Y = \langle d\rangle$. It follows that a subgroup $\langle d\rangle$ is normal in L. It follows that L is a Dedekind group. $\square$

Corollary 1. *Let G be a group whose subgroups of infinite special rank are transitively normal. If L is a periodic locally nilpotent subgroup of G with infinite special rank, and $H = N_G(L)$, then every subgroup of L is H-invariant.*

Proof. By Lemma 3 L is a Dedekind group. Let x be an arbitrary element of L. Using the arguments from the proof of Lemma 3, we can find in L two subgroups A, B with infinite special rank, such that $\langle x\rangle = \langle x\rangle A \bigcap \langle x\rangle B$. The subgroups $\langle x\rangle A$ and $\langle x\rangle B$ are normal in L, so that they are subnormal in H. Since the both subgroups $\langle x\rangle A$ and $\langle x\rangle B$ have infinite special rank, they are transitively normal in G. Being transitively normal and subnormal in H, $\langle x\rangle A$ and $\langle x\rangle B$ are normal in H. Then and subgroup $\langle x\rangle = \langle x\rangle A \bigcap \langle x\rangle B$ is normal in H. It follows that every cyclic subgroup of L is H-invariant, which follows that that every subgroup of L is H-invariant. $\square$

Corollary 2. *Let G be a periodic group whose subgroups of infinite special rank are transitively normal, and let L be a locally nilpotent radical of G. If L has infinite special rank, then every subgroup of L is G-invariant and $G/C_G(L)$ is abelian.*

Proof. Since $N_G(L) = G$, Corollary 1 shows that every subgroup of L is G-invariant. The fact, that $G/C_G(L)$ is abelian, follows, for example, from [13, Theorem 1.5.1]. $\square$

Corollary 3. *Let G be a periodic radical group, whose subgroups of infinite special rank are transitively normal, and let L be a locally nilpotent radical of G. If L has infinite special rank, then every subgroup of L is G-invariant and G/L is abelian.*

Proof. In fact, locally nilpotent radical of a radical group includes its centralizer [14, Theorem 7]. □

Corollary 4. *Let G be a locally finite group, whose subgroups of infinite special rank are transitively normal. If $\boldsymbol{M}$ is a family of normal subgroups of G, having infinite special rank and $H = \bigcap_{M \in \boldsymbol{M}} M$, then G/H is metabelian.*

Proof. Indeed, every subgroup of factor-group G/M, $M \in \mathbf{M}$, is transitively normal and Theorem 1 shows that this factor-group is metabelian. Using a Remak's theorem, we obtain that G/H is likewise metabelian. □

2. Proof of the main theorem

Lemma 4. *Let G be a group whose subgroups of infinite special rank are transitively normal. Suppose that G includes normal subgroups C, D such that $C \leqslant D$, C is a Chernikov subgroups, D/C is a p-group of infinite special rank, where p is a prime and $p \notin \Pi(C)$. Then G is metabelian.*

Proof. Let $K = C_D(C)$. Since D is periodic, D/K is a Chernikov group (see, for example, [15, Theorem 1.5.16]). Let $Z = K \bigcap C$, then K/Z is a p-group, having infinite special rank. By Lemma 1 and Lemma 3 K/Z is a Dedekind group. In particular, K/Z is nilpotent. An inclusion $Z \leqslant \zeta(K)$ implies that K is nilpotent. Then $K = Z \times P$ and P is an unique Sylow p-subgroup of K. Moreover, P has infinite special rank and is a Dedekind group. If P is not abelian, then $p = 2$ and $P = Q \times B$ where Q is a quaternion group and B is an infinite elementary abelian 2-subgroup [16]. Then $A = \zeta(Q) \times B$ is a G-invariant infinite elementary abelian 2-subgroup of P. If P is abelian, then the fact that P has infinite rank implies that $\Omega_1(P) = A$ is infinite. So in every case P includes a G-invariant infinite elementary abelian p-subgroup A. Then $A = A_1 \times A_2$ where the both subgroups A_1, A_2 are infinite. In particular, A_1, A_2 have infinite rank and hence transitively normal in G. On the other hand, A_1, A_2 are subnormal in G. Being transitively normal and subnormal, they are normal in G. Then Corollary 4 implies that G is metabelian. □

Corollary 5. *Let G be a group, whose subgroups of infinite special rank are transitively normal. Suppose that G includes the normal subgroups C, D such that $C \leqslant D$, C is a locally finite subgroups, having Chernikov Sylow q-subgroups for each prime q, D/C is a p-group of infinite rank, where p is a prime and $p \notin \Pi(C)$. Then G is metabelian.*

Proof. Let $R_q = O_q(C)$, then R_q is G-invariant and C/R_q is a Chernikov group (see, for example, [15, Theorems 2.5.12 and 3.5.15]). By Lemma 4 G/R_q is metabelian. The equation $\langle 1\rangle = \bigcap_{q\in\Pi(C)} O_q(C) = \langle 1\rangle$ together with Corollary 4 shows that G is a metabelian group. □

Lemma 5. *Let G be a group and H a normal subgroup of G. Suppose that H includes a finite G-invariant subgroup F such that H/F is a locally finite p-group for some prime p. Then H includes a G-invariant p-subgroup P such that H/P is finite.*

Proof. The subgroup $C = C_G(F)$ is normal in G and has finite index in G. Then $B = C \bigcap H$ is normal in G and has finite index in H. The center of B includes $E = C \bigcap F$. Since B/E is a p-group,B is locally nilpotent, so that $B = Q \times P$ where P (respectively Q) is a Sylow p-subgroup (respectively p'-subgroup) of B. In particular, P is a characteristic subgroup of B and therefore normal in G. Finiteness of indexes $|H : B|$ and $|B : P|$ implies that H/P is likewise finite. □

Corollary 6. *Let G be a group whose subgroups of infinite special rank are transitively normal. Suppose that G includes normal subgroups C, D such that $C \leqslant D$, C is a locally finite subgroup having Chernikov Sylow q-subgroups for each prime q, D/C is a p-group of infinite rank, where p is a prime. Then G is metabelian.*

Proof. Let $R = O_{p'}(C)$, then R is G-invariant and C/R is a Chernikov group (see, for example, [15, Theorems 2.5.12 and 3.5.15]). Moreover, the divisible part E/R is a p-subgroup. By Lemma 5 D/E includes a G-invariant p-subgroup K/E with finite index in D/E. It follows that K/E has infinite rank. Then and a p-group K/R has infinite rank. Now, we can apply Corollary 5. □

Corollary 7. *Let G be a soluble periodic group whose subgroups of infinite special rank are transitively normal. Suppose, that there exists a prime p such that G has a Sylow p-subgroup with infinite rank. Then G is metabelian.*

Proof. Being soluble, G has finite series of normal subgroups

$$\langle 1\rangle = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = G$$

whose factors are locally nilpotent. By our conditions there exist a positive integer k such that R_k has Chernikov Sylow q-subgroups for all primes q and R_{k+1}/R_k has a Sylow p-subgroup P/R_k of infinite rank. Since R_{k+1}/R_k is locally nilpotent, P is normal in G. Now we can apply Corollary 6. □

Lemma 6. *Let G be a group whose subgroups of infinite special rank are transitively normal. Suppose that G includes normal subgroups C, D, such that $C \leqslant D$, C is a Chernikov subgroup, D/C is a locally nilpotent group with Chernikov q-subgroup for all primes q. If D/C has infinite special rank, then G is metabelian.*

Proof. Let $K = C_D(C)$. Since D is periodic, D/K is a Chernikov group (see, for example, [15, Theorem 1.5.16]). Let $Z = K \bigcap C$, then K/Z is a locally nilpotent group, having infinite special rank. Since $Z \leqslant \zeta(K)$, K is locally nilpotent. We have $K = Dr_{p\in\Pi(K)}K_p$, where K_p is Chernikov Sylow p-subgroup of K, $p \in \Pi(K)$. Since K has infinite special rank, then the set $\Pi(K)$ is infinite and the set $\{r(K_p)|p \in \Pi(K)\}$ is not bounded. Then K includes two G-invariant subgroups R, T, having infinite special rank, such that $R \bigcap T = \langle 1\rangle$. Using Corollary 4, we obtain that G is metabelian. □

Lemma 7. *Let G be a group whose subgroups of infinite special rank are transitively normal. Suppose that G includes the normal subgroups C, D such that $C \leqslant D$, C is a locally finite subgroups, having Chernikov Sylow q-subgroups for each prime q, D/C is a locally nilpotent group with Chernikov p-subgroup for all primes p. If D/C has infinite special rank, then G is metabelian.*

Proof. Let $R_q = O_{q'}(C)$, then R_q is G-invariant and C/R_q is a Chernikov group (see, for example, [15, Theorems 2.5.12 and 3.5.15]). By Lemma 6 G/R_q is metabelian. The equation $\langle 1\rangle = \bigcap_{q\in\Pi(C)} O_{q'}(C) = \langle 1\rangle$ together with Corollary 4 shows that G is a metabelian group. □

Corollary 8. *Let G be a soluble periodic group of infinite special rank whose subgroups of infinite special rank are transitively normal. If G has infinite special rank, then G is metabelian.*

Proof. Being soluble, G has finite series of normal subgroups

$$\langle 1\rangle = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = G$$

whose factors are locally nilpotent. By our conditions there exist a positive integer k such that R_k has finite special rank and R_{k+1}/R_k has infinite special rank. Since R_{k+1}/R_k is locally nilpotent, $R_{k+1}/R_k = Dr_{p\in\pi}S_p/R_k$ where $\pi = \Pi(R_{k+1}/R_k)$ and S_p/R_k is the Sylow p-subgroup of R_{k+1}/R_k, $p \in \pi$. Since R_{k+1}/R_k has infinite special rank, then either there exists a prime p such that S_p/R_k is not Chernikov, or S_p/R_k is Chernikov for every prime $p \in \pi$, but the set π is infinite and the set $\{\mathbf{r}(S_p/R_k)|p \in \pi\}$ is not bounded. In first case G is metabelian by Corollary 6. In the second case we will apply Lemma 6. □

Lemma 8. *Let G be a periodic soluble group of infinite special rank whose subgroups of infinite special rank are transitively normal. If L is a locally nilpotent radical of G, then every subgroup of L is G-invariant.*

Proof. If L has infinite special rank, then we can apply Corollary 2. Therefore, suppose that L has finite special rank. Then a factor-group G/L has infinite rank. By Corollary 8 G/L is abelian. Since L is locally nilpotent, $L = Dr_{p\in\Pi(L)}L_p$, where L_p is a Sylow p-subgroup of L, $p \in \Pi(L)$. Let g be an arbitrary element of L_p. Put $\pi = \Pi(L)\{p\}$, for each $q \in \pi$ define the subgroup $M_q = Dr_{r\in\pi, r\neq q}L_r$. Then $L/M_q \cong L_p \times L_q$ is Chernikov. Since G/M_q is periodic, $(G/M_q)/C_{G/M_q}(L/M_q)$ is a Chernikov group (see, for example, [15, Theorem 1.5.16]). It follows that $C/M_q = C_{G/M_q}(L/M_q)$ has infinite special rank. Since the factor $(G/M_q)/((C/M_q)\bigcap(L/M_q))$ is abelian and the center of C/M_q includes $(C/M_q)\bigcap(L/M_q)$, C/M_q is nilpotent. L/M_q is hypercentral, so the product $(C/M_q)(L/M_q)$ is locally nilpotent [14]. By Corollary 1 every subgroup of $(C/M_q)(L/M_q)$ is G-invariant. In particular, $\langle g\rangle M_q$ is G-invariant. It is true for every prime $q \in \pi$, thus $\bigcap_{q\in\pi}\langle g\rangle M_q$ is G-invariant. But the choice of the subgroups M_q shows that $\langle g\rangle = \bigcap_{q\in\pi}\langle g\rangle M_q$. Thus, every primary cyclic subgroup of L is G-invariant. It follows that every cyclic subgroup of L is G-invariant. In turn out it follows that every subgroup of L is G-invariant. □

Corollary 9. *Let G be a soluble periodic group of infinite special rank whose subgroups of infinite special rank are transitively normal. Then G is hypercyclic.*

Proof. Let L be a locally nilpotent radical of G. By Lemma 8 every subgroup of L is G-invariant. Then L has an ascending series of G-invariant subgroups with cyclic factors. By Corollary 8 G/L is abelian. Hence the series of L can be extended to a series of a group G, whose factors are cyclic. □

Corollary 10. *Let G be a soluble periodic group of infinite special rank whose subgroups of infinite special rank are transitively normal. If R is a* *locally nilpotent residual of G, then R is abelian, every subgroup of R is G-invariant and $2 \notin \Pi(R)$.*

Proof. By Corollary 8 G is metabelian. It follows that R is abelian. Being normal and abelian, R contains a locally nilpotent radical L of G. Using Lemma 8 we obtain that every subgroup of R is G-invariant. Finally, by Corollary 9 G is hypercyclic. It follows that every finite subgroup of G is supersoluble. Then every finite subgroup of G has the normal Sylow $2'$-subgroup. It follows that G has the normal Sylow $2'$-subgroup D. Since G/D is a 2-group, $R \leqslant D$. Therefore, $2 \notin \Pi(R)$. □

Lemma 9. *Let A be an abelian p-group and G be a finite p'-group of automorphisms of A. If $\langle 1 \rangle \neq [A, G] \neq A$, then A includes a subgroup which is not G-invariant.*

Proof. By Corollary 8 we have $A = [A, G] \times C_A(G)$ (for example, [17, Proposition 5.19]). Suppose the contrary, let every subgroup of A is G-invariant. Choose the elements $1 \neq a \in \Omega_1([A, G])$, $1 \neq c \in \Omega_1(C_A(G))$. Since $a \notin C_A(G)$, there is an element $g \in G$ such that $a^g \neq a$. The fact that $\langle a \rangle$ is G-invariant implies that $a^g = a^k$ for some positive integer k such that $1 < k < p$. Since a cyclic subgroup $\langle ac \rangle$ is G-invariant, $(ac)^g = (ac)^m = a^m c^m$ for some positive integer m such that $1 < k < p$. On the other hand, $(ac)^g = a^g c^g = a^k c$. It follows that $m \equiv k(modp)$ and $m \equiv 1(modp)$, and we obtain a contradiction, which proves a result. □

Lemma 10. *Let G be a periodic soluble group of infinite special rank whose subgroups of infinite special rank are transitively normal. If R is a locally nilpotent residual of G, then $\Pi(R) \bigcap \Pi(G/R) = \varnothing$.*

Proof. Suppose the contrary, let $\Pi(R) \bigcap \Pi(G/R)$ is not empty and choose a prime $p \in \Pi(L) \bigcap \Pi(G/L)$. Corollary 10 shows that $p \neq 2$. Let R_p be a Sylow p-subgroup of R, then $R = R_p \times Q$ where Q is a Sylow p'-subgroup of R. Since G/R is locally nilpotent, it has a non-trivial normal Sylow p-subgroup P/R. Then P/Q is a normal Sylow p-subgroup of G/Q. Suppose first that P/Q has infinite special rank. Lemma 1 and Corollary 1 imply that every subgroup of P/Q is G-invariant. In particular, P/Q is a Dedekind group. The fact that $p \neq 2$ implies that P/Q is abelian [16]. Since G/R is locally nilpotent, $[P/Q, G/Q] \leqslant R_p/Q$. In particular, $[P/Q, G/Q] \neq P/Q$. If we suppose that $[P/Q, G/Q] = \langle 1 \rangle$, then G/Q is locally nilpotent, and we obtain a contradiction with a choice of Q. This contradiction proves that $\langle 1 \rangle \neq [P/Q, G/Q]$. But in this case Lemma 9 implies that P/Q includes a subgroup, which is not G-invariant. This contradiction shows that P/Q has a finite special rank.

It follows that R_p has finite special rank. Suppose that R has infinite special rank. Then a subgroup Q has infinite special rank. In this case every subgroup of a factor-group is transitively normal. Then Theorem 1 and the fact that R/Q is a locally nilpotent residual of G/Q give a contradiction.

Finally, consider the case when R has finite special rank. Being a normal p-subgroup, P/Q lies in a locally nilpotent radical of G/Q. By Lemma 1 and Lemma 8 every subgroup of P/Q is G-invariant. Repeating the above arguments, we obtain again a contradiction, which proves a result. □

Lemma 11. *Let G be a periodic soluble group of infinite special rank whose subgroups of infinite special rank are transitively normal. If R is a locally nilpotent residual of G, then G/R is a Dedekind group.*

Proof. If R has infinite special rank, then every subgroup of G/R is transitively normal. Since G/R is locally nilpotent, it must be Dedekind. Suppose now that R has a finite special rank. Then G/R has infinite special rank. Since it is locally nilpotent, Lemma 1 and Lemma 3 shows that G/R is a Dedekind group. □

Proof of the main result of the paper — Theorem 2.

Proof. Let G be a periodic soluble group of infinite special rank whose subgroups of infinite special rank are transitively normal. Denote by R the locally nilpotent residual of G. By Lemma 11 G/R is a Dedekind group. Lemma 10 shows that $\Pi(R) \bigcap \Pi(G/R) = \varnothing$. By Corollary 10 R is abelian, every subgroup of R is G-invariant and R is a $2'$-group. It follows that $G/C_G(R)$ is abelian (see, for example, [13, Theorem 1.5.1]). Finally Corollary 8 shows that G is metabelian. Thus, G satisfies all conditions of Theorem 1. According to this theorem every subgroup of G is transitively normal. □

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