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Automorphisms of homogeneous symmetric groups and hierarchomorphisms of rooted trees

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Dedicated to R. I. Grigorchuk on the occasion of his 50th birthday

Abstract. A representation of homogeneous symmetric groups by hierarchomorphisms of spherically homogeneous rooted trees are considered. We show that every automorphism of a homogeneous symmetric (alternating) group is locally inner and that the group of all automorphisms contains Cartesian products of arbitrary finite symmetric groups.

The structure of orbits on the boundary of the tree where investigated for the homogeneous symmetric group and for its automorphism group. The automorphism group acts highly transitive on the boundary, and the homogeneous symmetric group acts faithfully on every its orbit. All orbits are dense, the actions of the group on different orbits are isomorphic as permutation groups.

1. Introduction

The problem of classifications and investigation of direct limits of finite symmetric groups in certain sense are model problems for theory of locally finite groups. A number of results on such groups is contained in the last chapter of the monograph [KW], and later results are mentioned in survey of B.Hartley [Har]. A. E. Zalesskii in [Zal] introduced a certain class of embeddings of finite symmetric and alternative groups, the so called diagonal embeddings. He showed that they play an important role in investigation of the lattice of ideals in the group rings of direct limits of alternating groups (see also [Zal2], [HZ]).

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In the papers [KS1], [KS2] a certain type of diagonal embeddings was distinguished, namely *strictly diagonal embeddings*, and a full classification of direct limits of finite symmetric and alternating groups with respect to such embeddings was done. Such limits are called *homogeneous symmetric* (resp. *alternating*) groups. In the papers [KS1], [KS2] a number of properties of homogeneous symmetric and alternating groups were obtained. This includes their normal structure, description of the conjugacy classes and the centralizers of elements.

Although a number of problems concerning the structure of such groups are still open. In particular, professor O. Kegel and professor A. Zalesskii drew attention of the authors to the problem of characterization of the automorphism groups of homogeneous symmetric and alternating groups.

We approach this problem in our paper using a new technique of representing homogeneour symmetric group by hierarchomorphisms of spherically homogeneous rooted trees. A notion of *spherical hierarchomorphisms* of a spherically homogeneous rooted tree is introduced. We show that spherical hierarchomorphisms act on the boundary of the tree by local isometries and prove that the group of local isometries of a rooted tree is a product of two its subgroups. One is isomorphic to the homogeneous symmetric group defined by the tree type, and the second one is the group of all isometries of the tree boundary (i.e., the automorphism group of the tree).

Using result of M. Rubin [Rub], we prove that the automorphism group of the homogeneous symmetric group coincides with its normalizer in the homeomorphism group of the tree boundary. Making use of this fact we get (Theorem 13) that every automorphism of an arbitrary homogeneous symmetric (alternating) group is locally inner. We also get that the group of all automorphisms of a homogeneous symmetric group contains Cartesian products of arbitrary finite symmetric groups, i.e., an arbitrary countable residually finite group is embedded into the automorphism group.

The structure of orbits of the described action of the homogeneous symmetric group and of its automorphism group on the tree boundary is also investigated. In particular it is proved that automorphism group acts highly transitively on the tree boundary. The homogeneous symmetric group acts faithfully on every its orbit, all orbits are dense in the tree boundary, the actions of the group on different orbits are isomorphic as permutation groups.

We use standard terminology for groups acting on a rooted tree. For definitions see for example [GNS].

2. Preliminaries

2.1. Finite spherical hierarchomorphisms

The notion of a *hierarchomorphism* of a homogeneous rooted tree is a generalization of the notion of tree automorphism [Ner]. All hierarchomorphisms of a homogeneous rooted tree T form a group $Hier_0(T)$. This group is called the *large* hierarchomorphism group of T.

The notion of a hierarchomorphism can not be extended to an arbitrary spherically homogeneous rooted tree. But we can introduce a natural subgroup in the large hierarchomorphism group which can be defined for an arbitrary spherically homogeneous rooted tree.

Definition 1. *The bijection $V(T) \longrightarrow V(T)$ (where $V(T)$ is the set of vertices of the rooted tree T) is a* spherical hierarchomorphism *if for some $k \in \mathbb{N}$ it acts as a permutation on the vertices of level number k and preserves the incidence relation between all vertices from the levels of numbers $\geq k$.*

All spherical hierarchomorphisms of a spherically homogeneous rooted tree T form a group which we denote $LHier_0(T)$.

Every spherical hierarchomorphism of T can be represented in the form

$$u = (\alpha_1, \ldots, \alpha_{m_k})\sigma_k,$$

where σ_k is a permutation of the vertices of kth level and α_i is the automorphism of the subtree T_{v_i} of T with root v_i $(1 \leq i \leq m_k)$, where $v_1, \ldots, v_{m_k}$ are all vertices of kth level of T.

This decomposition is not unique. But there exists the minimal value of k for which such decomposition is possible. We call the decomposition of u for such k a *canonical* decomposition.

The spherical hierarchomorphism u is an automorphism of tree if and only if $k = 0$ in the canonical decomposition of u.

We say that a spherical hierarchomorphism u is *finite* if

$$u = (e, \ldots, e)\sigma_k,$$

where e is the identical automorphism of the subtrees.

Obviously, the product of two finite spherical hierarchomorphisms is finite and all finite spherical hierarchomorphisms of the tree T form a group, which will be denoted $LHier_{0f}(T)$.

Together with the large hierarchomorphism group, we consider the *small* group of hierarchomorphisms $Hier(T)$ of the tree T. This is the group of transformations of the boundary of T induced by the elements of $Hier_0(T)$ for a homogeneous rooted tree T. It is clear that we can define the *small* spherical hierarchomorphism group $LHier(T)$ and *small* finite spherical hierarchomorphism group $LHier_f(T)$ in the $Hier_0(T)$ for a spherically homogeneous rooted tree T, analogically to the "large" case.

2.2. Homogeneous symmetric and alternating group

At first we introduce the notion of a homogeneous symmetric group. We will do it as in the paper of N. Kroshko and the second named author [KS1].

By S_n we denote the symmetric group over the set $\{1, 2, \dots, n\}$.

Definition 2. *An embedding d of a transitive permutation group (G, X) into a permutation group (H, Y) is called* diagonal *if the restriction of $d(G)$ onto every orbit of length more than 1 is isomorphic to (G, X) as a permutation group. The diagonal embedding is called* strictly diagonal *if the length of every orbit of the image $d(G)$ on the set Y is greater than 1.*

Definition 3. *We say that the group G is a group of strictly diagonal type if G is a limit of an ascending chain of the symmetric groups S_{n_i} ($i \in \mathbb{N}$) where all inclusions $S_{n_i} \subset S_{n_{i+1}}$ are strictly diagonal.*

By x^α we denote the image of an element $x \in \{1, \dots, n\}$ under a permutation $\alpha \in S_n$.

Definition 4. *A permutation $d^r\alpha \in S_{nr}$ defined for $\alpha \in S_n$ by the rule*

$$(kn + i)^{d^r\alpha} = kn + i^\alpha \quad (0 \le k \le r - 1,\ 1 \le i \le n)$$

is called a homogeneous r-spreading *of the permutation α.*

It is clear that the map $d^r : \alpha \longrightarrow d^r\alpha$, $\alpha \in S_n$ is a strictly diagonal embedding of S_n into S_{nr}.

Let $\Omega = (a_1, a_2, \dots)$ be a sequence of natural numbers greater than 1, and let

$$f_\Omega(n) = a_1 \dots a_n.$$

Definition 5. *The limit of the following direct system*

$$S_{f_\Omega(1)} \xrightarrow{d^{a_2}} S_{f_\Omega(2)} \xrightarrow{d^{a_3}} \dots$$

$$(A_{f_\Omega(1)} \xrightarrow{d^{a_2}} A_{f_\Omega(2)} \xrightarrow{d^{a_3}} \dots)$$

is called homogeneous symmetric *(*alternating*) group and is denoted DS_Ω (DA_Ω).*

The sequence Ω determines a supernatural number $a_1a_2\ldots$ (we will denote this number Ω too). This number is called *characteristic* of Ω and and of the group DS_Ω, determined by this sequence.

Every homogeneous symmetric (alternating) group is naturally a subgroup of the group $S(\mathbb{N})$.

Proposition 1. *[KS1]*

1. *Two homogeneous symmetric (alternating) subgroup of $S(\mathbb{N})$ coincide if and only if their characteristics are equal.*

2. *Different homogeneous symmetric (alternating) subgroups of $S(\mathbb{N})$ are non-isomorphic.*

3. *If $G = \cup_{i\in\mathbb{N}} S_{a_i}$ ($G = \cup_{i\in\mathbb{N}} A_{a_i}$) is a group of strictly diagonal type and $\Omega = (a_1, a_2, \ldots)$ then G is isomorphic to DS_Ω (DA_Ω).*

2.3. Boundary of rooted tree and its homeomorphisms

We will study automorphisms of the homogeneous symmetric groups using representation of these groups by homeomorphisms of boundaries of rooted trees. We introduce now all required notions to define such representations.

Let T be a locally finite rooted tree with the rooted vertex v_0.

For every two vertices u, v of the tree T ($u, v \in V(T)$) we define the *distance* between u and v, denoted by $d(u, v)$, to be equal to the length of the shortest path connecting them.

For the rooted tree T with the root v_0 and an integer $n \geq 0$ we define the *level number* n (the sphere of the radius n) to be the set

$$V_n(T) = \{v \in V(T) : d(v_0, v) = n\}.$$

Let us say that a vertex v of the tree T lies under a vertex w, if the path connecting the vertex v and the root, contains the vertex w.

Let us denote by T_v the full subtree consisting of all vertices, that lie under the vertex v with the root v.

An end of the rooted tree is an infinite path without repetitions which starts in the root.

We will denote by ∂T the set of all ends of the tree T (its *boundary*).

We can introduce a natural ultrametric on ∂T putting

$$\rho(\gamma_1, \gamma_2) = 1/(n+1),$$

where n is the length of the maximal common part of the paths γ_1 and γ_2. The topology introduced by the metric ρ is compact, totally disconnected and has a base of open sets

$$P_{ni} = \{\gamma \in \partial T \mid \quad i \in \gamma\}, \quad i \in V_n(T).$$

Note that $P_{ni} = \partial T_{v_i}$ and P_{ni} is a ball with center in any end $\gamma \ni v_i$ and radius equal to $1/(n+1)$, where $v_i \in V_n$. Every ball in ∂T is clopen. We we have $P_{ni} \cap P_{mj} \neq \emptyset$ if and only if $P_{ni} = P_{mj}$ that is $n = m$ and $i = j$.

If the degree of a vertex $v \in V_n(T)$ depends only on n, then the tree T is called *spherically homogeneous. Characteristic* of a spherically homogeneous tree T is the sequence $\Omega = (a_0, a_1, \ldots)$, where a_0 is degree of the root and $a_n + 1$ is degree of any vertex of nth level.

Let T be a spherically homogeneous rooted tree with root v_0 and characteristic Ω. All such trees are isomorphic to the tree T_Ω whose set of vertices is the set of all finite sequences $(i_0, i_1, \ldots, i_{n-1})$, where $i_k \in \{1, 2, \ldots, a_k\}$ and $n \geq 0$ is an integer. We include also the empty sequence (corresponding to $n = 0$) and two vertices are adjacent if and only if they are of the form $(i_0, \ldots, i_{n-1}), (i_0, \ldots, i_{n-1}, i_n)$.

The full automorphism group $\operatorname{Aut} T_\Omega$ of T_Ω acts on ∂T_Ω by isometries. Moreover $\operatorname{Aut} T_\Omega = \operatorname{Iso} T_\Omega$.

The subgroup of $\operatorname{Aut} T_\Omega$ of all automorphisms fixing all vertices of the level number n is denoted by $\operatorname{Stab}_{\operatorname{Aut} T_\Omega}(n)$ and is called the *level stabilizer*.

Let T^k be the finite tree that is obtained from the tree T by truncation of the vertices belonging to the levels of numbers greater than k.

By $\operatorname{Hom} \partial T_\Omega$ we denote the group of all homeomorphisms of the boundary of the rooted tree T_Ω.

Let H_n be the group of homeomorphisms of the boundary ∂T_Ω, which permute the balls P_{ni} only, i.e., do not change the coordinates i_k of the vertices $(i_0, \ldots, i_m)$ for all $k \geq n$. Clearly, H_n is isomorphic to the symmetric group $S_{f_\Omega(n)}$, and $H_n \leq H_k$ for $n \leq k$. Let us define a subgroup H_Ω of the full homeomorphism group ∂T_Ω, as the union of the subgroups H_n, $n \in \mathbb{N}$.

Let $AH_n \leq H_n$ be the subgroup isomorphic to the alternating group $Alt_{f_\Omega(n)}$. Clearly, $AH_n \leq AH_k$ for $n \leq k$. Let us define a subgroup AH_Ω of the full homeomorphism group ∂T_Ω, as the union of the subgroups AH_n, $n \in \mathbb{N}$.

Also we need some facts on homeomorphism group of ∂T.

Theorem 2 ([Rub] Corollary 3.13c). *Let X be locally compact Hausdorff space, G_1, G_2 be subgroups and for every open $D \subseteq X$, $x \in D$ and*

$i = 1, 2$ the set $\{g(x) \mid g \in G_i$ and restriction of g on $X \backslash D$ is identity$\}$ be somewhere dense. If $\varphi : G_1 \longrightarrow G_2$ is an isomorphism then there is a homeomorphism $h \in \operatorname{Hom} X$ such that for every $g \in G_1$ the following equality holds $\varphi(g) = hgh^{-1}$.

It is easy to see that

Remark 1. *The space ∂T is a locally compact Hausdorff space, and the groups H_Ω and AH_Ω satisfy conditions of Rubin's Theorem.*

3. Main results

3.1. Local isometries

One more reason for studying the groups H_Ω is that these groups appear as natural subgroups in the local isometry group of rooted tree boundary.

Let us consider the concept of a *local isometry*.

Let (X, ρ) be a metric space.

Definition 6. *A bijection $\alpha : X \longrightarrow X$ is called* local isometry *if for every $x \in X$ there exists a neighborhood U_x of x such that for every $x_1, x_2 \in U_x$ the equality*

$$\rho(x_1^\alpha, x_2^\alpha) = \rho(x_1, x_2)$$

holds.

Definition 7. *A bijection $\alpha : X \longrightarrow X$ is called* uniformly local isometry *if there exists $\delta > 0$ such that*

$$\rho(x_1^\alpha, x_2^\alpha) = \rho(x_1, x_2)$$

for all $x_1, x_2 \in X$ such that $\rho(x_1, x_2) < \delta$.

Obviously, these two definitions are different. But in some special cases they coincide.

Lemma 3. *For a compact metric space (X, ρ) every local isometry of (X, ρ) is a uniformly local isometry.*

Proof of this lemma is straightforward.

Let us denote the group of all local isometries by $LI(\partial T_\Omega)$.

Lemma 4. *Let g be a local isometry of ∂T_Ω. There exist $\alpha \in \operatorname{Aut} T_\Omega$ and $\beta \in H_\Omega$ such that $g = \alpha\beta$.*

Proof. First of all, the homeomorphism g is a uniformly local isometry. Take some $\delta > 0$ such that the homeomorphism g preserves the distances between ends, which are at distance less than δ. Then the homeomorphism g preserves distances between ends from each ball P_{ni}, $i \in V_n(T_\Omega)$, where n is big enough. It means that g acts as a permutation on $V_n(T_\Omega)$. Let $\beta^{-1} \in H_\Omega$ be a homeomorphism which acts on $V_n(T_\Omega)$ in the same way as g. We have that $\alpha = g\beta^{-1}$ acts trivially on $V_n(T_\Omega)$. Therefore α acts as an isometry on each ball P_{ni}, $i \in V_n(T_\Omega)$. Thus α is an automorphism of T_Ω. □

Conversely, it is easy to see that $\operatorname{Aut} T_\Omega H_\Omega < LI(\partial T_\Omega)$. So we have

Theorem 5. *The group $LI(\partial T_\Omega)$ is decomposed in product of* $\operatorname{Aut} T_\Omega$ *and H_Ω.*

Theorem 6.

1. *The group $LHier(T_\Omega)$ is isomorphic to the group $LI(\partial T_\Omega)$.*
2. *The group $LHier_f(T_\Omega)$ is isomorphic to the group H_Ω.*
3. *The group H_Ω (AH_Ω) is isomorphic to the group DS_Ω (DA_Ω).*

Proof. The first and the second parts of the theorem follow immediately from Theorem 5 and definitions of the groups $LHier(T_\Omega)$ and $LHier_f(T_\Omega)$.

The subgroup $H_n < H_\Omega$ acts naturally on the level number n of the tree T_Ω for every $n \in \mathbb{N}$. Note that the embedding of (H_n, V_n) into (H_k, V_k), where $k > n$, is strictly diagonal. Hence the group H_Ω is of a strictly diagonal type. So, the third part of the theorem follows from Proposition 1, item 3. □

Corollary 1. *The groups H_{Ω_1} (AH_{Ω_1}) and H_{Ω_2} (AH_{Ω_2}) are isomorphic iff sequences Ω_1 and Ω_2 define the same supernatural number.*

Proposition 7. *All finitely generated subgroups in the group $LI(\partial T_\Omega)$ are residually finite.*

Proof. For every finitely generated subgroup G of the group $LI(\partial T_\Omega)$ we can choose another Ω_1 such that there exists a natural embedding of the group G into $\operatorname{Aut} T_{\Omega_1}$. And the group $\operatorname{Aut} T_{\Omega_1}$ is residually finite (see for instance [GNS]).

Really, there exists $t > 0$ such that for all $g \in G$ the equality

$$\rho(x^g, y^g) = \rho(x, y)$$

holds for $x, y \in T_\Omega$, $\rho(x, y) < t$.

Let $n = [\frac{1}{t}]$. Define Ω_1 in such a way that

$$a'_1 = a_1 a_2 \cdots a_n, a'_k = a_{n+k-1}, k > 1.$$

It is easy to see that elements of G act onto ∂T_{Ω_1} as isometries. □

3.2. Automorphisms

Lemma 8. *The group H_Ω has trivial center.*

Proof. Since H_Ω is union of subgroups $H_n \simeq S_{f_\Omega(n)}$, which have trivial centers, thus H_Ω also has trivial center. □

Let us denote $N_\Omega = N_{\mathrm{Hom}\,\partial T_\Omega}(H_\Omega)$

Theorem 9. $N_\Omega \simeq \mathrm{Aut}\, H_\Omega$.

Proof. From Rubin's Theorem 2 and Remark 1 we get that every automorphism of $H_\Omega < \mathrm{Hom}\,\partial T_\Omega$ is induced by a homeomorphism of ∂T_Ω. Taking into account Lemma 8 we get the required statement. □

Proposition 10. *Every automorphism of the group H_Ω (AH_Ω) is locally inner.*

Proof. The proofs for the group H_Ω and for the group AH_Ω are similar. So we will prove this proposition for H_Ω only.

Let $\alpha \in \mathrm{Aut}\, H_\Omega$ and $g \in H_\Omega$. We have $g \in H_n$ for some $n \in \mathbb{N}$. Since H_Ω is union of its subgroups H_n ($n \in \mathbb{N}$), there exists $k \in \mathbb{N}$ such that

$$\alpha(H_n) \leq H_k.$$

Let us show that $\alpha|_{H_n}$ is induced by an inner automorphism of H_k. By Theorem 9 the automorphism α is induced by some homeomorphism of ∂T_Ω.

Let $\gamma \in \mathrm{Hom}\,\partial T_\Omega$ induce the automorphism α. Suppose, that for some $1 \leq i, j \leq f_\Omega(n)$, $i \neq j$ and $1 \leq l \leq f_\Omega(k)$ holds

$$\gamma^{-1}(P_{kl}) \cap P_{ni} \neq \emptyset, \tag{1}$$

$$\gamma^{-1}(P_{kl}) \cap P_{nj} \neq \emptyset. \tag{2}$$

Let $g \in H_n$ be such that

$$g(P_{ni}) = P_{ni}, \tag{3}$$

$$g(P_{nj}) \neq P_{nj}. \tag{4}$$

Since $g^\gamma \in H_k$ we get

$$g^\gamma(P_{kl}) = P_{km}$$

where $1 \le m \le f_\Omega(k)$. Taking into account (1) and (3) we get

$$g^\gamma(P_{kl}) = P_{kl}$$

and by $g^\gamma \in H_k$ we have $g^\gamma(x) = x$ for all $x \in P_{kl}$. Also taking into account (2) and (4) we get that there is $x_0 \in P_{kl} \cap \gamma(P_{nj})$ and

$$g^\gamma(x_0) \neq x_0.$$

This is a contradiction. Hence,

$$\gamma^{-1}(P_{kl}) \subset P_{ni}$$

for some i.

Let

$$\gamma(P_{ni}) = \{P_{kl_{i1}}, \dots, P_{kl_{i,r(i)}}\}$$

where $1 \le l_{i1}, \dots, l_{i,r(i)} \le f_\Omega(k)$.

It follows from proved above that for different i and j the sets $\gamma(P_{ni})$ and $\gamma(P_{nj})$ don't intersect. Additionally, $r(i)$ doesn't depend on i, because $g^\gamma \in H_k$ for all $g \in H_n$ and H_n is transitive.

Let $g_i \in H_n$, $1 \le i \le f_\Omega(n)$ such that $g_i(P_{ni}) = P_{n1}$. Since $g_i^\gamma \in H_k$ then for every $1 \le m \le r$ the following equality holds

$$g_i(P_{kl_{im}}^{\gamma^{-1}}) = P_{kl_{1t}}^{\gamma^{-1}}$$

for some $1 \le t \le r$.

So the sets

$$\{g_i(P_{kl_{i1}}^{\gamma^{-1}}), \dots, g_i(P_{kl_{ir}}^{\gamma^{-1}})\}$$

not depend on i. Therefore there exists a homeomorphism $\delta \in \operatorname{Hom} \partial T_\Omega$ such that

$$\delta\gamma|_{H_n} \in H_k$$

and for every $1 \le j \le r$ the following equation is valid

$$P_{kl_{ij}}^{\gamma^{-1}\delta^{-1}} = P_{ks} \subset P_{ni},$$

for some $1 \le s \le r$.

Let us note that δ acts on H_n nontrivially only within every ball P_{ni}. Moreover on all balls δ acts in the same way. Therefore δ belongs to the centralizer of H_n in $\operatorname{Hom} \partial T_\Omega$. Hence, $g^{\delta^{-1}\gamma^{-1}} = g^\gamma$ for all $g \in H_n$.

So we proved that $\alpha|_{H_n}$ is induced by an inner automorphism of H_k. Consequently every automorphism H_Ω is locally inner. □

The next fact follows from the proof of the theorem.

Corollary 2. *Let* $\alpha(H_n) \le H_k$ $(\alpha(AH_n) \le AH_k)$. *Then* $\alpha|_{H_n} \in \operatorname{Inn} H_k$ $(\alpha|_{AH_n} \in \operatorname{Inn} AH_k)$.

We will construct below a set of subgroups of $\operatorname{Aut} H_\Omega$ and will show that every residually finite group can be embedded into some subgroup from this set.

Let $N = \{n_0 = 0, n_i \ge 3 \mid i \in \mathbb{N}\}$ be an increasing sequence of non-negative integers. We will consider automorphisms $\alpha \in \operatorname{Aut} H_\Omega$ satisfying the condition

$$\alpha(H_{n_i}) = H_{n_i}, \tag{5}$$

for all natural i. Obviously, all such automorphisms form subgroup A_N of the group $\operatorname{Aut} H_\Omega$.

Let $q_\Omega(n_k, n_{k+1}) = a_{n_k+1} \dots a_{n_{k+1}}$.

Lemma 11. *The centralizer* $C_{H_{n_{k+1}}}(H_{n_k})$ *is isomorphic to the symmetric group* $S(q_\Omega(n_k, n_{k+1}))$.

Proof. Elements of the centralizer $C_{H_\Omega}(H_{n_k})$ act nontrivially only inside the balls $P_{n_k i}$ $(1 \le i \le f_\Omega(n_k))$. Moreover every element of this centralizer acts equally on all the balls $P_{n_k i}$. Therefore we get

$$H_{n_{k+1}} \cap C_{H_\Omega}(H_{n_k}) \simeq S(q_\Omega(n_k, n_{k+1})).$$

□

Proposition 12. *The group* A_N *is isomorphic to the Cartesian product of the groups* $S_{q_\Omega(n_k,n_{k+1})}$, $k \in \mathbb{N}$.

Proof. Clearly an automorphism defined by an element of $C_{H_{n_k}}(H_{n_{k-1}})$, $k \in \mathbb{N}$ belongs to A_N. Let us denote by D_k the subgroup $C_{H_{n_k}}(H_{n_{k-1}})$ of A_N, $k \in \mathbb{N}$. By Lemma 11 the groups D_k and $S_{q_\Omega(n_{k-1},n_k)}$, $k \in \mathbb{N}$ are isomorphic.

Every H_{n_i} is isomorphic to some symmetric group of degree greater than 6 (because $n_1 \ge 3$). Hence a restriction of the automorphism from A_N onto H_{n_i} is an inner automorphism of H_{n_i}.

Let $\alpha_1 = \alpha|_{H_{n_1}}$ be an inner automorphism defined by an element of H_{n_1}. We get $\alpha_1 \in A_N$ and that $\alpha_1^{-1}\alpha$ acts trivially on H_{n_1}.

Let $\alpha_2 = \alpha_1^{-1}\alpha|_{H_{n_2}}$ be an inner automorphism defined by an element of H_{n_2}. We get $\alpha_2 \in A_N$ and that $\alpha_2^{-1}\alpha_1^{-1}\alpha$ acts trivially on H_{n_2}. Moreover, $\alpha_2 \in C_{H_{n_2}}(H_{n_1})$. Hence α_2 commutes with α_1.

The automorphisms $\alpha_k \in A_N$, $k \ge 3$ are analogically defined. These automorphisms have the next properties:

1. $\alpha_k^{-1} \ldots \alpha_1^{-1} \alpha|_{H_{n_k}} = Id$;

2. $\alpha_k \in C_{H_{n_k}}(H_{n_{k-1}})$;

3. $\alpha_k \alpha_n = \alpha_n \alpha_k$,

for all $n, k \in \mathbb{N}$.

Let us note that only a finite number of automorphisms α_k act non-trivially on ∂T_Ω. So the infinite product

$$\alpha_1 \alpha_2 \ldots \alpha_k \ldots$$

gives a well defined automorphism of ∂T_Ω. It follows from the properties of automorphisms α_k that

$$\alpha = \alpha_1 \alpha_2 \ldots,$$

i.e., that the group A_N is the product of its subgroups D_k, $k \in \mathbb{N}$. The subgroups D_k commute pairwise. Therefore all of them are normal subgroups of A_N. Since D_k has trivial center and $D_{k+1} D_{k+2} \ldots$ centralizes D_k, the intersection of D_k and $D_{k+1} D_{k+2} \ldots$ is trivial. Further, $D_1 \ldots D_{k-1} \subset \operatorname{Inn} H_{n_k}$ and H_{n_k} has trivial center and D_k centralizes H_{n_k}. Hence intersection of D_k and $D_1 \ldots D_{k-1}$ is trivial. So intersection $D_k \cap D_1 \ldots D_{k-1} D_{k+1} D_{k+2} \ldots$ is trivial. Taking into account the proved above we get that A_N is Cartesian product of its subgroups D_k, $k \in \mathbb{N}$. □

We have the following corollary of this proposition.

Corollary 3. *Every residually finite group can be embedded into* $\operatorname{Aut} H_\Omega$.

Proof. Since we can choose the sequence N arbitrarily, we can make the number $q_\Omega(n_k, n_{k+1})$ arbitrarily big for all $k \in \mathbb{N}$. Therefore for a residually finite group G we can choose such a sequence N than G is embedded into A_N. So G is embedded into $\operatorname{Aut} H_\Omega$. □

So, we can conclude

Theorem 13. *The automorphism group of the group H_Ω (AH_Ω) has the following properties*

1. *Every automorphism of the group H_Ω (AH_Ω) is locally inner.*

2. *The group* $\operatorname{Aut} H_\Omega$ *contains countable Cartesian product of finite symmetric groups of arbitrary great degree.*

3. *Every countable residually finite group can be embedded into* $\operatorname{Aut} H_\Omega$

3.3. Orbits

Theorem 14. *The group N_Ω is highly transitive on ∂T_Ω.*

Proof. Let $\Sigma_\nu = \{\xi_1, \ldots, \xi_n, \nu\} \subset \partial T_\Omega$ be a set of arbitrary points of ∂T_Ω. Also let $\Sigma = \{\xi_1, \ldots, \xi_n\}$.

Let us show that there exists $\alpha \in N_\Omega$ such that

$$\alpha : \xi_i \longrightarrow \xi_i,\ 1 \leq i \leq n-1;$$

$$\alpha : \xi_n \longrightarrow \nu.$$

Let k be the minimal natural number such that every point of Σ_ν belongs to one of the balls $\{P_{k1}, \ldots, P_{kt}\} = P_k$. Let us choose $\{m_j | j \geq 1\}$ such that the inequality

$$|P_{m_{j+1}1}| < 2^{-1}|P_{m_j1}| < |P_{k1}|$$

holds. We have

$$|P_{m_j1}| < 2^{-j}. \tag{6}$$

There exists an element α_{m_11} of H_{m_11} such that

$$\alpha_{m_1} : P_{m_1i} \longrightarrow P_{m_1i},\ 1 \leq i \leq n-1$$

$$\alpha_{m_1} : P_{m_1n} \longrightarrow P_{m_10}.$$

After application of $\alpha_{m_{j-1}}$ the images of the points of Σ and the corresponding points $\xi_1, \ldots, \xi_{n-1}, \nu$ belong to the same balls from P_{j-1}.

Therefore, by induction there exists an element α_{m_j} from H_{m_j} such that

$$\alpha_{m_j} : P_{m_ji} \longrightarrow P_{m_ji},\ 1 \leq i \leq n-1$$

$$\alpha_{m_j} : P_{m_jn} \longrightarrow P_{m_j0}$$

where $\xi_i \in P_{m_ji}, \nu \in P_{m_j0}$ and

$$\alpha_{m_j}\alpha_{m_{j-1}}^{-1} \in C_{H_{m_j}}(H_{m_{j-1}}) \tag{7}$$

By (7) and taking into account the choice of $\alpha_{m_j}(\alpha_{m_j} \in H_{m_j})$ we get that the sequence $\{\alpha_{m_j}\}$ generates an automorphism $\alpha \in A_\Omega \leq N_\Omega$. Let us show that α is satisfies the required properties.

We have $\xi_i, \alpha(\xi_i) \in P_{m_ji}$ and $\nu, \alpha(\xi_n) \in P_{m_j0}$ by construction for $j \in \mathbb{N}$. By (6), we get $\alpha(\xi_i) = \xi_i; \alpha(\xi_n) = \nu$. □

Corollary 4. *The group* Aut H_Ω *is continual.*

Theorem 15.

1. *Every orbit of the group H_Ω is a dense subset of ∂T_Ω.*
2. *The group H_Ω acts faithfully on every its orbit.*
3. *For every orbit O, the groups (H_Ω, O) and $(DS_\Omega, \mathbb{N})$ are isomorphic as permutation groups.*

Proof. The groups (H_Ω, O_1) and (H_Ω, O_2) are isomorphic as permutation groups for arbitrary orbits O_1 and O_2 of the group $(H_\Omega, \partial T_\Omega)$. Really, N_Ω acts transitively on ∂T_Ω. So there is $g \in N_\Omega$ such that $g(O_1) = O_2$.

Moreover we have that the sequence of the points $\{\alpha_{m_j}(\xi_n) \in H_{m_j}\} \subset \partial T_\Omega$ has the limit $\nu \in \partial T_\Omega$ by proof of Theorem 14. Since the points $\xi_n, \nu \in \partial T_\Omega$ are arbitrary, we get that every orbit of the group H_Ω is a dense subset of ∂T_Ω. Taking into account continuity of the acting of $H_\Omega < \operatorname{Hom} \partial T_\Omega$ on ∂T_Ω we have that the group H_Ω acts faithfully on every its orbit.

Let O_0 be the orbit of $(H_\Omega, \partial T_\Omega)$ which contains the point $0 = 00\ldots$. In order to finish the proof of the theorem, it is sufficient to prove that the groups (H_Ω, O_0) and $(DS_\Omega, \mathbb{N})$ are isomorphic as permutation groups.

It is easy to see that

$$O_0 = \{j_1 j_2 \ldots j_n 000\ldots \mid 0 \le j_k \le a_k - 1, k \le n, n \in \mathbb{N}\}.$$

Let us enumerate the vertices of tree T_Ω as follows: vertex j_1 from the first level is numbered by $i_1 = j_1 + 1$; further, inductively the vertex $(j_1, j_2, \ldots, j_n)$ of V_n which is adjacent to the vertex from V_{n-1} with number i_{n-1} is numbered by

$$i_n = i_{n-1} + j_n f_\Omega(n-1).$$

Note that the vertices from V_n are numbered by numbers from 1 to $f_\Omega(n)$.

Let $u = j_1 j_2 \ldots j_k 00\ldots \in O_0$. We number u and get

$$i_1 i_2 \ldots i_{k-1} i_k i_k i_k \ldots, \quad i_{k-1} \ne i_k.$$

Let φ maps u to i_k. Then φ is bijective mapping from O_0 to $\mathbb{N}$.

The group $(DS_n, \mathbb{N})$ permutes the ordered sets

$$\{i + k f_\Omega(n) \mid k \in \mathbb{N} \cup 0\}$$

for $1 \le i \le f_\Omega(n)$.

The group (H_n, O_0) acts by permutations on the set of balls $\{P_{n1}, \ldots, P_{nf_\Omega(n)}\}$ and acts trivially within each of these balls. Let j be a vertex from the level number n. And let $\varphi(j) = i_n$, $I_0 = \{i_n\}$, $I_{n+l} = \{i + kf_\Omega(n+l) \mid 0 \leq k \leq a_{n+l+1} - 1, i \in I_{n+l-1}\}$ for $l \in \mathbb{N}$. Note that

$$\varphi(P_{nj}) = \bigcup_{l=0}^{\infty} I_l =$$

$$\{i_n + f_\Omega(n) \sum_{l=0}^{m} k_l q_\Omega(n, n+l) \mid 0 \leq k_l \leq a_{n+l+1} - 1, m \in \mathbb{N}\} =$$

$$\{i_n + f_\Omega(n)k \mid k \in \mathbb{N} \cup 0\}.$$

Therefore $(DS_n, \mathbb{N})$ and (H_n, O_0) are isomorphic as permutation groups. Moreover this isomorphism $\varphi_n^* : (H_n, O_0) \longrightarrow (DS_n, \mathbb{N})$ is determined by φ. Since the isomorphism φ does not depend on n we get $\varphi_n^*|_{H_k} = \varphi_k^*$ for all $k < n$. Hence $(DS_\Omega, \mathbb{N}) = \cup_{n=1}^{\infty}(DS_n, \mathbb{N})$ and $(H_\Omega, O_0) = \cup_{n=1}^{\infty}(H_n, O_0)$ are isomorphic as permutation groups.

□

3.4. Intersections

Lemma 16. *[L, LN] Automorphism group of the group $FA(T_\Omega)$ coincides with the normalizer of this group in* $\mathrm{Aut}\, T_\Omega$.

The group $\mathrm{Aut}\, T_\Omega$ *is complete, i.e.* $\mathrm{Aut}\, T_\Omega \simeq \mathrm{Aut}(\mathrm{Aut}\, T_\Omega)$.

Let $\gamma \in \mathrm{Aut}\, T_\Omega$ and $\gamma_k = \pi_k(\gamma)$ where $\pi_k : \mathrm{Aut}\, T_\Omega \longrightarrow \mathrm{Aut}\, T_\Omega^k$.
Obviously $\mathrm{Aut}(T_\Omega) < \mathrm{Hom}(\partial T_\Omega)$.

Lemma 17. *Intersection $H_\Omega \cap \mathrm{Aut}(T_\Omega)$ coincides with $FA(T_\Omega)$.*

Proof. This assertion follows from the equality $H_n \cap \mathrm{Aut}\, T_\Omega = \mathrm{Aut}\, T_\Omega^n$. □

Lemma 18. *Let $\gamma \in \mathrm{Aut}\, FA(T_\Omega)$. There exists an integer $k \geq n$ such that $\gamma\gamma_k^{-1}$ commutes with $\mathrm{Aut}\, T_\Omega^n$.*

Proof. There exists $k \in \mathbb{N}$ such that $(\mathrm{Aut}\, T_\Omega^n)^\gamma < \mathrm{Aut}\, T_\Omega^k$. By Lemma 16, we can consider $\gamma_k \in \mathrm{Aut}\, T_\Omega^k$.The equality

$$a^\gamma = a^{\gamma_k}$$

holds for all $a \in \mathrm{Aut}\, T_\Omega^n$.

Therefore $a^{\gamma\gamma_k^{-1}} = a$ for all $a \in \mathrm{Aut}\, T_\Omega^n$. □

In fact, the condition of Lemma 18 is a criterion.

Lemma 19. *An element γ belongs to $\mathrm{Aut}\, FA(T_\Omega)$ iff there exists an integer k such that $\gamma\gamma_k^{-1}$ commutes with $\mathrm{Aut}\, T_\Omega^n$.*

Proof. For an element $g \in FA(T_\Omega)$ there exists an integer n such that $g \in \operatorname{Aut} T_\Omega^n$. Taking into account $\gamma_k \in FA(T_\Omega)$ we end the proof. □

In case of a homogeneous tree we will have that $\operatorname{Aut} FA(T_m)$ consists of the automorphisms which are defined by automata with finite memory [NS] ("forgetful" automata). This fact is well-known.

Theorem 20. *The intersection* $\operatorname{Aut} H_\Omega$ *and* $\operatorname{Aut}(T_\Omega)$ *coincides with* $\operatorname{Aut} FA(T_\Omega)$.

Proof. Suppose that there exists $\alpha \in \operatorname{Aut} H_\Omega$ such that $\alpha \in \operatorname{Aut} T_\Omega \setminus \operatorname{Aut} FA(T_\Omega)$. Let $g \in FA(T_\Omega)$. We have $g^\alpha \in \operatorname{Aut} T_\Omega \setminus FA(T_\Omega)$, but this is a contradiction with Lemma 17.

So

$$\operatorname{Aut} H_\Omega \cap \operatorname{Aut}(T_\Omega) \subseteq \operatorname{Aut} FA(T_\Omega). \tag{8}$$

Let us prove that in (8) the equality takes place.

Let $\beta \in H_n$ for some $n \in \mathbb{N}$ and $\gamma \in \operatorname{Aut} FA(T_\Omega)$. By Lemma 18, we can assume that $\gamma = (\gamma_1, \ldots, \gamma_{f_\Omega(n)}) \in \operatorname{Stab}(n)$ and that γ commutes with $\operatorname{Aut} T_\Omega^n$. Let us show that β commutes with γ.

Let us consider the rooted tree T_ξ with characteristic $\xi = (f_\Omega(n), a_{n+1}, a_{n+2}, \ldots)$. For this tree $\operatorname{Aut} T_\xi < \operatorname{Hom} \partial T_\Omega$ and the automorphisms from $\operatorname{Aut} T_\Omega$ act naturally on the tree T_ξ. Therefore, there exists an embedding $\varphi : \operatorname{Aut} T_\Omega \longrightarrow \operatorname{Aut} T_\xi$ which is determined by the embedding of $\operatorname{Aut} T_\Omega^n$ into $\operatorname{Aut} T_\xi^1$. In addition $FA(T_\Omega)^\varphi < FA(T_\xi)$ and $(\operatorname{Aut} T_\Omega^n)^\varphi < \operatorname{Aut} T_\xi^n = H_n$. An automorphism $\pi \in H_n$ acts on $\operatorname{Stab}_{\operatorname{Aut} T_\xi}(n)$ by permutation of indices. In particular,

$$[\pi, \gamma] = (\gamma_1^{-1}\gamma_{\pi(1)}, \ldots, \gamma_{f_\Omega(n)}^{-1}\gamma_{\pi(f_\Omega(n))}). \tag{9}$$

The commutator from (9) is equal to the identity for all $\pi \in (\operatorname{Aut} T_\Omega^n)^\varphi$. Also $(\operatorname{Aut} T_\Omega^n)^\varphi$ acts transitively on $V_n(T_\xi)$. Therefore, $\gamma_1 = \ldots = \gamma_{f_\Omega(n)}$ and hence $[\pi, \gamma] = 1$ for all $\pi \in H_n$. So we get $\beta^\gamma = \beta$. □

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