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On 2-state Mealy automata of polynomial growth

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Abstract. We consider the sequence of 2-state Mealy automata over the finite alphabets, that have polynomial growth orders and define the infinitely presented automatic transformation semigroups.

1. Introduction

The notion of growth was introduced in the middle of last century [12, 19] and was applied to various geometrical and algebraic objects [1, 20]. Growth of Mealy automata have been studied since the 80th of last century [4, 6], and it is close interrelated with growth of automatic transformation semigroups (groups), defined by Mealy automata [6].

Mainly, attention of researchers are attracted to investigations of growth of invertible Mealy automata (see, for example, [3, 7, 8, 10]). Invertibility of the Mealy automaton allows to consider automatic transformation group, defined by this automaton. Investigations of growth of arbitrary Mealy automata (see, for example, [17], [13], and the research of all 2-state Mealy automata over the 2-symbol alphabet [15]) produce results, which show principal distinctions between the cases of invertible and arbitrary Mealy automata.

For example, there was found the smallest possible Mealy automaton of intermediate growth, which has 2 states and is considered over the 2-symbol alphabet [18], [15]. On the other hand, the smallest invertible

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Mealy automata of intermediate growth either have 3 states or is considered over the 3-symbol alphabet [2, 3]. There were found Mealy automata of polynomial growth such, that growth functions of the automaton and the automatic transformation semigroup have different growth orders [15]. There was constructed the set of Mealy automata, which define the free semigroup [14].

As follows from [9] (see also [10]), the automatic transformation group, defined by any invertible Mealy automaton of the polynomial growth order, is "virtually nilpotent" and cannot be infinitely presented. In the paper we consider the sequence $\{A_m,\, m \geq 3\}$ of 2-state Mealy automata of polynomial growth, which define the infinitely presented automatic transformation semigroups for all $m \geq 4$. These automata were announced on IV International Algebraic Conference in Ukraine, Lviv, 2003 [16].

In section 2 we formulate theorem 1, where presentations of the automatic transformation semigroups, defined by the automata from $\{A_m,\, m \geq 3\}$, are described, and theorem 2, where the growth functions are described. Some numerical properties of the growth functions are proved in corollary 1. In section 3 necessary definitions and properties of Mealy automata, finitely generated semigroups and growth functions are provided. Section 4 is devoted to the investigation of the automatic transformations, defined by A_m, and to the proof of theorem 1. Theorem 2 and corollary 1 are proved in section 5.

Another sequence $\{B_m,\, m \geq 3\}$ of Mealy automata of polynomial growth such that for all $m \geq 3$ the automaton B_m has the polynomial growth order $\left[n^{m-2}\right]$ and defines the infinitely presented semigroup, is provided in section 6. Hence, for any positive integer $d > 0$ there exist Mealy automata of growth order $\left[n^{d}\right]$, which define infinitely presented automatic transformation semigroups.

2. Main results

Let $X_m = \{x_0, x_1, \ldots, x_{m-1}\}$ be the m-symbol alphabet, and let $Q_2 = \{q_0, q_1\}$ be the 2-element set of internal states. Let $A_m = (X_m, Q_2, \pi_m, \lambda_m)$, $m \geq 3$, be the 2-state Mealy automaton over the m-symbol alphabet (figure 1), and the transition function π_m and the output function λ_m are defined in the following way:

$$\begin{aligned} &\pi_m(x_0, q_j) = q_0, &&\pi_m(x_1, q_j) = q_1, &&\pi_m(x_i, q_j) = q_0,\\ &\lambda_m(x_0, q_j) = x_0, &&\lambda_m(x_1, q_j) = x_j, &&\lambda_m(x_i, q_j) = x_{i-j}, \end{aligned}$$

where $i = 2, 3, \ldots, m-1$ and $j = 0, 1$. Let us denote S_m the automatic transformation semigroup, defined by A_m, and let γ_{A_m} and γ_{S_m} are the

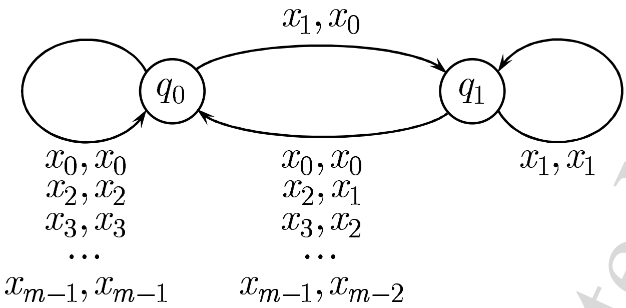

Figure 1: The automaton A_m

growth functions of the automaton A_m and the semigroup S_m, respectively.

Theorem 1. *For any $m \geq 3$ the semigroup S_m has the following presentation:*

$$S_m = \left\langle \; f_0, f_1 \; \middle| \; \begin{array}{l} f_1 f_0^{p_1} f_1 \prod\limits_{i=2}^{m-2} (f_0^{p_i} f_1) = f_0^{p_1+1} f_1 \prod\limits_{i=2}^{m-2} (f_0^{p_i} f_1) \\ p_1 = 1, 2; p_2, p_3, \ldots, p_{m-2} \geq 0 \end{array} \; \right\rangle . \quad (1)$$

All semigroups S_m for $m \geq 4$ are infinitely presented.

Theorem 2. *For $m \geq 3$ the growth functions γ_{A_m} and γ_{S_m} are defined by the following equalities:*

$$\gamma_{A_m}(n) = \sum_{i=0}^{m-1} \binom{n}{i}, \quad (2)$$

$$\gamma_{S_m}(n) = \sum_{i=0}^{m} \binom{n+1}{i} - 2, \quad (3)$$

for all $n \geq 1$.

Corollary 1. *1. For all $m \geq 3$ the functions γ_{A_m} and γ_{S_m} have the growth orders $[n^{m-1}]$ and $[n^m]$, respectively.*

2. The pointwise limit of the sequence $\{\gamma_{A_m}, m \geq 3\}$ of polynomial growth functions is the exponential function 2^n, that is for any positive integer $n \geq 1$ the equality holds

$$\lim_{m \to \infty} \gamma_{A_m}(n) = 2^n.$$

3. The growth functions γ_{A_m} satisfy the equalities:

$$\gamma_{A_m}(n) = 2 + \sum_{i=1}^{n-1} \gamma_{A_{m-1}}(i),$$

where $m \geq 4$, $n \geq 1$.

3. Preliminaries

3.1. Growth functions

Let us consider the set of positive non-decreasing functions of a natural argument $\gamma : \mathbb{N} \to \mathbb{N}$; in further such functions will be called the growth functions.

Definition 1. *Let $\gamma_i : \mathbb{N} \to \mathbb{N}$, $i = 1, 2$, are growth functions. The function γ_1 has no greater growth order (notation $\gamma_1 \preceq \gamma_2$) than the function γ_2, if there exist numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that*

$$\gamma_1(n) \leq C_1 \gamma_2(C_2 n)$$

for any $n \geq N_0$.

Definition 2. *The growth functions γ_1 and γ_2 are equivalent or have the same growth order (notation $\gamma_1 \sim \gamma_2$), if the following inequalities hold:*

$$\gamma_1 \preceq \gamma_2 \quad \textit{and} \quad \gamma_2 \preceq \gamma_1.$$

The equivalence class of the function γ is called *the growth order* and is denoted by the symbol $[\gamma]$. The growth order $[\gamma]$ is called *polynomial*, if $[\gamma] = \left[n^d\right]$ for some $d > 0$.

3.2. Mealy automata

Let us denote the set of all finite words over X_m, including the empty word ε, by the symbol X_m^*, and denote the set of all infinite (to right) words by the symbol X_m^ω.

Let $A = (X_m, Q_n, \pi, \lambda)$ be a *non-initial Mealy automaton* with finite set of states $Q_n = \{q_0, q_1, \ldots, q_{n-1}\}$, input and output alphabets are the same and are equal to X_m, $\pi : X_m \times Q_n \to Q_n$ and $\lambda : X_m \times Q_n \to X_m$ are its transition and output functions, respectively. The function λ can be extended in a natural way to the mapping $\lambda : X_m^* \times Q_n \to X_m^*$ or to the mapping $\lambda : X_m^\omega \times Q_n \to X_m^\omega$ (see, for example, [5], etc).

Definition 3 ([5]). *For any state* $\mathsf{q} \in Q_n$ *the transformation* $f_{\mathsf{q}} : X_m^* \to X_m^*$ $(f_{\mathsf{q}} : X_m^\omega \to X_m^\omega)$, *defined by the equality*

$$f_{\mathsf{q}}(u) = \lambda(u, \mathsf{q}),$$

where $u \in X_m^*$ $(u \in X_m^\omega)$, *is called the automatic transformation, defined by* A *at the state* q.

Let us consider the transformation σ_{q} over the alphabet X_m, $\mathsf{q} \in Q_n$, defined by the output function λ:

$$\sigma_{\mathsf{q}} = \begin{pmatrix} \lambda(x_0, \mathsf{q}) & \lambda(x_1, \mathsf{q}) & \ldots & \lambda(x_{m-1}, \mathsf{q}) \end{pmatrix}.$$

Let q be an arbitrary state. The image of the word $u = u_0u_1u_2\ldots \in X_m^\omega$ under the action of the automatic transformation f_{q} can be written in the following way:

$$f_{\mathsf{q}}(u_0u_1u_2\ldots) = \lambda(u_0, \mathsf{q}) \cdot f_{\pi(u_0,\mathsf{q})}(u_1u_2\ldots) = \sigma_{\mathsf{q}}(u_0) \cdot f_{\pi(u_0,\mathsf{q})}(u_1u_2\ldots).$$

It means that f_{q} acts on the first symbol of u by the transformation σ_{q} over X_m, and acts on the rest of the word without first symbol by the transformation $f_{\pi(u_0,\mathsf{q})}$. Therefore the transformations defined by the automaton A can be written in *the unrolled form*:

$$f_{q_i} = \left(f_{\pi(x_0,q_i)}, f_{\pi(x_1,q_i)}, \ldots, f_{\pi(x_{m-1},q_i)}\right) \sigma_{q_i},$$

where $i = 0, 1, \ldots, n-1$.

The Mealy automaton $A = (X_m, Q_n, \pi, \lambda)$ defines the set

$$F_A = \left\{f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right\}$$

of automatic transformations over X_m^ω. The Mealy automaton A is called *invertible* if all transformations from the set F_A are bijections. It's easy to show (see, for example, [7]) that A is invertible if and only if the transformation σ_{q} is a permutation of X_m for each state $\mathsf{q} \in Q_n$.

Definition 4 ([5]). *The Mealy automata* $A_i = (X_m, Q_n, \pi_i, \lambda_i)$, $i = 1, 2$, *such that there exist permutations* $\xi, \psi \in Sym(X_m)$ *and* $\theta \in Sym(Q_n)$ *such, that the following equalities hold*

$$\theta\pi_1(\mathsf{x}, \mathsf{q}) = \pi_2(\xi\mathsf{x}, \theta\mathsf{q}), \qquad \psi\lambda_1(\mathsf{x}, \mathsf{q}) = \lambda_2(\xi\mathsf{x}, \theta\mathsf{q})$$

for all $\mathsf{x} \in X_m$ *and* $\mathsf{q} \in Q_n$, *are called isomorphic automata.*

Definition 5 ([5]). *The Mealy automata* $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$, $i = 1, 2$, *are called equivalent, if* $F_{A_1} = F_{A_2}$.

Theorem 3 ([5]). *Each class of equivalent Mealy automata over the alphabet X_m contains, up to isomorphism, a unique reduced or minimal (by the number of states) automaton.*

The minimal automaton can be found using standard algorithm of minimization.

Definition 6 ([4]). *Let $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$, $i = 1, 2$, be arbitrary Mealy automata. The automaton $A = (X_m, Q_{n_1} \times Q_{n_2}, \pi, \lambda)$ such that its transition and output functions are defined in the following way:*

$$\pi(\mathsf{x}, (\mathsf{q}_1, \mathsf{q}_2)) = (\pi_1(\lambda_2(\mathsf{x}, \mathsf{q}_2), \mathsf{q}_1), \pi_2(\mathsf{x}, \mathsf{q}_2)),$$
$$\lambda(\mathsf{x}, (\mathsf{q}_1, \mathsf{q}_2)) = \lambda_1(\lambda_2(\mathsf{x}, \mathsf{q}_2), \mathsf{q}_1),$$

where $\mathsf{x} \in X_m$, $(\mathsf{q}_1, \mathsf{q}_2) \in Q_{n_1} \times Q_{n_2}$, is called the product of the automata A_1 and A_2.

Proposition 1 ([4]). *For any states $\mathsf{q}_1 \in Q_{n_1}$, $\mathsf{q}_2 \in Q_{n_2}$ and an arbitrary word $u \in X_m^*$ ($u \in X_m^\omega$) the following equality holds:*

$$f_{(\mathsf{q}_1, \mathsf{q}_2), A}(u) = f_{\mathsf{q}_1, A_1}(f_{\mathsf{q}_2, A_2}(u)).$$

It follows from proposition 1 that for the transformations f_{q_1, A_1} and f_{q_2, A_2}, $\mathsf{q}_1 \in Q_{n_1}$, $\mathsf{q}_2 \in Q_{n_2}$, the unrolled form of the product $f_{(\mathsf{q}_1, \mathsf{q}_2), A_1 \times A_2}$ is defined by the equality:

$$f_{(\mathsf{q}_1, \mathsf{q}_2), A_1 \times A_2} = f_{\mathsf{q}_1, A_1} f_{\mathsf{q}_2, A_2} = (g_0, g_1, \ldots, g_{m-1})\, \sigma_{\mathsf{q}_1, A_1} \sigma_{\mathsf{q}_2, A_2},$$

where $g_i = f_{\pi_1\left(\sigma_{\mathsf{q}_2, A_2}(x_i), \mathsf{q}_1\right), A_1} f_{\pi_2(x_i, \mathsf{q}_2), A_2}$, $i = 0, 1, \ldots, m-1$.

The power A^n is defined for any automaton A and any positive integer n. Let us denote $A^{(n)}$ the minimal Mealy automaton, equivalent to A^n. It follows from definition 6, that $\left|Q_{A^{(n)}}\right| \le |Q_A|^n$.

Definition 7. *[6] The function γ_A of a natural argument, defined by*

$$\gamma_A(n) = \left|Q_{A^{(n)}}\right|, \; n \in \mathbb{N},$$

is called the growth function of the Mealy automaton A.

3.3. Semigroups

Let S be a semigroup with the finite set of generators $G = \{s_0, s_1, \ldots, s_{k-1}\}$. Let us denote the free semigroup with the set G of generators by the symbol G^+. Obviously (see, for example, [11]), if the semigroup S does not contain the identity, then S is a homomorphic image of the free semigroup G^+. Similarly, the monoid $S = sg\,(G)$ is a homomorphic image of the free monoid G^*.

The elements of the free semigroup G^+ are called *semigroup words.* In the sequel, we identify them with corresponding elements of S. The semigroup words s_1 and s_2 are called [11] *equivalent relative to the system G of generators in the semigroup S*, if in S the equality $\mathsf{s}_1 = \mathsf{s}_2$ holds.

Definition 8. *Let s be an arbitrary element of S. The length $\ell(\mathsf{s})$ of s is the minimal possible number of the generators in decomposition*

$$\mathsf{s} = s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_l},$$

where $s_{i_j} \in G$, $1 \le j \le l$, $l > 0$.

Let us sort the generators of S according to their index; and introduce a linear order on the set of elements of G^+: semigroup words are ranked on length, and then words of the same length are arranged lexicographically. The representative of the class of the introduced above equivalence is the minimal semigroup word in the sense of this order.

Definition 9. *Let $\mathsf{s} \in S$ be an arbitrary element. The normal form of this element is the representative of the class of the equivalence of semigroup words, which is mapped on the element s.*

Definition 10. *The function γ_S of a natural argument such that*

$$\gamma_S(n) = \left|\left\{\ s \in S \ \middle|\ \ell(s) \le n\ \right\}\right|, \; n \in \mathbb{N},$$

is called the growth function of S relative to the system G of generators.

Definition 11. *The function $\widehat{\gamma}_S$ of a natural argument such that*

$$\widehat{\gamma}_S(n) = \left|\left\{\ s \in S \ \middle|\ s = s_{i_1} s_{i_2} \ldots s_{i_n},\ s_{i_j} \in G,\ 1 \le j \le n\ \right\}\right|, \; n \in \mathbb{N},$$

is called the spherical growth function of S relative to the system G of generators.

Definition 12. *The function δ_S of a natural argument such that*

$$\delta_S(n) = \left|\left\{\ s \in S \ \middle|\ \ell(s) = n\ \right\}\right|, \; n \in \mathbb{N},$$

is called the word growth function of S relative to the system G of generators.

The following proposition is well-known, and is proved in many papers (see, for example, [7]).

Proposition 2. *Let S be an arbitrary finitely generated semigroup, and let G_1 and G_2 be systems of generators of S. Let us denote the growth function of S relative to the set G_i of generators by the symbol γ_{S_i}, $i = 1, 2$. Then $[\gamma_{S_1}] = [\gamma_{S_2}]$.*

From the definitions 10, 11 and 12, the following inequalities hold for $n \in \mathbb{N}$:

$$\delta_S(n) \leq \widehat{\gamma}_S(n) \leq \gamma_S(n) = \sum_{i=0}^{n} \delta_S(i). \tag{4}$$

Proposition 3. *Let S be an arbitrary finitely generated monoid. Then*

$$[\gamma_S] = [\widehat{\gamma}_S] \geq [\delta_S].$$

Let S be a semigroup without the identity. Then the growth function and the spherical growth function may have different growth orders. For example, let $S = \mathbb{N}$ be the additive semigroup, $S = sg\,(1)$. Then $\gamma_S(n) = n$, $\widehat{\gamma}_S(n) = 1$, and these functions have different growth orders, $[1] < [n]$.

There are many results concerning the growth of groups. For references see the survey [7], the book [10], other papers.

3.4. Growth of Mealy automata

Definition 13. *Let $A = (X_m, Q_n, \pi, \lambda)$ be a Mealy automaton. The semigroup*

$$S_A = sg\left(f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right)$$

is called the semigroup of automatic transformations, defined by A.

For an invertible Mealy automaton, let us examine *the group of transformations* it defines.

Let A be the Mealy automaton, γ_A be its growth function; let S_A be the semigroup, defined by A, and γ_{S_A} and $\widehat{\gamma}_{S_A}$ are the growth function and the spherical growth function of S_A, respectively. From definition 13 it follows that

Proposition 4 ([6]). *For any $n \in \mathbb{N}$ the value $\gamma_A(n)$ equals the number of those elements of S_A, that can be presented as a product of length n of the generators $\left\{f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right\}$, i.e.*

$$\gamma_A(n) = \widehat{\gamma}_{S_A}(n),\ n \in \mathbb{N}.$$

From this proposition and (4) it follows, that $\gamma_A(n) \leq \gamma_{S_A}(n)$ for any $n \in \mathbb{N}$.

4. Semigroups S_{A_m}, $m \geq 3$

Let us fix the number $m \geq 3$ in this section.

4.1. Automatic transformations, defined by A_m

Let us denote the automatic transformations f_{q_0} and f_{q_1}, defined by the automaton A_m, by the symbols f_0 and f_1, and let us study their properties. Their unrolled forms are the following:

$$\begin{aligned} f_0 &= (f_0, f_1, f_0, f_0, \ldots, f_0)\,(x_0, x_0, x_2, x_3, \ldots x_{m-1})\,, \\ f_1 &= (f_0, f_1, f_0, f_0, \ldots, f_0)\,(x_0, x_1, x_1, x_2, \ldots x_{m-2})\,. \end{aligned} \tag{5a}$$

For any integer $p > 0$ from (5a) it follows

$$f_0^p = \left(f_0^p, f_0^{p-1} f_1, f_0^p, f_0^p, \ldots, f_0^p\right)(x_0, x_0, x_2, x_3, \ldots x_{m-1})\,,$$

and, similarly, for $0 < p \leq m-2$ we have

$$f_1^p = \left(f_0^p, f_1^p, f_1^{p-1} f_0, f_1^{p-2} f_0^2, \ldots, f_1 f_0^{p-1}, f_0^p, f_0^p, f_0^p, \ldots, f_0^p\right)$$
$$(x_0, x_1, x_1, x_1, \ldots, x_1, x_1, x_2, x_3, \ldots x_{m-1-p})\,.$$

Thus for any numbers p_1, p_2, p_3 such that $p_1, p_3 > 0$ and $0 < p_2 \leq m-2$, the following equalities hold:

$$\begin{aligned} f_0^{p_1} f_1^{p_2} = \Big(& f_0^{p_1+p_2}, f_0^{p_1-1} f_1^{p_2+1}, f_0^{p_1-1} f_1^{p_2} f_0, f_0^{p_1-1} f_1^{p_2-1} f_0^2, \\ & \ldots, f_0^{p_1-1} f_1 f_0^{p_2}, f_0^{p_1+p_2}, f_0^{p_1+p_2}, \ldots, f_0^{p_1+p_2}\Big) \\ & (x_0, x_0, x_0, x_0, \ldots, x_0, x_2, x_3, \ldots x_{m-1-p_2})\,, \end{aligned} \tag{5b}$$

$$\begin{aligned} f_1^{p_2} f_0^{p_3} = \Big(& f_0^{p_2+p_3}, f_0^{p_2+p_3-1} f_1, f_1^{p_2-1} f_0^{p_3+1}, f_1^{p_2-2} f_0^{p_3+2}, \\ & \ldots, f_1 f_0^{p_2+p_3-1}, f_0^{p_2+p_3}, f_0^{p_2+p_3}, f_0^{p_2+p_3}, \ldots, f_0^{p_2+p_3}\Big) \\ & (x_0, x_0, x_1, x_1, \ldots, x_1, x_1, x_2, x_3, \ldots x_{m-1-p_2})\,, \end{aligned}$$

$$\begin{aligned} f_0^{p_1} f_1^{p_2} f_0^{p_3} = \Big(& f_0^{p_1+p_2+p_3}, f_0^{p_1+p_2+p_3-1} f_1, f_0^{p_1-1} f_1^{p_2} f_0^{p_3+1}, f_0^{p_1-1} f_1^{p_2-1} f_0^{p_3+2}, \\ & \ldots, f_0^{p_1-1} f_1 f_0^{p_2+p_3}, f_0^{p_1+p_2+p_3}, f_0^{p_1+p_2+p_3}, \ldots, f_0^{p_1+p_2+p_3}\Big) \\ & (x_0, x_0, x_0, x_0, \ldots, x_0, x_2, x_3, \ldots x_{m-1-p_2})\,. \end{aligned}$$

From these equations for arbitrary integers $p_1, p_3 \geq 0$, $p_2 > 0$ the equality follows

$$f_0^{p_1} f_1^{p_2} f_0^{p_3}(x_2^*) = x_0^{p_1} x_1^{p_2} x_2^*. \tag{5c}$$

Proposition 5. *In the semigroup S_m the following relations hold:*

$$f_1 f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} = f_0^{1+p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}}, \qquad (6)$$

where $k \geq 1$, $p_i > 0$, $1 \leq i \leq 2k$, $\sum\limits_{i=1}^{k} p_{2i} = m-2$.

Proof. Let $\mathsf{s} = f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots {f_0}^{p_{2k-1}} f_1^{p_{2k}}$ be an arbitrary element of S_m such, that parameters p_i, $1 \leq i \leq 2k$, satisfy the requirements of the proposition. As s defines some automatic transformation over X_m^ω, then there exist elements $\mathsf{s}_{x_0}, \mathsf{s}_{x_1}, \dots, \mathsf{s}_{x_{m-1}}$, that for any word $u \in X_m^\omega$ the equalities hold:

$$\mathsf{s}(x_i u) = \mathsf{s}(x_i) \cdot \mathsf{s}_{x_i}(u),$$

where $i = 0, 1, \dots, m-1$. Moreover, from (5a) the equalities follow

$$f_j(x_0 u) = x_0 \cdot f_0(u),$$

where $j = 0, 1$. Let us consider the transformation σ_{s} over X_m. Using the requirement $\sum\limits_{i=1}^{k} p_{2i} = m-2$ and equation (5b), we have

$$\sigma_{\mathsf{s}} = (x_0, x_0, \dots, x_0, x_2, x_3, \dots x_{m-1-p_2}) \cdot$$
$$\cdot (x_0, x_0, \dots, x_0, x_2, x_3, \dots x_{m-1-p_4}) \cdot \dots \cdot (x_0, x_0, \dots, x_0, x_2, x_3, \dots x_{m-1-p_{2k}}) =$$
$$= \left(x_0, x_0, \dots, x_0, x_2, x_3, \dots x_{\left(m-1-\sum\limits_{i=1}^{k} p_{2i}\right)} \right) = (x_0, x_0, \dots x_0),$$

where transformations are applied right-to-left. Thus, for an arbitrary word $u = u_0 u_1 u_2 \dots \in X_m^\omega$ holds

$$f_i\, \mathsf{s}(u_0 u_1 u_2 \dots) = f_i(\sigma_{\mathsf{s}}(u_0) \cdot \mathsf{s}_{u_0}(u_1 u_2 \dots)) =$$
$$= f_i(x_0 \cdot \mathsf{s}_{u_0}(u_1 u_2 \dots)) = x_0 \cdot f_0\, \mathsf{s}_{u_0}(u_1 u_2 \dots),$$

where $i = 0, 1$. Hence,

$$f_1 f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots {f_0}^{p_{2k-1}} f_1^{p_{2k}}(u) = f_0^{p_1+1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots {f_0}^{p_{2k-1}} f_1^{p_{2k}}(u).$$

□

Proposition 6. *The relations* (6) *in the semigroup S_m follow from the relations*

$$\begin{aligned} f_1 f_0^1 f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} &= f_0^2 f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}}, \\ f_1 f_0^2 f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} &= f_0^3 f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}}, \end{aligned} \qquad (7)$$

where $k \geq 1$, $p_i > 0$, $2 \leq i \leq 2k$, $\sum\limits_{i=1}^{k} p_{2i} = m-2$.

Proof. Let us prove the proposition by an induction on p_1. For $p_1 = 1$ and $p_1 = 2$ the assertion of the proposition immediately follows from proposition 5. Let us assume that the proposition is proved for $p_1 \geq 2$. Using the induction hypothesis, for $(p_1 + 1)$ we have

$$\begin{aligned}
f_1 f_0^{p_1+1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} &= \\
&= f_1 f_0 \cdot f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} = \\
&= f_1 f_0 \cdot f_1 f_0^{p_1-1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} = \\
&= f_1 f_0 f_1 f_0^{p_1-1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_1^{p_{2k-2}} f_0^{p_{2k-1}} f_1^{p_{2k}-1} \cdot f_1 = \\
&= {f_0}^2 \cdot f_1 f_0^{p_1-1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} = \\
&= f_0^{2+1+p_1-1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} = \\
&= f_0^{p_1+2} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}}.
\end{aligned}$$

□

Proposition 7. *For any $m \geq 4$ the infinite system (7) of relations is irreducible, that is no one of relations follows from others.*

Proof. Let r_1 and r_2 are arbitrary relations of (7):

$$\begin{aligned}
r_1 : f_1 f_0^{j_1} f_1 \mathsf{s}_1 f_1 &= f_0^{j_1+1} f_1 \mathsf{s}_1 f_1, \\
r_2 : f_1 f_0^{j_2} f_1 \mathsf{s}_2 f_1 &= f_0^{j_2+1} f_1 \mathsf{s}_2 f_1,
\end{aligned}$$

where $j_1, j_2 \in \{1, 2\}$, $\mathsf{s}_1, \mathsf{s}_2$ are semigroup words, which includes $(m-4)$ symbols f_1. Relation r_1 can be applied to r_2 if and only if $\mathsf{s}_1 = \mathsf{s}_2$ and $j_1 = 1$, $j_2 = 2$. From relation r_1 the equalities follow

$$f_1^2 f_0^1 f_1 \mathsf{s}_2 f_1 = f_1 f_0^2 f_1 \mathsf{s}_2 f_1, \qquad f_0 f_1 f_0 f_1 \mathsf{s}_2 f_1 = f_0^3 f_1 \mathsf{s}_2 f_1,$$

but the relation r_2 is necessary, because it sets up the equality between left and right equations. Hence, no one relation of (7) can be output from other relations. □

4.2. Proof of theorem 1

Proposition 8. *An arbitrary element $\mathsf{s} \in S_m$ admits a unique minimal-length representation as a word of the form*

$$f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} {f_0}^{p_{2k+1}}, \tag{8}$$

where at $k = 0$ let $p_1 > 0$, and at $k \geq 1$ let $p_i > 0$, $2 \leq i \leq 2k$, $p_1, p_{2k+1} \geq 0$, $\sum_{i=2}^{k} p_{2i} \leq m - 3$.

Proof. Let $\mathsf{s} \in S_m$ be an arbitrary element. It can be written as a product of the generators of S_m, that is

$$\mathsf{s} = f_0^{p_1} f_1^{p_2} f_0^{p_3} \dots f_0^{p_{2l-1}} f_1^{p_{2l}} f_0^{p_{2l+1}},$$

where $l \geq 0$, $p_1, p_{2l+1} \geq 0$, $p_i > 0$, $1 < i \leq 2l$, and $\sum\limits_{i=1}^{2l+1} p_i = \ell(s) > 0$.

If the semigroup word s contains no great than $(m-3)$ symbols f_1 or s has been already written as (8), then the relations (7) can not be applied and the assertion of the proposition is true. Let us assume that the semigroup word s contains at least $(m-2)$ symbols f_1 and is not written in the form (8). Then let us find in s the maximal right subword

$$\widetilde{s} = f_0^{p_{2i+1}} f_1^{p_{2i+2}} f_0^{p_{2i+3}} f_1^{p_{2i+4}} \dots f_0^{p_{2l-1}} f_1^{p_{2l}} f_0^{p_{2l+1}}$$

such that it has the form (8), that is

$$\sum_{j=i+2}^{l} p_{2j} \leq m-3, \qquad \sum_{j=i+1}^{l} p_{2j} \geq m-2.$$

Let us note, that the word $\widetilde{s}$ can not be reduced by the relations (7). From the proof of proposition 6 and the condition $p_{2i+1} > 0$ it follows that

$$\mathsf{s} = f_0^{p_1} f_1^{p_2} \dots f_0^{p_{2i-1}} f_1^{p_{2i}} \cdot \widetilde{s} = f_0^{\sum\limits_{i=1}^{2i+1} p_i} f_1^{p_{2i+2}} f_0^{p_{2i+3}} f_1^{p_{2i+4}} \dots f_0^{p_{2l-1}} f_1^{p_{2l}} f_0^{p_{2l+1}}.$$

Hence, s can be reduced to the form (8). □

Let $\mathsf{s} = f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} f_0{}^{p_{2k+1}}$ be an arbitrary element of S_m, written in the form (8), and let us denote $P_a^b = \sum\limits_{i=a}^{b} p_i$ and $R_a^b = \sum\limits_{i=a}^{b} p_{2i}$. Let us write the unrolled form of s by using the equations (5*).

If the requirement $R_1^k < m-2$ is satisfied, then

$$\begin{aligned}
\mathsf{s} = \Big(& f_0^{P_1^{2k+1}}, f_0^{P_1^{2k+1}-1} f_1, \\
& f_0^{P_1^{2k-1}-1} f_1^{p_{2k}} f_0^{P_{2k+1}^{2k+1}+1}, f_0^{P_1^{2k-1}-1} f_1^{p_{2k}-1} f_0^{P_{2k+1}^{2k+1}+2}, \dots, \\
& \qquad f_0^{P_1^{2k-1}-1} f_1 f_0^{P_{2k+1}^{2k+1}+p_{2k}}, \\
& f_0^{P_1^{2k-3}-1} f_1^{p_{2k-2}} f_0^{P_{2k-1}^{2k+1}+1}, f_0^{P_1^{2k-3}-1} f_1^{p_{2k-2}-1} f_0^{P_{2k-1}^{2k+1}+2}, \dots, \\
& \qquad f_0^{P_1^{2k-3}-1} f_1 f_0^{P_{2k-1}^{2k+1}+p_{2k-2}}, \\
& \dots \\
& f_0^{P_1^1-1} f_1^{p_2} f_0^{P_3^{2k+1}+1}, f_0^{P_1^1-1} f_1^{p_2-1} f_0^{P_3^{2k+1}+2}, \dots, f_0^{P_1^1-1} f_1 f_0^{P_3^{2k+1}+p_2}, \\
& \qquad f_0^{P_1^{2k+1}}, f_0^{P_1^{2k+1}}, \dots, f_0^{P_1^{2k+1}} \Big) \\
& \Big(x_0, x_0, x_0, \dots, x_0, x_2, x_3, \dots, x_{m-1-R_1^k} \Big).
\end{aligned} \tag{9a}$$

Otherwise, if $R_1^k \geq m-2$, we have

$$\begin{aligned}
\mathsf{s} = \Big(& f_0^{P_1^{2k+1}}, f_0^{P_1^{2k+1}-1} f_1, \\
& f_0^{P_1^{2k-1}-1} f_1^{p_{2k}} f_0^{P_{2k+1}^{2k+1}+1}, f_0^{P_1^{2k-1}-1} f_1^{p_{2k}-1} f_0^{P_{2k+1}^{2k+1}+2}, \dots, \\
& \qquad f_0^{P_1^{2k-1}-1} f_1 f_0^{P_{2k+1}^{2k+1}+p_{2k}}, \\
& f_0^{P_1^{2k-3}-1} f_1^{p_{2k-2}} f_0^{P_{2k-1}^{2k+1}+1}, f_0^{P_1^{2k-3}-1} f_1^{p_{2k-2}-1} f_0^{P_{2k-1}^{2k+1}+2}, \dots, \\
& \qquad f_0^{P_1^{2k-3}-1} f_1 f_0^{P_{2k-1}^{2k+1}+p_{2k-2}}, \\
& \dots \\
& f_0^{P_1^1-1} f_1^{p_2} f_0^{P_3^{2k+1}+1}, f_0^{P_1^1-1} f_1^{p_2-1} f_0^{P_3^{2k+1}+2}, \dots, \\
& \qquad f_0^{P_1^1-1} f_1^{p_2-\left(m-3-R_2^k\right)} f_0^{P_3^{2k+1}+m-2-R_2^k} \Big) \\
& (x_0, x_0, x_0, \dots, x_0).
\end{aligned} \tag{9b}$$

Proof of theorem 1.

In proposition 6 is proved, that the relations (7) hold in the semigroup S_m. Using these relations, an arbitrary element $\mathsf{s} \in S_m$ can be reduced to the form (8). It is necessary to prove, that two semigroup elements, written in different forms (8), define different transformations over X_m^ω.

Then the system (7) of the relations is the set of the defining relations, and the semigroup S_m has the presentation by generators and defining relations (1).

Let s_1 and s_2 are arbitrary semigroup elements, which have different forms (8),

$$\begin{aligned} \mathsf{s}_1 &= f_0^{p_1} f_1^{p_2} f_0^{p_3} f_1^{p_4} \dots f_0^{p_{2k-1}} f_1^{p_{2k}} f_0{}^{p_{2k+1}}, \\ \mathsf{s}_2 &= f_0^{t_1} f_1^{t_2} f_0^{t_3} f_1^{t_4} \dots f_0^{t_{2l-1}} f_1^{t_{2l}} f_0{}^{t_{2l+1}}. \end{aligned} \tag{10}$$

Let us assume by contradiction, that the elements s_1 and s_2 define the same transformation over X_m^ω, but have different values of the parameters in (10). Then for any word $u \in X_m^\omega$ the equality

$$\mathsf{s}_1(u) = \mathsf{s}_2(u) \tag{11}$$

holds. Let us keep notations P_a^b, R_a^b, and denote $T_a^b = \sum\limits_{i=a}^{b} t_i$.

From equations (9*) and (5*) follows, that for word $u_1 = x_0 x_1^*$ the equalities hold:

$$\begin{aligned} \mathsf{s}_1(x_0 x_1^*) &= x_0 \cdot f_0^{P_1^{2k+1}}(x_1^*) = x_0^2 \cdot f_0^{P_1^{2k+1}-1} f_1(x_1^*) = \dots = \\ &\qquad = x_0^{P_1^{2k+1}+1} \cdot f_1^{P_1^{2k+1}}(x_1^*) = x_0^{P_1^{2k+1}+1} x_1^*, \\ \mathsf{s}_2(x_0 x_1^*) &= x_0 \cdot f_0^{T_1^{2l+1}}(x_1^*) = x_0^2 \cdot f_0^{T_1^{2l+1}-1} f_1(x_1^*) = \dots = \\ &\qquad = x_0^{T_1^{2l+1}+1} \cdot f_1^{T_1^{2l+1}}(x_1^*) = x_0^{T_1^{2l+1}+1} x_1^*. \end{aligned}$$

Using the assumption $\mathsf{s}_1(u_1) = \mathsf{s}_2(u_1)$, we obtain the requirement

$$\ell(\mathsf{s}_1) = \sum_{i=1}^{2k+1} p_i = \sum_{i=1}^{2l+1} t_i = \ell(\mathsf{s}_2).$$

Not restricting a generality let us assume, that $k \geq l$.

Let $l = 0$. Then $\mathsf{s}_2 = f_0^{t_1}$, but $k \geq 1$; and from (9*) and (5c) for the word $u_2 = x_2^*$ we have

$$\begin{aligned} \mathsf{s}_1(x_2^*) &= x_0 \cdot f_0^{P_1^{2k-1}-1} f_1^{p_{2k}} f_0^{p_{2k+1}+1}(x_2^*) = x_0^{P_1^{2k-1}} x_1^{p_{2k}} x_2^*, \\ \mathsf{s}_2(x_2^*) &= f_0^{t_1}(x_2^*) = x_2^*, \end{aligned}$$

that contradicts the assumption (11).

Let now $l \geq 1$. As elements s_1 and s_2 have the same length, then let us choice the minimal possible index i, $0 \leq i < 2l$, such that $p_{2k-i} \neq t_{2l-i}$, and denote $i_0 = \left[\frac{i}{2}\right]$, $i_1 = i - 2i_0$. Obviously, the equality

$$\sum_{h=l-i_0+1}^{l} t_{2h} = \sum_{h=k-i_0+1}^{k} p_{2h} = R^k_{k-i_0+1},$$

hold. As the elements s_1 and s_2 are written in the form (8), then $R^k_{k-i_0+1} \leq m-3$, and from equations (9*) and (5c) for the word $u_3 = x_{\left(2+R^k_{k-i_0+1}\right)} x_2^*$ follows

$$\mathsf{s}_1\left(x_{\left(2+R^k_{k-i_0+1}\right)} x_2^*\right) = x_0 \cdot f_0^{P_1^{2k-2i_0-1}-1} f_1^{p_{2k-2i_0}} f_0^{P^{2k+1}_{2k-2i_0+1}+1}(x_2^*) =$$
$$= x_0^{P_1^{2k-2i_0-1}} x_1^{p_{2k-2i_0}} x_2^*,$$
$$\mathsf{s}_2\left(x_{\left(2+R^k_{k-i_0+1}\right)} x_2^*\right) = x_0 \cdot f_0^{T_1^{2l-2i_0-1}-1} f_1^{t_{2l-2i_0}} f_0^{T^{2l+1}_{2l-2i_0+1}+1}(x_2^*) =$$
$$= x_0^{T_1^{2l-2i_0-1}} x_1^{t_{2l-2i_0}} x_2^*.$$

From the assumption (11) it follow the equalities $p_{2k-2i_0} = t_{2l-2i_0}$ and

$$\sum_{h=1}^{2k-2i_0-1} p_h = \sum_{h=1}^{2l-2i_0-1} t_h. \tag{12}$$

If i is the even integer, then $i_1 = 0$, $i = 2i_0$, and we obtain an inconsistency with the choice of i. Now let i be the odd number, $i_1 = 1$. If $i = 2l - 1$, then in the case $k = l$ the choice of i contradicts with the requirement (12). In sequel, let it be either $i < 2l - 1$ or $k > l$. As s_1 is written in the form (8), then

$$\sum_{h=l-i_0}^{l} t_{2h} = \sum_{h=k-i_0}^{k} p_{2h} = R^k_{k-i_0} \leq m - 3.$$

Similarly to the previous speculations, for the word $u_4 = x_{\left(2+R^k_{k-i_0}\right)} x_2^*$ we have

$$\mathsf{s}_1\left(x_{\left(2+R^k_{k-i_0}\right)} x_2^*\right) = x_0 \cdot f_0^{P_1^{2k-2i_0-3}-1} f_1^{p_{2k-2i_0-2}} f_0^{P^{2k+1}_{2k-2i_0-1}+1}(x_2^*) =$$
$$= x_0^{P_1^{2k-i-2}} x_1^{p_{2k-i-1}} x_2^*,$$
$$\mathsf{s}_2\left(x_{\left(2+R^k_{k-i_0}\right)} x_2^*\right) = x_0 \cdot f_0^{T_1^{2l-2i_0-3}-1} f_1^{t_{2l-2i_0-2}} f_0^{T^{2l+1}_{2l-2i_0-1}+1}(x_2^*) =$$
$$= x_0^{T_1^{2l-i-2}} x_1^{t_{2l-i-1}} x_2^*,$$

whence follows the requirements

$$\sum_{h=1}^{2k-i-2} p_h = \sum_{h=1}^{2l-i-2} t_h \quad \text{and} \quad p_{2k-i-1} = t_{2l-i-1}. \tag{13}$$

Joining the requirements (12) and (13), it is follows the equality of the parameters $p_{2k-i} = t_{2l-i}$, that contradicts the choice of the index i.

In proposition 7 it is proved that for all $m \geq 4$ the set of defining relations of the semigroup S_m is irreducible. Hence, the semigroup S_m is infinitely presented. □

5. Growth functions

Proof of theorem 2. Let us fix $m \geq 3$, and calculate the growth functions of the automaton A_m and the semigroup S_m. Using the proved in section 4, for any $n \in \mathbb{N}$ the value $\delta_{S_m}(n)$ is equal to the number of elements in the form (8) such, that $\sum\limits_{i=1}^{2k+1} p_i = n$. Let us separate the set of elements of the form (8) into three subsets:

$$f_0^{p_1}, \quad p_1 > 0, \tag{14a}$$

$$f_0^{p_1} f_1^{p_2}, \quad p_1 \geq 0, p_2 > 0 \tag{14b}$$

and

$$f_0^{p_1} f_1^{p_2-1} \left(f_1 f_0^{p_3}\right) \left(f_1 f_0^{p_4}\right) \ldots \left(f_1 f_0^{p_{l-1}}\right) \left(f_1 f_0^{p_l}\right), \tag{14c}$$

where $3 \leq l \leq m$, $p_1 \geq 0$, $p_2, p_3 > 0$, $p_i \geq 0$, $3 \leq i \leq l$. There is a unique word f_0^n of length n, which has the form (14a); the count of semigroup elements of sort (14b) of length n equals $\begin{pmatrix} n-1+1 \\ 1 \end{pmatrix}$, and the count of elements of sort (14c) of length n equals

$$\sum_{l=3}^{m} \begin{pmatrix} n-(1+l-2)+(l-1) \\ l-1 \end{pmatrix} = \sum_{l=2}^{m-1} \begin{pmatrix} n \\ l \end{pmatrix}.$$

Thus, for any $n \in \mathbb{N}$ the equality hold:

$$\delta_{S_m}(n) = 1 + \begin{pmatrix} n \\ 1 \end{pmatrix} + \sum_{l=2}^{m-1} \begin{pmatrix} n \\ l \end{pmatrix} = \sum_{l=0}^{m-1} \begin{pmatrix} n \\ l \end{pmatrix}.$$

As the defining relations of the semigroup S_m do not change the length of semigroup words, then

$$\gamma_{A_m}(n) = \delta_{S_m}(n) = \sum_{i=0}^{m-1} \binom{n}{i},$$

and

$$\gamma_{S_m}(n) = \sum_{j=1}^{n} \delta_{S_m}(j) = \sum_{i=0}^{m-1} \sum_{j=1}^{n} \binom{j}{i} =$$
$$= \sum_{i=0}^{m-1} \binom{n+1}{i+1} - \sum_{i=0}^{m-1} \binom{1}{i+1} = \sum_{i=0}^{m} \binom{n+1}{i} - 2,$$

that holds for all $n \geq 1$. The theorem 2 is completely proved. □

Proof of corollary 1. 1. For all $n \geq m$ from (2) it follows, that the value $\gamma_{A_m}(n)$ includes the largest binomial coefficient $\binom{n}{m-1}$, which gives the polynomial growth order $[n^{m-1}]$. Similarly, for the growth function of S_m from (3) it follows, that γ_{S_m} has the polynomial growth order $[n^m]$.

2. Let us fix $n \geq 1$. From (2) it follows, that for $m \geq n+1$ the equality holds

$$\gamma_{A_m}(n) = \sum_{i=0}^{m-1} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} = 2^n,$$

whence

$$\lim_{m\to\infty} \gamma_{A_m}(n) = 2^n.$$

3. Let us fix $m \geq 4$. From the properties of binomial coefficients and the formula (2) it follows

$$2 + \sum_{i=1}^{n-1} \gamma_{A_{m-1}}(i) = 2 + \sum_{i=1}^{n-1} \sum_{j=0}^{m-2} \binom{i}{j} =$$
$$2 + \sum_{j=0}^{m-2} \left(\binom{n}{j+1} - \binom{1}{j+1} \right) = \sum_{j=0}^{m-1} \binom{n}{j} = \gamma_{A_m}(n),$$

that completes the proof of corollary 1. □

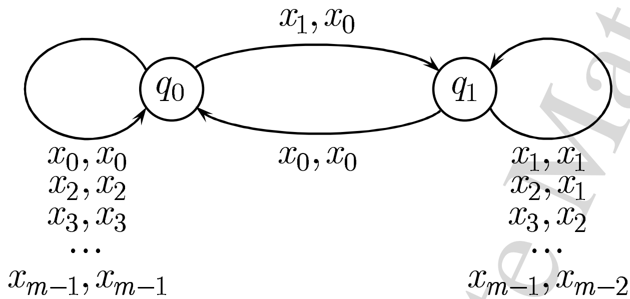

Figure 2: The automaton B_m

6. Sequence $\{B_m,\, m \geq 3\}$

As was mentioned in introduction, let us consider the sequence $\{B_m,\, m \geq 3\}$ of 2-state Mealy automata, where the automaton B_m, $m \geq 3$, is shown on figure 2 by its Moore diagram. The following theorem describes the main properties of the automaton B_m.

Theorem 4. *For any $m \geq 3$ the automaton B_m has the growth order $\left[n^{m-2}\right]$ and defines the infinitely presented automatic transformation semigroup.*

Theorem 4 may be proved similarly to theorems 1 and 2, but its proof is required a bit more technical details. Indeed, we have

Proposition 9. *In the semigroup S_{B_m}, defined by B_m, the following relations hold:*

$$\begin{aligned} f_1^{m-1} f_0 &= f_1^{m-2} f_0^2, \\ f_1^{p_1} \prod_{i=2}^{m-p_1} \left(f_0^{p_i} f_1\right) &= f_1^{p_1-1} f_0 \prod_{i=2}^{m-p_1} \left(f_0^{p_i} f_1\right), \end{aligned} \tag{15}$$

where $0 < p_1 \leq m-2$, $0 < p_2$, $0 \leq p_3, p_4, \dots, p_{m-p_1}$.

On the one hand, the set of relations (15) is not irreducible system of relations. For example, for $m = 3$ these relations are

$$\begin{aligned} f_1^2 f_0 &= f_1 f_0^2, \\ f_1 f_0^{p_2} f_1 &= f_0^{p_2+1} f_1, \end{aligned}$$

where $p_2 \geq 1$. They can be reduced to the irreducible system of relations

$$\begin{aligned} f_1^2 f_0 &= f_1 f_0^2, \\ f_1 f_0^{2^p} f_1 &= f_0^{2^p+1} f_1, \end{aligned}$$

where $p > 0$.

On the other hand, for all $m \geq 3$ the relations (15) includes infinite irreducible set of relations, and therefore semigroup S_{B_m} is infinitely presented. Moreover, the growth functions of the automaton B_m and the semigroup S_{B_m} are described in the following proposition.

Proposition 10. *For $m \geq 3$ the growth functions γ_{B_m} and $\gamma_{S_{B_m}}$ are defined by the following equalities:*

$$\gamma_{B_m}(n) = \sum_{i=0}^{m-2} \binom{n}{i} + \max(0, n-m+2),$$

$$\gamma_{S_{B_m}}(n) = \sum_{i=0}^{m-1} \binom{n+1}{i} - 2 + \frac{1}{2}\max(0, (n-m+2)(n-m+3)),$$

for all $n \geq 1$.

Obviously, the growth functions γ_{B_m} and $\gamma_{S_{B_m}}$ have polynomial growth orders.

References

[1] I.K. Babenko, *The problems of the growth and the rationality in algebra and topology*, Uspehi Math. Nauk 41, (1986), 95-142

[2] Laurent Bartholdi, *Croissance de groupes agissant sur des arbres*, University of Genéve, Genéve, (2000)

[3] Jacek Fabrykowski, Narain D. Gupta, *On groups with sub-exponential growth functions. II*, J. Indian Math. Soc. 56, (1991), 217-228

[4] Ferenc Gecseg, *Products of automata*, Springer-Verlag, Berlin ets., (1986), 107

[5] V.M. Glushkov, *Abstract theory of automata*, Uspehi Math. Nauk 16, (1961), 3-62

[6] R.I. Grigorchuk, *On cancellation semigroups of the exponential growth*, Mathematicheskie Zametki, 43, (1988), 305-319

[7] R.I.Grigorchuk, V.V. Nekrashevich, V.I. Sushchansky, *Automata, dynamical systems, and groups*, Proceedings of the Steklov Institute of Mathematics, 231, (2000), 128-203

[8] Rostislav I. Grigorchuk, Andrzej Żuk, *The Lamplighter Group as a Group Generated by a 2-State Automaton and Its Spectrum* Forschungsinstitut Für Mathematik Eth Zürich, (November 1999), 35

[9] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. Inst. Hautes Etudes Sci. 53, (1981), 53-73

[10] Pierre de la Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press, Chicago & London, (2000), 310

[11] Gerard Lallement, *Semigroups and combinatorial applications*, John Willey & Sons, New York, Chichester, Brisbane, Toronto, (1979), 400

[12] Milnor John, *Growth of finitely generated solvable groups*, 2, (1968), 447-451

[13] Andrey S. Oliynyk, Illya I. Reznykov, Vitaliy I. Sushchansky *The transformation semigroups, defined by Mealy automata over the finite alphabet*, Algebraic Structures and Their Applications: Proceedings of Ukrainian Mathematical Congress-2001, Institute of Mathematics of NAS of Ukraine, Kiev, (2002), 80-99

[14] Illya I. Reznykov, *The two-state Mealy automata over the two-symbol alphabet, which define the free semigroup*, Bulletin of the University of Kiev, ser.: Physics & Mathematics Sciences, (2001), 58-65

[15] Illya I. Reznykov, *The growth functions of two-state Mealy automata over the two-symbol alphabet and the semigroups, defined by them*, Kyiv Taras Schevchenko University, Kiev, Ukraine, (2002), 135

[16] Illya I. Reznykov, *On the series of Mealy automata with polynomial growth functions*, Proceedings of IV International Algebraic Conference in Ukraine, Lviv, August 2003, 193-194

[17] Illya I. Reznykov, Vitaliy I. Sushchansky, *The growth functions of 2-state automata over the 2-symbol alphabet*, Reports of the NAS of Ukraine, (2002), 76-81

[18] Illya I. Reznykov, Vitaliy I. Sushchansky, *The two-state Mealy automata over the two-symbol alphabet of the intermediate growth*, Mathematicheskie Zametki, 72, (2002), 102-117

[19] A.S. Švarc, *A volume invariant of coverings*, Dokladi Akademii Nauk SSSR, 105, (1955), 32-34

[20] V.A. Ufnarovskiy, *Combinatorial and asymptotical methods in algebra*, Itogi nauki i tehniki, vol. 57, VINITI, Moscow, 57, (1990), 5-177

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