ON THE COORDINATED APPROXIMATION METHOD FOR NONLINEAR ILL-POSED PROBLEMS

A generalization of the method of coordinated approximation suggested by Yu. Gaponenko [1] for the space $L_2(0, 1)$ is developed for abstract Hilbert spaces. In particular, it is shown that, for $L_2(0, 1)$, some assumptions concerning an exact solution can be weaken.

1. Introduction. Consider the operator equation

$$A(u) = f,$$

where $A$ is a possibly nonlinear operator over a real Hilbert space $H$. Suppose that, for an exact right-hand member $f$, problem (1) has an exact solution $u^*$, satisfying the a priori estimate

$$\|u^*\| \leq R.$$  \hspace{1cm} (2)

If the right-hand member $f_\delta \in H$ differs from $f$ at most by $\delta$, i.e., $\|f_\delta - f\| \leq \delta$, then we are interested in a regularization method for finding approximate solutions $u_\delta = R_\delta(f_\delta)$ such that the rate of convergence $u_\delta \to u^*$ ($\delta \to 0$) can be effectively estimated.

The coordinated approximation method. For the case where $H = L_2[0, 1]$, Gaponenko [1] suggested the method of coordinated approximation under the hypothesis that the exact solution $u^*(t)$ satisfies an a priori estimate

$$|u^*(t)| \leq R \quad \forall \ t \in [0, 1].$$  \hspace{1cm} (3)

Gaponenko's constructive method uses many special properties of $L_2[0, 1]$ and is complicated enough, so even its extension to the multidimensional case $L_2(G)$, where $G \subseteq \mathbb{R}^n$, is rather difficult. Therefore, it may be interesting to describe a general scheme of the method of coordinated approximation in abstract Hilbert spaces. Besides, it will be shown that, when $H = L_2[0, 1]$, Gaponenko's hypothesis (3) can be replaced by the weaker assumption (2). Moreover, when $u^*(t)$ is smooth enough, the convergence of regularized solutions can be improved.

2. Weak convergence in Hilbert spaces. Let $(X_1, X_0, H)$ be a triple of spaces, where $(X_1, \|\cdot\|_1)$, $(X_0, \|\cdot\|_0)$ are real separable Banach spaces and $H$ is a real Hilbert space with a scalar product $(\cdot, \cdot)$ and the corresponding norm $\|x\| = (x, x)^{1/2}$. Further, suppose that $X_1$ is densely, continuously, and compactly imbedded in $X_0$, and $X_0$ is densely and continuously embedded in $H$, i.e.,

i) $\forall x \in X_1 \Rightarrow x \in X_0$, $\|x\|_0 \leq C_1 \|x\|_1$; $\forall x \in X_0 \Rightarrow x \in H$, $\|x\| \leq C_0 \|x\|_0$;

ii) $\forall x \in X_0 \exists \{x_n\} \subseteq X_1$: $\|x_n - x\|_0 \to 0$, $n \to \infty$;

iii) $\forall w \in H \exists \{w_n\} \subseteq X_0$: $\|w_n - w\|_0 \to 0$, $n \to \infty$;

iii) every set bonded in the norm of $X_1$ is relatively compact in $X_0$. 
We introduce in $H$ a variational norm
\[
\|x\|_H = \sup \{ \langle x, v \rangle : v \in X_1; \|v\|_1 \leq 1 \}, \quad x \in H.
\] (4)

The following theorem shows that the weak convergence in $H$ can be described by means of the variational norm (4):

Theorem 1. A sequence $\{x_n\} \subset H$ converges weakly to $x \in H$, $x_n \to x$, if and only if the following two conditions hold:

\[
\exists R > 0: \|x_n\|, \|x\| \leq R, \quad \forall n \geq 1,
\] (5)

\[
\|x_n - x\|_H \to 0, \quad n \to \infty.
\] (6)

Proof. 1) Suppose that $(x_n \to x)$, $n \to \infty$. Then the sequence $\{x_n\}$ is bounded by a constant $R$. Since $\|x\| = \lim \|x_n\| \leq R$, condition (5) is satisfied. By definition (4), we can choose $\{v_n\} \subset S_1 = \{ x \in X_1 : \|x\|_1 \leq 1 \}$ and $\varepsilon_n \to +0$ such that

\[
\|x_n - x\|_H = \sup_{v \in S_1} \langle x_n - x, v \rangle \leq \langle x_n - x, v_n \rangle + \varepsilon_n.
\] (7)

Since $|\langle x_n - x, v_n \rangle| \leq \|x_n - x\| \|v_n\| \leq 2RC_0C_1 \|v_n\|_1 \leq 2RC_0C_1$, the sequence $\{x_n - x, v_n\}$ is compact. Let $l$ be an arbitrary limit point of $\langle x_n - x, v_n \rangle$. Then there exists a subsequence $\langle x_{n_k} - x, v_{n_k} \rangle$ convergent to $l$. Since $S_1 \subset X_0$ is compact, we can select a convergent subsequence $v_{n_k} \to v \in X_0$. Taking into account that $\langle x_{n_k} - x, v \rangle \to 0$ because $x_{n_k} \to x$ and

\[
\langle x_{n_k} - x, v_{n_k} - v \rangle \leq \|x_{n_k} - x\| \|v_{n_k} - v\| \leq 2RC_0 \|v_{n_k} - v\| \to 0,
\]

we get $\langle x_{n_k} - x, v_{n_k} \rangle = \langle x_{n_k} - x, v \rangle + \langle x_{n_k} - x, v_{n_k} - v \rangle \to 0$. Thus, we have

\[
l = \lim_{k} \langle x_{n_k} - x, v_{n_k} \rangle = \lim_{k} \langle x_{n_k} - x, v_{n_k} \rangle = 0.
\]

Since $l$ is an arbitrary limit point of $\langle x_n - x, v_n \rangle$, it follows that $\langle x_n - x, v_n \rangle \to 0$ and (7) implies (6).

2) It is evident that $X_1$ is dense in $H$; hence, $\forall v \in H \exists \tilde{v} \in X_1 (\|\tilde{v}\|_1 = r): \|v - \tilde{v}\| \leq \delta(r)$, where $\delta(r) \to 0$, $r \to +\infty$. Suppose that $\|x_n - x\|_H \leq \varepsilon_n$, where $\varepsilon_n$ is a sequence decreasing monotonically to zero. For any fixed $v \in H$, we choose $v_n \in X_1$ such that $\|v_n\|_1 = \varepsilon_n^{1/2}$ and $\|v_n - v\|_H \leq \delta(\varepsilon_n^{-1/2})$. It follows from

\[
\langle x_n - x, v \rangle = \langle x_n - x, v_n \rangle + \langle x_n - x, v - v_n \rangle \leq \varepsilon_n^{1/2} \|x_n - x\| \|v_n\|_H + 2R \delta(\varepsilon_n^{-1/2}) \leq \varepsilon_n^{1/2} \|x_n - x\|_H + 2R \delta(\varepsilon_n^{-1/2}) \leq \varepsilon_n^{1/2} + 2R \delta(\varepsilon_n^{-1/2}) \to 0, \quad n \to \infty,
\]

that $x_n \to x$. Theorem 1 is proved.

3. Finite dimensional approximations. Suppose that $X_0$ and $H$ possess a common basis $\{e_i\}_1^n \subset X_0$. Without loss of generality, we can assume that $\langle e_i, e_j \rangle = \delta_{ij}$, $i, j \geq 1$. Consider a bounded linear projection $P_N : H \to X_0$ defined by
\[ P_N x = \sum_{i=1}^{N} \langle x, e_i \rangle e_i. \]

Since \( \{ e_i \} \) is a common basis of \( X_0 \) and \( H \), we have
\[ \forall x \in H \quad \| (I - P_N) x \| \to 0, \quad N \to \infty, \tag{8} \]
\[ \forall x \in X_0 \quad \| (I - P_N) x \|_0 \to 0, \quad N \to \infty, \tag{9} \]
where \( I \) is the identity operator. Further,
\[ \| P_N x \|_0 \leq \sum_{i=1}^{N} \| \langle x, e_i \rangle \|_0 \| e_i \|_0 \leq \left( \sum_{i=1}^{N} \| e_i \|_0 \right) \| x \| = \tilde{C}_N \| x \|. \]

Thus,
\[ \| P_N x \|_0 \leq \tilde{C}_N \| x \|. \tag{10} \]

**Theorem 2.** For any \( x \in H \), the following estimate holds:
\[ \| (I - P_N) x \|_V \leq C_0 \varepsilon_N \| x \|, \tag{11} \]
where \( \varepsilon_N \to 0, \quad N \to \infty. \)

**Proof.** Since
\[ \| \langle x - P_N x, v \rangle \| = \| \langle x - P_N x, P_N v + v - P_N v \rangle \| \leq \| x \| \| (I - P_N) v \| \leq C_0 \| x \| \| (I - P_N) v \|_0 \]
for every \( x \in H \) and \( v \in S_1 = \{ v \in X_1 : \| v \|_1 \leq 1 \} \), we have \( \| (I - P_N) x \|_V \leq \leq C_0 \varepsilon_N \| x \| \), where
\[ \varepsilon_N = \sup_{v \in S_1} \| (I - P_N) v \|_0. \]

Since \( \| (I - P_N) v \|_0 \to 0 \) for every \( v \in X_0 \), we have \( \| (I - P_N) \|_{X_0 \to X_0} \leq C \) by the Banach–Steinhaus theorem. Further, because \( S_1 \subset X_0 \) is compact, for any \( \varepsilon > 0 \), there exists an \( \varepsilon(2C)^{-1} \)-finite net of \( S_1 \), denote it by \( \{ v_1, v_2, v_m \} \), such that \( \forall v \in S_1 \exist v_i : \| v - v_i \|_0 \leq \varepsilon(2C)^{-1}. \) On the other hand, there exists \( N = N_0(\varepsilon) \) such that
\[ \forall N \geq N_0 \quad \| (I - P_N) v_i \|_0 < \frac{\varepsilon}{2}, \quad i = 1, 2, \ldots, m. \]

By combining the last two inequalities, we have
\[ \forall v \in S_1, \forall N \geq N_0 \quad \| (I - P_N) v \|_0 \leq \| (I - P_N) v_i \|_0 + \| (I - P_N)(v - v_i) \|_0 \leq \frac{\varepsilon}{2} + C\varepsilon/2C = \varepsilon. \]

Thus, \( \varepsilon_N \to 0 \) as \( N \to \infty. \) Theorem 2 is proved.
In what follows, we suppose that \( \{e_i\}_1^\infty \subset X_1 \). Then
\[
\|P_N x\|_0 \leq \sum_{i=1}^N |\langle x, e_i \rangle| \|e_i\|_0 = \sum_{i=1}^N \|e_i\|_0 \|e_i\|_1 \left| \langle x, e_i / \|e_i\|_1 \rangle \right| \leq \left( \sum_{i=1}^N \|e_i\|_0 \|e_i\|_1 \right) \|x\|_V = C_N \|x\|_V.
\]

Thus, we get the important inequality
\[
\forall \ x \in H \quad \|P_N x\|_0 \leq C_N \|x\|_V. \tag{12}
\]

Let \( S = \{ x \in H : \|x\| \leq R \} \). Since \( S \) is a bounded closed set in the Hilbert space \( H \), it is weakly compact. By Theorem 1, \( (S, \| \cdot \|_V) \) is a compact metric space.

By setting
\[
S_N = P_N(S) = \left\{ x_N = \sum_{i=1}^N \langle x, e_i \rangle e_i : x \in S \right\},
\]
and
\[
S_N^{(L)} = \left\{ x = \sum_{i=1}^N c_i e_i, \|x\| \leq R ; \ c_i \in \left\{ 0, \pm \frac{R}{NL}, \ldots, \pm \frac{(NL-1)R}{NL}, \pm R \right\} \right\},
\]
we obtain the following result:

**Lemma 1.** \( S_N^{(L)} \) is a \( C_0 R (e_N + C_1 L^{-1}) \)-finite net for the compact set \( (S, \| \cdot \|_V) \).

**Proof.** First, we note that, by Theorem 2, \( \|x - x_N\|_V = \|x - P_N x\|_V \leq C_0 R e_N \). Further, since \( |\langle x, e_i \rangle| \leq \|x\| \leq R \), we can choose \( C_i^{(L)} \) from the finite set \( \{ 0, \pm R/\|e_i\| \} \) such that \( |C_i^{(L)} - \langle x, e_i \rangle| \leq R/\|e_i\| \) and \( |C_i^{(L)}| \leq |\langle x, e_i \rangle|, \ i = 1, 2, \ldots, N \). Now let
\[
x_N^{(L)} = \sum_{i=1}^N C_i^{(L)} e_i.
\]
A simple calculation shows that \( \|x_N^{(L)}\| \leq R \), we have
\[
\|x_N^{(L)} - x_N\|_V = \left\| \sum_{i=1}^N (C_i^{(L)} - \langle x, e_i \rangle) e_i \right\|_V \leq \sum_{i=1}^N |C_i^{(L)} - \langle x, e_i \rangle| \|e_i\|_V \leq \frac{R}{\|e_i\|} \sum_{i=1}^N \|e_i\|_V.
\]

Since \( \langle e_i, v \rangle \leq \|e_i\| \|v\| \leq C_0 \|v\|_0 \leq C_0 C_i \|v\|_1 \leq C_0 C_1 \) for any \( v \in S_1 = \{ v \in X_1 : \|v\|_1 \leq 1 \} \), we get \( \|e_i\|_V \leq C_0 C_1 \); hence, \( \|x_N^{(L)} - x_N\|_V \leq C_0 C_1 R L^{-1} \)

It is clear that \( S_N^{(L)} \) is a \( C_0 R (e_N + C_1 L^{-1}) \)-finite net for \( (S, \| \cdot \|_V) \). Indeed, for any \( x \in S \), there exists \( x_N^{(L)} \in S_N^{(L)} \) such that
\[
\|x_N^{(L)} - x\|_V \leq \|x - x_N\|_V + \|x_N - x\|_V \leq C_0 R (e_N + C_1 L^{-1}). \tag{13}
\]
Definition. $x^{(L)}_N$ is called a projection of $x \in S$ onto $S^{(L)}_N$.

4. Coordinated approximation method. In this section, we suppose that the nonlinear operator $\mathcal{A}: H \to H$ satisfies the following hypotheses:

h1) $\mathcal{A}$ is a one-to-one and strongly continuous mapping (i.e., if $x_N \to x$, then $\mathcal{A} x_N \to \mathcal{A} x$);

h2) The modulus of continuity of $\mathcal{A}$ in the compact set $(S, \| \cdot \|_V)$ is supposed to be known, $\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\| \leq \omega(\|v_1 - v_2\|_V), \forall v_1, v_2 \in S$;

h3) The exact solution $u^*$ of problem (1) satisfies the a priori estimate (2).

First, we choose the regularization parameters $N = N(\delta)$ and $L = L(\delta)$ from the conditions

$$\omega(C_0 R [\varepsilon_N + C_1 / l]) > \delta \quad \text{for} \quad n = 1, \ldots, N - 1, l = 1, \ldots, L - 1,$$

$$\omega(C_0 R [\varepsilon_N + C_1 / L]) \leq \delta.$$  

It is obvious that $N(\delta), L(\delta) \to +\infty$ as $\delta \to 0$.

Let $W(\delta) = \{ v \in S^{(L)}_N : \|\mathcal{A}(v) - f_\delta\| \leq 2\delta \}$, where $N = N(\delta)$ and $L = L(\delta)$ are the regularization parameters defined above.

**Lemma 2.** The set $W(\delta)$ is nonempty and $W(\delta) \to 0$ as $\delta \to 0$.

**Proof.** Denote by $u^{(L)}_N$ a projection of $u^* \in S$ onto $S^{(L)}_N$. It follows from (13) and (15) that

$$\|\mathcal{A}(u^{(L)}_N) - f_\delta\| \leq \|\mathcal{A}(u^{(L)}_N) - \mathcal{A}(u^*)\| + \|f - f_\delta\| \leq$$

$$\leq \omega(C_0 R [\varepsilon_N + C_1 / L]) + \delta \leq 2\delta.$$

Thus, $u^{(L)}_N \in W(\delta)$. Since $\mathcal{A}: (S, \| \cdot \|_V) \to \mathcal{A}(S)$ is a one-to-one and continuous mapping on the compact set $(S, \| \cdot \|_V)$, by the Tikhonov's lemma [3], the inverse operator $\mathcal{A}^{-1}: \mathcal{A}(S) \to (S, \| \cdot \|_V)$ is continuous. Let $\overline{\omega}(t)$ be the modulus of continuity of $\mathcal{A}^{-1}$, i.e., $\|\mathcal{A}^{-1}(f_1) - \mathcal{A}^{-1}(f_2)\|_V \leq \overline{\omega}(\|f_1 - f_2\|_V)$, $f_1, f_2 \in \mathcal{A}(S)$, where $\overline{\omega}(t) = \overline{\omega}(t, R) \to 0$ as $t \to 0$ and $R$ is fixed. Since $\|\mathcal{A}(v) - \mathcal{A}(\tilde{v})\| \leq \|\mathcal{A}(v) - f_\delta\| + \|\mathcal{A}(\tilde{v}) - f_\delta\| \leq 4\delta$ for any $v, \tilde{v} \in W(\delta)$, we have $\|v - \tilde{v}\|_V \leq \overline{\omega}(4\delta)$. Thus, $\text{diam } W(\delta) \leq \overline{\omega}(4\delta) \to 0$ as $\delta \to 0$.

Now we are ready to prove the main theorems.

**Theorem 3.** Suppose that all hypotheses h1)--h3) are satisfied. Then, for every $w^\delta \in W(\delta)$, the following estimate holds:

$$\|w^\delta - u^*\|_V \leq \text{diam } W(\delta) + C_0 R (\varepsilon_N(\delta) + C_1 / L(\delta)) = \mu(\delta, R).$$

**Proof.** For any $w^\delta \in W(\delta)$, we have $\|w^\delta - u^{(L)}_N\|_V \leq \text{diam } W(\delta)$. It follows from Lemma 2, estimate (13), and the last inequality that

$$\|w^\delta - u^*\|_V \leq \|w^\delta - u^{(L)}_N\|_V + \|u^{(L)}_N - u^*\|_V \leq$$

$$\leq \text{diam } W(\delta) + C_0 R (\varepsilon_N(\delta) + C_1 / L(\delta)) \leq$$

$$\leq \overline{\omega}(4\delta) + C_0 R (\varepsilon_N(\delta) + C_1 / L(\delta)) \to 0, \quad \delta \to 0.$$

Let $w^\delta_k = P_k w^\delta$ and $u^*_k = P_k u^*$ be the finite dimensional approximations of $w^\delta$ and $u^*$, respectively.
Theorem 3 and estimate (12) lead us to the following practically useful relation:
\[ \| w^\delta_k - u^* \|_0 \leq c_k \| w^\delta - u^* \|_V \leq c_k \{ \text{diam } W(\delta) + C_0 R [ \varepsilon N(\delta) + C_1 / L(\delta)] \}. \]

**Theorem 4.** In addition to hypotheses h1)–h3), assume that \( u^* \in X_0 \). Then \( \| u^\delta - u^* \|_0 \to 0, \delta \to 0 \), where \( u^\delta = P_k w^\delta \), and \( K = K(\delta) \) satisfies the relations
\[ C_k \mu(\delta, R) \leq 1 / k, \quad k = 1, \ldots, k, \quad C_k \mu(\delta, R) > 1 / k, \quad k = K + 1. \]

**Proof.** It is clear that \( K = K(\delta) \to \infty \) as \( \delta \to 0 \). By letting \( u_k = P_k u^* \), we have
\[ \| u^\delta - u^* \|_0 \leq \| P_k u^* \|_0 + \| P_k u^\delta - u^* \|_0. \]
Since \( u^* \to X_0 \), we have \( \| P_k u^\delta - u^* \|_0 \to 0, \delta \to 0 \). Further, by (12),
\[ \| u^\delta - P_k u^* \|_0 = \| P_k (w^\delta - u^*) \| \leq C_k \| w^\delta - u^* \|_V \leq C_k(\delta) \mu(\delta, R) \leq 1 / K(\delta). \]
Thus,
\[ \| u^\delta - u^* \|_0 \leq 1 / k(\delta) + \| P_k(\delta) u^* - u^* \|_0 \to 0, \delta \to 0. \]

Now let us return to the case \( H = L_2[0, 1] \). Define \( X_0 = H^m_0[0, 1] \) to be a subspace consisting of all functions \( u(t) \) in \( C^{m-1}[0, 1] \) with \( u^{m-1}(t) \) absolutely continuous on \( [0, 1] \), \( u^{(m)} \in L_2[0, 1] \), and \( u^{(i)}(0) = u^{(i)}(1) \), \( i = 0, m-1 \). Then \( X_0 \) is a Banach space with the norm
\[ \| u \|_0 = \sum_{i=0}^{m-1} \max_{0 \leq t \leq 1} |u^{(i)}(t)| + \left( \int_0^1 \| u^{(m)}(t) \|^2 \, dt \right)^{1/2}. \]
Further, denote by \( X_1 \) the space \( C^l_0[0, 1] \) of \( l \)-times continuously differentiable functions satisfying the boundary conditions: \( u^{(i)}(0) = u^{(i)}(1), \quad i = 0, l \). Let
\[ \| u \|_1 = \sum_{i=0}^l \max_{0 \leq t \leq 1} |u^{(i)}(t)|. \]
Suppose that \( l \geq m + 1 \). Then \( X_1 \) is continuously and compactly imbedded in \( X_0 \) and \( X_0 \) is continuously imbedded in \( H \). By using well-known facts on Fourier series [2], we can prove that the common basis of \( X_0 \) and \( H \) consists of trigonometric functions, which also belong to \( X_1 \),
\[ \{ e_n \}_{n=0}^{\infty} = \{ 1, \sqrt{2} \text{ sin } 2\pi n t, \sqrt{2} \text{ cos } 2\pi n t \}. \]
Hence, \( X_1 \) is dense in \( X_0 \) and \( X_0 \) is dense in \( H \). By applying Theorems 3 and 4 we come to the following conclusions:
1. The method of coordinated approximation is still convergent if estimate (3) is replaced by the weaker assumption (2).
2. If the exact solution \( u^*(t) \) is smooth enough, i.e., \( u^* \in H^m_0[0, 1] \), then the regularized solutions \( u^\delta \) converge to \( u^* \) in the norm of \( H^m_0[0, 1] \).


Received 02.12.92