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Generators and ranks in finite partial transformation semigroups

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Abstract. We extend the concept of path-cycles, defined in [2], to the semigroup $\mathcal{P}_n$, of all partial maps on $X_n = \{1, 2, \ldots, n\}$, and show that the classical decomposition of permutations into disjoint cycles can be extended to elements of $\mathcal{P}_n$ by means of path-cycles. The device is used to obtain information about generating sets for the semigroup $\mathcal{P}_n \setminus \mathcal{S}_n$, of all singular partial maps of X_n. Moreover, by analogy with [3], we give a definition for the (m, r)-rank of $\mathcal{P}_n \setminus \mathcal{S}_n$ and show that it is $\frac{n(n+1)}{2}$.

1. Introduction

Since the work of Howie [7], establishing that every singular map in the full transformation semigroup $\mathcal{T}_n$ on the finite set $X_n = \{1, 2, \ldots, n\}$ is expressible as a product (that is composition) of idempotent singular maps, there have been many articles concerned with this idea in $\mathcal{T}_n$ (see for example, [1–3, 8–10, 12, 13, 15]).

Evseev and Podran [5] established that even in the larger semroup $\mathcal{P}_n$, consisting of all partial maps on X_n, all elements (other than permutations) are expressible as products of idempotents. Garba [6] extended all the results of [9–11, 15] to $\mathcal{P}_n$ using a result of Vagner [16] quoted in [4, p.254].

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In analysing elements of $\mathcal{T}_n$, there are many variations in notations. Lipscomb [14] developed what might be called a linear notation for elements of $\mathcal{P}_n$. Recently, Ayik et al. [2] described an alternative approach, to the Lipscomb's linear notation for elements of $\mathcal{T}_n$, which generalised the concept of cycle notation for permutations in the symmetric group $\mathcal{S}_n$. In this paper we show that this idea can be further generalise to the larger semigroup $\mathcal{P}_n$ via Vagner's result. The technique is used to obtain information about generators for $\mathcal{P}_n \setminus \mathcal{S}_n$.

It is known (see [6, Theorem 4.1]) that the rank of $\mathcal{P}_n \setminus \mathcal{S}_n$, defined by

$$\operatorname{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \min\{|A| : \langle A\rangle = \mathcal{P}_n \setminus \mathcal{S}_n\},$$

is equal to $n(n+1)/2$. The idempotent rank of $\mathcal{P}_n \setminus \mathcal{S}_n$ is the cardinality of a smallest generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$ consisting solely of idempotents, and this too equals $n(n+1)/2$. For any fixed m and r such that $2 \leqslant r \leqslant m \leqslant n$, we give a definition for (m, r)-rank of $\mathcal{P}_n \setminus \mathcal{S}_n$, analogous to the definition given in [3] for $\mathcal{T}_n \setminus \mathcal{S}_n$, and show that it is once again equal to $\frac{n(n+1)}{2}$.

This article is a direct translation of the results in [2,3] for $\mathcal{T}_n$ to similar results concerning $\mathcal{P}_n$. Thus, many of our proofs are direct modifications of the corresponding proofs in [2, 3].

2. Preliminaries

Let $X_n = \{1, \dots, n\}$ and let $\mathcal{P}_n$ be the partial transformation semigroup on X_n. For a subset $\{x_1, \dots, x_m\}$ of X_n let $\alpha \in \mathcal{P}_n$ be such that $x_i\alpha = x_{i+1} (1 \leqslant i \leqslant m-1)$ and $x\alpha = x$ $(x \in X_n \setminus \{x_1, \dots, x_m\})$. If:

i) $x_m\alpha = x_r$ for some $1 \leqslant r \leqslant m$, α is called an *(m, r)-path-cycle* and is denoted by $\alpha = [x_1, \dots, x_m|x_r]$;

ii) $x_m \notin \operatorname{dom}(\alpha)$, α is called an *$(m, 0)$-path-cycle*, or an *m-chain* and is denoted by $\alpha = [x_1, \dots, x_m]$.

An element of $\mathcal{P}_n$ is called a *path-cycle* of size m if it is either an (m, r)-path-cycle or an m-chain. An (m, r)-path-cycle is called: an *r-cycle* if $r = 1$; a *proper path-cycle* if $r \neq 1$; and an *m-path* if $m = r$.

We let $X_n^0 = X_n \cup \{0\}$ and denote the semigroup of all full transformations of X_n^0 by $\mathcal{T}_{X_n^0}$. For each $\alpha \in \mathcal{P}_n$ the map α^*, defined by

$$\alpha^* = \begin{cases} x\alpha & \text{if } x \in \operatorname{dom}(\alpha), \\ 0 & \text{if } x \notin \operatorname{dom}(\alpha), \end{cases}$$

belongs to $\mathcal{T}_{X_n^0}$. Let $\mathcal{P}_n^*$ be the set of all elements in $\mathcal{T}_{X_n^0}$ that fixed 0 and let $\mathcal{S}_n^*$ be the set of all permutations in $\mathcal{P}_n^*$. It is clear that $\mathcal{P}_n^*$ is a subsemigroup of $\mathcal{T}_{X_n^0}$ and from [6, Lemma 2.4] it is regular.

For convenience we record the following result due to Vagner [16] (also to be found in [4, p.254]).

Theorem 1. *For each $\alpha \in \mathcal{P}_n$ and each $\beta \in \mathcal{P}_n^*$, the mappings $\alpha \mapsto \alpha^*$ and $\beta \mapsto \beta|_{X_n}$ (the restriction of β to X_n) are mutually inverse isomorphisms of $\mathcal{P}_n$ onto $\mathcal{P}_n^*$ and vice-verse.*

Here we make the following important remark which will be effectively used throughout the next sections.

Remark 1. i) For $1 \leqslant r < m \leqslant n$, an (m, r)-path-cycle $[x_1, \ldots, x_m|x_r]$ in $\mathcal{P}_n^*$ corresponds in these isomorphisms to an (m, r)-path-cycle $[x_1, \ldots, x_m|x_r]$ in $\mathcal{P}_n$, while an m-path $[x_1, \ldots, x_m|x_m]$ in $\mathcal{P}_n^*$ corresponds either to an m-path $[x_1, \ldots, x_m|x_m]$ in $\mathcal{P}_n$ if $x_m \neq 0$, or to an $(m-1)$-chain $[x_1, \ldots, x_{m-1}]$ in $\mathcal{P}_n$ if $x_m = 0$.

ii) A set of elements in $\mathcal{P}_n$ generates $\mathcal{P}_n$ if and only if its image under the isomorphisms generates $\mathcal{P}_n^*$ and vice-verse.

3. Generating sets

In this section we identify many generating sets of path-cycles for the semigroup $\mathcal{P}_n \setminus \mathcal{S}_n$. First, we start by generating $\mathcal{P}_n$ using path-cycles.

Theorem 2. *Each element of $\mathcal{P}_n$ is expressible as a product of path-cycles in $\mathcal{P}_n$.*

Proof. Let $\alpha \in \mathcal{P}_n$. The associated map $\alpha^* \in \mathcal{P}_n^*$ is expressible as a product $\alpha^* = \alpha_1 \cdots \alpha_P$ of path-cycles in $\mathcal{T}_{X_n^0}$ using the algorithm described in [2]. Since $0\alpha^* = 0$, the algorithm ensures that $0\alpha_i = 0$ for all i. Hence, $\alpha_i = \delta_i^*$ for some path-cycle δ_i in $\mathcal{P}_n$. Therefore, by the isomorphism $\alpha = \delta_1 \cdots \delta_p$. □

As in [2], the integer p is called the *path-cycle rank* of α and is denoted by $pcr(\alpha)$. By [2, Theorem 2], we have that $pcr(\alpha^*) = def(\alpha^*) + cycl(\alpha^*)$, where $def(\alpha^*) = |X_n^0 \backslash im(\alpha^*)|$, the defect of α^* and $cycl(\alpha^*)$ is the number of cycles in the decomposition. It has also been observed in [6, Lemma 2.2 & 2.3] that $cycl(\alpha^*) = cycl(\alpha)$ and $def(\alpha^*) = def(\alpha)$ for all $\alpha \in \mathcal{P}_n$. Thus, we have the following observation.

Lemma 1. *Let $\alpha \in \mathcal{P}_n$. Then $pcr(\alpha) = def(\alpha) + cycl(\alpha)$.*

Next, we have

Theorem 3. *For each $\alpha \in \mathcal{P}_n \setminus \mathcal{S}_n$, there exists proper path-cycles $\gamma_1, \ldots, \gamma_k$ in $\mathcal{P}_n \setminus \mathcal{S}_n$ such that $\alpha = \gamma_1 \cdots \gamma_k$.*

Proof. Let $\alpha \in \mathcal{P}_n \setminus \mathcal{S}_n$. By [2, Theorem 4], the associated map $\alpha^* \in \mathcal{P}_n^* \setminus \mathcal{S}_n^*$ is expressible as a product $\alpha^* = \beta_1 \cdots \beta_k$ of proper path-cycles in $\mathcal{T}_{X_n^0}$ and since $0\alpha^* = 0$, the method of factorisation ensures that each of the proper path-cycles β_i is in $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$. Hence, by the isomorphism, $\alpha = \gamma_1 \cdots \gamma_k$, where for each i, $\gamma_i^* = \beta_i$ and each γ_i is a path-cycle in $\mathcal{P}_n \setminus \mathcal{S}_n$. It is also clear that each γ_i is a proper path-cycle. □

Theorem 4. *The set of all 2-paths and 1-chains in $\mathcal{P}_n \setminus \mathcal{S}_n$ together generates $\mathcal{P}_n \setminus \mathcal{S}_n$.*

Proof. By [2, Theorem 5], each element of $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ is a product of 2-paths in $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$. Thus, the result follows from the Isomorphisms between $\mathcal{P}_n \setminus \mathcal{S}_n$ and $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$, and Remark 1. □

Theorem 5. *For each $m \in \{2, \ldots, n\}$, the semigroup $\mathcal{P}_n \setminus \mathcal{S}_n$ can be generated by path-cycles of size m or $m - 1$.*

Proof. Since, for each $m \in \{2, \ldots, n\}$, the semigroup $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ is generated by its path-cycles of size m. It remains to show that each path-cycle of size m in $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ corresponds to path-cycles of size m or $m - 1$ under the isomorphism. But this is the content of Remark 1. □

Theorem 6. *Let $m \in \{2, \ldots, n\}$. Then the set of all m-paths and all m-chains in $\mathcal{P}_n \setminus \mathcal{S}_n$ generates $\mathcal{P}_n \setminus \mathcal{S}_n$.*

Proof. For any $x_1, x_2 \in X_n$, we observe that

$$\begin{aligned}[x_1, x_2|x_2] &= [x_m, x_{m-1}, \ldots, x_3, x_1, x_2|x_2][x_1, x_3, x_4, \ldots, x_m, x_2|x_2], \\ [x_1] &= [x_m, x_{m-1}, \ldots, x_1][x_1, x_2, \ldots, x_m].\end{aligned}$$

Thus the result follows from Theorem 4. □

Theorem 7. *Let $m \in \{2, \ldots, n\}$ and $r \in \{2, \ldots, m\}$. Then the set of all (m, r)-path-cycles and all m-chains in $\mathcal{P}_n \setminus \mathcal{S}_n$ generates $\mathcal{P}_n \setminus \mathcal{S}_n$.*

Proof. By Theorem 4 it suffices to show that each 2-path $[x, y|y]$ and each 1-chain $[x]$ in $\mathcal{P}_n \setminus \mathcal{S}_n$ can be expressed as a product of (m, r)-path-cycles and m-chains $\mathcal{P}_n \setminus \mathcal{S}_n$ respectively. But, as in [3, Theorem 5], we have

$$[x, y|y] = [x_1, x_2, \dots, x_m|x_r][x_{r-1}, x_{r-2}, \dots, x_1, x_m, x_{m-1}, \dots, x_r|x_m]$$

where $\{x_1, x_2, \dots, x_m\} \subseteq X_n$, $x_{r-1} = x$ and $x_m = y$. Also, as in Theorem 6,

$$[x] = [x_m, x_{m-1}, \dots, x_1][x_1, x_2, \dots, x_m]$$

where $x_1 = x$. □

Remark 2. Each 1-chain $[x]$ in $\mathcal{P}_n \setminus \mathcal{S}_n$ can be expressed as a product of 2 k-paths, for each $k \in \{2, \dots, n\}$, simply by choosing $k-1$ distinct points $x_2, x_3, \dots, x_k \in X_n \setminus \{x\}$ and observing that

$$[x] = [x_k, x_{k-1}, \dots, x][x, x_2, \dots, x_k].$$

Thus, for any fixed $k, m \in \{2, \dots, n\}$ and $r \in \{2, \dots, m\}$, the set of all (m, r)-path-cycles and all k-chains in $\mathcal{P}_n \setminus \mathcal{S}_n$ generates $\mathcal{P}_n \setminus \mathcal{S}_n$.

4. Rank properties

For any fixed m and r such that $2 \leqslant r \leqslant m \leqslant n$, we define the (m, r)-*rank* of $\mathcal{P}_n \setminus \mathcal{S}_n$, denoted by $\operatorname{rank}_{m,r}(\mathcal{P}_n \setminus \mathcal{S}_n)$, to be the cardinality of a smallest generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$ consisting solely of (m, r)-path-cycles and $(m-1)$-chains. In the light of Remarks 1 and 2, the corresponding (m, r)-*rank* of $\mathcal{P}^*_n \setminus \mathcal{S}^*_n$, denoted by $\operatorname{rank}_{m,r}(\mathcal{P}^*_n \setminus \mathcal{S}^*_n)$, is define to be the cardinality of a smallest generating set for $\mathcal{P}^*_n \setminus \mathcal{S}^*_n$ consisting solely of (m, r)-path-cycles and m-paths. In this section, we show that $\operatorname{rank}_{m,r}(\mathcal{P}_n \setminus \mathcal{S}_n)$ is equal to $n(n+1)/2$. Since $\operatorname{rank}_{m,r}(\mathcal{P}_n \setminus \mathcal{S}_n)$ is at least as large as $\operatorname{rank}(\mathcal{P}_n \setminus \mathcal{S}_n)$, it is sufficient to prove that $\operatorname{rank}_{m,r}(\mathcal{P}_n \setminus \mathcal{S}_n) \leqslant n(n+1)/2$.

A digraph Γ with n vertices is called *complete* if, for all $i \neq j$ in the set of vertices, either $i \to j$ or $j \to i$ is an edge. It is called *strongly connected* if, for any two vertices i and j, there is a path from i to j. A vertex i in a digraph is called a *sink* if, for all vertices j, $j \to i$ is an edge and $i \to j$ is not an edge.

In the semigroup $\mathcal{P}^*_n$, idempotents of defect 1 are 2-paths of type $[i, j|j]$ where $i, j \in X^0_n$ and $0 \neq i \neq j$. There are n^2 such 2-paths in $\mathcal{P}^*_n$. To each set I^* of 2-paths in $\mathcal{P}^*_n$ we associate a digraph $\triangle(I^*)$ with $n+1$

vertices, in which $i \to j$ is a directed edge if and only if $[i, j|j] \in I^*$. First, we prove the following.

Theorem 8. *A set I^*, of 2-paths in $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ ($n \geqslant 3$), is a generating set for $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ if and only if 0 is a sink in $\triangle(I^*)$ and the digraph $\triangle(I^*) - 0$ is strongly connected and complete.*

Proof. Suppose that I^* is a set of 2-paths in $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ that generates $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$. First, we observe that, for all $i = 1, \ldots, n$, the 2-paths $[0, i|i]$ cannot be in I^* since $[0, i|i] \notin \mathcal{P}_n^* \setminus \mathcal{S}_n^*$. Thus, for all $i = 1, \ldots, n$, $0 \to i$ is not an edge in $\triangle(I^*)$. Therefore, $\deg_{out}(0) = 0$. Also, by Remark 1, the image set I of I^* (under the isomorphisms in Theorem 1) is a generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$, consisting of 2-paths and 1-chains. Since each 2-path and each 1-chain is an idempotents of defect 1, by [10, Lemma 1], we must have $[i] \in I$, for all $i = 1, \ldots, n$. Thus, again by Remark 1, $[i, 0|0] \in I^*$ for all $i = 1, \ldots, n$ and so, $i \to 0$ is an edge in $\triangle(I^*)$ for all $i = 1, \ldots, n$. Therefore 0 is a sink in $\triangle(I^*)$.

Now, we show that $\triangle(I^*) - 0$ is strongly connected and complete. It is not difficult to observe that the image set $I \setminus \{[i] : i = 1, \ldots, n\}$ of $I^* \setminus \{[i, 0|0] : i = 1, \ldots, n\}$ (under the isomorphisms in Theorem 1) is a generating set for the semigroup $\mathcal{T}_n \setminus \mathcal{S}_n$, of all singular full transformations of X_n. Thus, by Howie (1078, Theorem 1), $\triangle(I \setminus \{[i] : i = 1, \ldots, n\}) = \triangle(I^* \setminus \{[i, 0|0] : i = 1, \ldots, n\}) = \triangle(I^*) - 0$ must be strongly connected and complete.

Conversely, suppose that 0 is a sink in $\triangle(I^*)$ and that the digraph $\triangle(I^*) - 0$ is strongly connected and complete. Observe that each map $\alpha^* \in \mathcal{P}_n^* \setminus \mathcal{S}_n^*$ can be expressed as

$$\alpha = [i_1, 0|0][i_2, 0|0] \cdots [i_m, 0|0]\alpha_1,$$

where $i_1, i_2, \ldots, i_m \in X_n$ are non-zero pre-images of 0 under α^*, and α_1 is a map in $\mathcal{P}_n^*$ defined by

$$x\alpha_1 = \begin{cases} x & \text{if } x \in \{0, i_1, \ldots, i_m\}, \\ x\alpha & \text{if } x \notin \{0, i_1, \ldots, i_m\}. \end{cases}$$

Now, since $i\alpha_1 = 0$ if and only if $i = 0$, it is clear that for any $\beta_1, \beta_2, \ldots, \beta_k \in I^*$,

$$\alpha_1 = \beta_1\beta_2 \cdots \beta_k \quad \text{if and only if} \quad \alpha_1|_{X_n} = \beta_1|_{X_n}\beta_2|_{X_n} \cdots \beta_k|_{X_n}. \qquad (1)$$

But, since $\triangle(I^*) - 0$ is strongly connected and complete, it follows from [8, Lemma 1] that $I \setminus \{[i] : i = 1, 2, \ldots, n\}$ is a generating set for $\mathcal{T}_n \setminus \mathcal{S}_n$. Thus, by (2) and the isomorphisms, α_1 is a product of element in I^* and so α is generated by I^*. □

Next, we make use of the following result from [6, Theorem 4.1].

Theorem 9. *For $n \geqslant 3$, $\operatorname{rank}_{2,2}(\mathcal{P}_n^* \setminus \mathcal{S}_n^*) = n(n+1)/2$.*

It follows from Theorems 8 and 9 that a digraph associated with a minimal generating set of 2-paths in $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ is complete and contains $n(n+1)/2$ edges. Consequently, the underlying (undirected) graph of such a generating set is, upto isomorphism, the complete graph K_n^* with vertices $0, 1, \ldots, n$.

The following definition is from [3].

Definition 1. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $|E(G)|$ is even, let A and B be disjoint subsets of $E(G)$ such that $|A| = |B| = |E(G)|/2$; the triple (A, B, φ) is called a *pairing* of G if $\varphi : A \to B$ is a bijection such that, for each $e \in A$, e and $\varphi(e)$ have no vertices in common. If $|E(G)|$ is odd, a pairing of G is defined to be a pairing of $G - e$, for some $e \in E(G)$.

From [3, Lemma 3] we deduce the following.

Lemma 2. *For all $n \geqslant 3$, there exists a pairing of K_n^*.*

Proof. For each $n \geqslant 3$, form a pairing (A, B, φ) of the complete graph K_{n+1} on the vertex set $\{1, 2, \ldots, n+1\}$ using the construction described in [3, Lemma 3]. In each of the disjoint subsets A, B of $E(K_{n+1})$ replace each edge (i, j) by $(i, j)^* = (i-1, j-1)$ to obtain subsets A^*, B^* of $E(K_n^*)$. Then (A^*, B^*, φ^*), where $\varphi^*(i-1, j-1) = (\varphi(i, j))^*$, is a pairing of K_n^*. □

Before we prove our next theorem stating that $\operatorname{rank}_{m,r}(\mathcal{P}_n^* \setminus \mathcal{S}_n^*) = \frac{n(n+1)}{2}$, it is convenient to deal with two particular cases.

Lemma 3. *For each $n \geqslant 3$ and each $2 \leqslant m \leqslant n$,*

$$\operatorname{rank}_{m,2}(\mathcal{P}_n^* \setminus \mathcal{S}_n^*) = \frac{n(n+1)}{2}.$$

Proof. From Theorem 9, we know that the result holds when $m = 2$. Let I^* be a generating set for $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$ consisting of 2-paths with $|I^*| = n(n+1)/2$. Them, from Theorem 8, $[i, 0|0] \in I^*$, for all $i = 1, 2, \ldots, n$. If n is even, then we form $n/2$ distinct pairs of $\{[i, 0|0] : i = 1, 2, \ldots, n\}$ and corresponding to each pair $[i, 0|0] \leftrightarrow [j, 0|0]$ (with $i \neq j$) define m-paths

$$\alpha = [j, x_2, x_3, \ldots, x_{m-2}, i, 0|0]\,, \tag{2}$$

$$\beta = [i, x_{m-2}, x_{m-3}, \ldots, x_2, j, 0|0]\,, \tag{3}$$

where the $m-3$ elements $x_2, x_3, \ldots, x_{m-2}$ are distinct elements in $X_n \setminus \{i, j\}$. Then $\alpha\beta = [i, 0|0]$ and $\beta\alpha = [j, 0|0]$. For each $[i, j|j] \in I^* \setminus \{[i, 0|0] : i = 1, 2, \ldots, n\}$ we associate an $(m, 2)$-path-cycle

$$\alpha_{ij} = [i, x_2, x_3, \ldots, x_{m-1}, j|x_2]\,. \tag{4}$$

Then $\alpha_{ij}^{m-1} = [i, j|j]$. Thus, in equalities (2), (3) and (4), we have found $n(n+1)/2$ $(m, 2)$-path-cycles and m-paths that generate elements in I^*.

Now, if n is odd, then we form $(n-1)/2$ distinct pairs of $\{[i, 0|0] : i = 1, 2, \ldots, n-1\}$ and corresponding to each pair define m-paths α and β as in equalities (2) and (3) respectively. For the 2-path $[n, 0|0]$, we choose a 2-path $[k, l|l] \in I^* \setminus \{[i, 0|0] : i = 1, 2, \ldots, n\}$ and define m-paths

$$\gamma = [k, x_2, x_3, \ldots, x_{m-2}, n, 0|0]\,, \tag{5}$$

$$\delta = [n, x_{m-2}, x_{m-3}, \ldots, x_2, k, l|l]\,. \tag{6}$$

Then, $\gamma\delta = [n, 0|0]$ and $\delta\gamma = [k, l|l]$. Lastly, for each $[i, j|j] \in I^* \setminus \{[k, l|l], [i, 0|0] : i = 1, 2, \ldots, n\}$ we associate an $(m, 2)$-path-cycle α_{ij} given in equality (4). Thus, again, in equalities (2-6), we found $n(n+1)/2$ $(m, 2)$-path-cycles and m-paths that generate elements in I^*. □

Lemma 4. $\mathrm{rank}_{3,3}(\mathcal{P}_3^* \setminus \mathcal{S}_3^*) = 6.$

Proof. From Theorems 8 and 9, we know that

$$I^* = \{[1, 0|0], [2, 0|0], [3, 0|0], [1, 3|3], [2, 1|1], [3, 2|2]\}$$

is a minimal generating set for $\mathcal{P}_3^* \setminus \mathcal{S}_3^*$. Define $(3, 3)$-path-cycles as $\alpha_1 = [2, 1, 0|0]$, $\alpha_2 = [3, 2, 0|0]$, $\alpha_3 = [1, 3, 0|0]$, $\beta_1 = [1, 2, 3|3]$, $\beta_2 = [2, 3, 1|1]$ and $\beta_3 = [3, 1, 2|2]$. Then the set $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$ is a minimal generating set for $\mathcal{P}_3^* \setminus \mathcal{S}_3^*$, since $\alpha_1\beta_1 = [1, 0|0]$, $\alpha_2\beta_2 = [2, 0|0]$, $\alpha_3\beta_3 = [3, 0|0]$, $\beta_2\beta_3\beta_1 = [1, 3|3]$, $\beta_3\beta_1\beta_2 = [2, 1|1]$ and $\beta_1\beta_2\beta_3 = [3, 2|2]$. □

Theorem 10. *For each $n \geqslant 3$ and each $2 \leqslant r \leqslant m \leqslant n$,*

$$\operatorname{rank}_{m,r}(\mathcal{P}_n^* \setminus \mathcal{S}_n^*) = \frac{n(n+1)}{2}.$$

Proof. By virtue of Lemmas 3 and 4, we only need to consider the case when $n \geqslant 4$ and $r \geqslant 3$. Thus, suppose that $n \geqslant 4$ and $3 \leqslant r \leqslant m \leqslant n$. Let

$$P\{[1,n|n],[1,n-1|n-1],[m-r+2,n|n]\}$$

and

$$Q = \{[n,1|1],[n-1,1|1],[n,m-r+2|m-r+2]\}.$$

Then define

$$I^* = \{[i,0|0] : 1 \leqslant i \leqslant n\} \cup (\{[i,j|j] : 1 \leqslant i < j \leqslant n\} \setminus P) \cup Q.$$

Since $|P| = |Q| = 3$, it is clear that

$$|I^*| = n + |\{[i,j|j] : 1 \leqslant i < j \leqslant n\}| = n + \binom{n}{2} = \frac{n(n+1)}{2},$$

and that 0 is a sink in the associated digraph $\triangle(I^*)$. Also, observe that, when $m - r + 2 \neq n - 1$, the digraph $\triangle(I^*) - 0$ has a Hamiltonian cycle

$$1 \to 2 \to \cdots \to n-1 \to n \to 1$$

and, when $m - r + 2 = n - 1$, the digraph $\triangle(I^*) - 0$ has a Hamiltonian cycle

$$n \to n-1 \to 1 \to 2 \to \cdots \to n-3 \to n-2 \to n.$$

Thus in both cases the digraph $\triangle(I^*) - 0$ is strongly connected. It is easy to see that the digraph is complete, and so, by Theorem 8, $\triangle(I^*)$ is a generating set for $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$.

Suppose that $|I^*|$ is even. By Lemma 2, we can pair elements of I^* in such a way that

$$[i,j|j] \leftrightarrow [k,l|l] \implies \{i,j\} \cap \{k,l\} = \varnothing. \tag{7}$$

There are two cases: (i) $r = m$; (ii) $3 \leqslant r \leqslant m - 1$. In case (i), for each pair of type (7), let

$$\alpha = [i, x_2, x_3, \ldots, x_{m-2}, k, l|l] \tag{8}$$

and

$$\beta = [k, x_{m-2}, x_{m-3}, \ldots, x_2, i, j|j]\,, \tag{9}$$

where the $m-3$ elements $x_2, x_3, \ldots, x_{m-2}$ are fixed distinct elements of $\in X_n \setminus \{i, j, k, l\}$. Then

$$\alpha\beta = [k, l|l] \quad \text{and} \quad \beta\alpha = [i, j|j],$$

and so, in equalities (8) and (9), we have found $\frac{n(n+1)}{2}$ m-paths that generate $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$.

In case (ii), where $3 \leqslant r \leqslant m-1$, if both $j \neq 0$ and $l \neq 0$ hold, we define, for each pair of type (7),

$$\gamma = \begin{cases} [i, k, j, x_4, \ldots, x_{m-1}, l|j] & \text{if } r = 3 \\ [i, x_2, \ldots, x_{m-3}, k, j, l|j] & \text{if } r = m-1 \\ [i, x_2, \ldots, x_{r-2}, k, j, x_{r+1}, \ldots, x_{m-1}, l|j] & \text{if } 3 < r < m-1 \end{cases} \tag{10}$$

and

$$\delta = \begin{cases} [k, i, l, x_{m-1}, \ldots, x_4, j|l] & \text{if } r = 3 \\ [k, x_{m-3}, \ldots, x_2, i, l, j|l] & \text{if } r = m-1 \\ [k, x_{r-2}, \ldots, x_2, i, l, x_{m-1}, \ldots, x_{r+1}, j|l] & \text{if } 3 < r < m-1, \end{cases} \tag{11}$$

where the $m-4$ elements $x_2, \ldots, x_{r-2}, x_{r+1}, \ldots, x_{m-1}$ are fixed distinct elements of $X_n \setminus \{i, j, k, l\}$. Then, in all the situations,

$$\gamma\delta = [k, l|l] \quad \text{and} \quad \delta\gamma = [i, j|j].$$

And so, we have found $\frac{n(n+1)}{2}$ (m, r)-path-cycles and/or m-paths that generate $\mathcal{P}_n^* \setminus \mathcal{S}_n^*$.

So far we have dealt with the case where $|I^*|$ is even. Suppose now that $|I^*|$ is odd. By Lemma 2, we have a pairing of the elements of $J = I^* \setminus \{[n, 1|1]\}$. By the above argument we can ensure that, for all $3 \leqslant r \leqslant m$, all elements of J are products of (m, r)-path-cycles and m-paths of the forms (8) and (9), or (10) and (11). In particular with those generators, we obtain $\xi = [n, m-r+2|m-r+2]$. We now define

$$\eta = \begin{cases} [2, 3, \ldots, m-1, 1, n|n] & \text{if } r = m \\ [m-r+2, m-r+3, \ldots, m-1, 1, 2, \ldots, m-r+1, n|2] & \text{if } r < m. \end{cases}$$

Then

$$(\eta\xi)^{m-1} = [n, 2, 3, \ldots, m-1, 1|2]^{m-1} = [n, 1|1].$$

The (m, r)-path-cycle η (if $r < m$) or m-path η (if $r = m$) does not appear in the list of elements (8), (9), (10) and (11); for otherwise we would have found a generating set with fewer than $\frac{n(n+1)}{2}$ elements. Hence, adding η to the generating elements already described gives a generating set consisting of $\frac{n(n+1)}{2}$ (m, r)-path-cycles and m-paths. □

Now, using Theorem 10 and Remark 1, we have proved the next theorem.

Theorem 11. *For each $n \geqslant 3$ and each $2 \leqslant r \leqslant m \leqslant n$,*

$$\operatorname{rank}_{m,r}(\mathcal{P}_n \setminus \mathcal{S}_n) = \frac{n(n+1)}{2}.$$

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