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ON THE EXISTENCE OF A CYCLIC VECTOR OF SOME FAMILIES OF OPERATORS*

ПРО ІСНУВАННЯ ЦИКЛІЧНОГО ВЕКТОРА ДЕЯКИХ СІМЕЙ ОПЕРАТОРІВ*

It is proved that, under some restrictions, a family of selfadjoint commuting operators $A = (A_\varphi)_{\varphi \in \Phi}$ where Φ is a nuclear space, has a cyclic vector iff there exists a Hilbert space $H \subset \Phi'$ of full operator-valued measure E, where Φ' is the dual of Φ, E is the joint resolution of the identity of the family A.

Доведено, що при деяких припущеннях сім'я самоспряжених комутуючих операторів $A = (A_\varphi)_{\varphi \in \Phi}$ де Φ – ядерний простір, має циклічний вектор тоді і лише тоді, коли існує гільбертів простір $H \subset \Phi'$ повної операторнозначної міри E, де Φ' – спряжений до Φ простір, E – сумісний розклад одиниці сім'ї A.

1. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a family of selfadjoint operators in $\mathcal{H}$, commuting in the sense of resolutions of the identity (r. i.) and indexed by a set X. Given an operator M, denote by $\mathcal{D}(M)$ the domain of M. A vector $\Omega \in \mathcal{H}(\| \Omega \|_{\mathcal{H}} = 1)$ is called a cyclic vector of the family $\mathcal{A}$ if, for every collection of distinct points $x_1, \dots, x_p \in X$ and numbers $m_1, \dots, m_p \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ $(p \in \mathbb{N})$, we have $\Omega \in \mathcal{D}(\mathcal{A}_{x_1}^{m_1} \dots \mathcal{A}_{x_p}^{m_p})$ and the linear span of all the vectors of the form $\mathcal{A}_{x_1}^{m_1} \dots \mathcal{A}_{x_p}^{m_p} \Omega$ is dense in $\mathcal{H}$.

Assume that $\Phi = \operatorname{pr} \lim_{\tau \in T} H_\tau$ is a real separable nuclear space, where $(H_\tau)_{\tau \in \mathcal{T}}$ is a collection of real Hilbert spaces H_τ, and Φ is dense in every H_τ. Denote $H_0 := H_{\tau_0}$, where τ_0 is a fixed index from $\mathcal{T}$. We may suppose all the spaces $H_\tau (\tau \in \mathcal{T})$ to be embedded into H_0 topologically. On the space $\Phi' = \operatorname{ind} \lim_{\tau \in T} H_{-\tau}$, which is the dual of Φ ($H_{-\tau}$ is the dual of H_τ with respect to (w. r. t.) H_0), we define a σ-algebra $\mathfrak{C}_\sigma(\Phi')$ generated by the cylinder sets in Φ' of the form

$$C(\varphi_1, \dots, \varphi_n, \Delta) = \{x \in \Phi' \mid ((x, \varphi_1)_{H_0}, \dots, (x, \varphi_n)_{H_0}) \in \Delta\}$$

$$(\varphi_1, \dots, \varphi_n \in \Phi, \ \Delta \in \mathcal{B}(\mathbb{R}^n), \ n \in \mathbb{N}).$$

Let $\mu(\cdot)$ be a probability measure on this σ-algebra.

In what follows, we will deal with the family $A = (A_\varphi)_{\varphi \in \Phi}$ of selfadjoint commuting operators A_φ that are defined in the Hilbert space $L_2(\Phi', d\mu(\cdot))$ as operators of multiplication by the function $(x, \varphi)_{H_0}$ $(x \in \Phi')$ and study the problem of the relationship between the existence of a cyclic vector of the family A and the existence of some space $H_{-\tau'}$ $(\tau' \in \mathcal{T})$ of full measure $\mu(\cdot)$.

This problem was studied in [1–3], where it was shown that, for the space $\Phi' = \mathbb{R}^\infty$, which is the dual of the nuclear space $\Phi = \mathbb{R}_0^\infty$ of real finite sequences w. r. t. l_2, and

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for an arbitrary probability measure $\tilde{\mu}(\cdot)$ on $\mathbb{R}^{\infty}$, there exists a Hilbert space $H_- = l_2((p_j)_{j=1}^{\infty}) \subset \mathbb{R}^{\infty}$ of full measure $\tilde{\mu}(\cdot)$, and the vector $\exp\{-\|x\|_{H_-}^2\}$ is a cyclic vector of the family $(A\varphi)_{\varphi\in\mathbb{R}_0^{\infty}}$. By using this, it is easy to verify that the existence of a space $H_{-\tau'}$ of full measure $\mu(\cdot)$ implies the existence of a cyclic vector of the family $(A\varphi)_{\varphi\in\Phi}$; in [2], this statement was proved for $\Phi' = S'(\mathbb{R}^1)$, i.e., for the Schwartz space of tempered distributions.

Let a nuclear space Φ be, in addition, an LF-space, i.e. $\Phi = \operatorname{ind}\lim_{\tau\in\mathbb{N}} \Phi_n$ is a strict inductive limit of F-spaces Φ_n (it may be shown that the spaces Φ_n are, indeed, separable [4]). Under such restriction, we will show that the existence of a cyclic vector of the family A implies the existence of a space $H_{-\tau'}$ $(\tau' \in \mathcal{T})$ of full measure $\mu(\cdot)$. Note that the Schwartz space $S(\mathbb{R}^n)$ and the space $\mathcal{D}(\mathbb{R}^n)$ of infinitely-differentiable functions on $\mathbb{R}^n$ with compact support belong to this class of nuclear spaces.

2. Theorem. *Let Φ be a real nuclear LF-space. Then there exists a cyclic vector $\Omega(\cdot) \in L_2(\Phi', d\mu(\cdot))$ of the family $A = (A_\varphi)_{\varphi\in\Phi}$ iff there exists $\tau' \in \mathcal{T}$ such that $H_{-\tau'}$ is a set of full measure $\mu(\cdot)$.*

Proof. *"If" part.* The intersections $\alpha \cap H_{-\tau'}$ $(\alpha \in \mathfrak{C}_\sigma(\Phi'))$ form a σ-algebra in $H_{-\tau'}$, which coincides with the σ-algebra $\mathfrak{C}_\sigma(H_{-\tau'})$ generated by the cylinder sets in $H_{-\tau'}$:

$$C(l_1, \ldots, l_n, \Delta) = \{x \in H_{-\tau'} \mid ((x, l_1)_{H_{-\tau'}}, \ldots$$

$$\ldots, (x, l_n)_{H_{-\tau'}}) \in \Delta\} \quad (\Delta \in \mathcal{B}(\mathbb{R}^n),\ n \in \mathbb{N}),$$

where $(l_j)_{j=1}^{\infty}$ is an arbitrary orthonormal basis of $H_{-\tau'}$ [5]. Since $H_{-\tau'}$ is a set of full measure $\mu(\cdot)$, we can define the probability measure $\mu'(\cdot)$ on $\mathfrak{C}_\sigma(H_{-\tau'})$ by

$$\mathfrak{C}_\sigma(H_{-\tau'}) \ni (\alpha \cap H_{-\tau'}) \mapsto \mu'(\alpha \cap H_{-\tau'}) = \mu(\alpha) \quad (\alpha \in \mathfrak{C}_\sigma(\Phi')).$$

Therefore, to prove the "if" part it suffices to show that the family $A' = (A'_\varphi)_{\varphi\in\Phi}$ of operators A'_φ, which act in the Hilbert space $L_2(H_{-\tau'}, \mathfrak{C}_\sigma(H_{-\tau'}), d\mu'(\cdot))$ as operators of multiplication by the function $(x, \varphi)_{H_0} (x \in H_{-\tau'})$, has a cyclic vector.

We can suppose the space $H_{\tau'}$ to be embedded into H_0 quasinuclearly, i.e. the inclusion $H_{\tau'} \to H_0$ is a Hilbert–Schmidt operator. Indeed, if it is not the case, we take a space $H_{\tau''} (\tau'' \in \mathcal{T})$ such that the inclusion $H_{\tau''} \subset H_{-\tau'}$ is quasinuclear, and, therefore, the inclusion $H_{\tau''} \subset H_0$ is also quasinuclear; furthermore, the space $H_{-\tau''}$ is a set of full measure $\mu(\cdot)$.

This is why there exists an orthonormal basis $(e_j)_{j=1}^{\infty}$ of H_0 such that the vectors $((p_j)^{-1/2} e_j)_{j=1}^{\infty}$, where p_j are some positive numbers, form an orthonormal basis of $H_{\tau'}$. Then, in the sense of the unitary isomorphism $V\colon H_0 \to l_2$, $Ve_j = \delta_j = (\delta_{i,j})_{i=1}^{\infty}$ $(j \in \mathbb{N})$, where $\delta_{i,j}$ is the Kronecker delta, we have the following inclusions

$$\mathbb{R}^{\infty} \supset l_2((1/p_j)_{j=1}^{\infty}) = H_{-\tau'} \supset l_2 = H_0 \supset l_2((p_j)_{j=1}^{\infty}) = H_{\tau'},$$

where

$$l_2((q_j)_{j=1}^{\infty}) = \left\{ (x_j)_{j=1}^{\infty} \in \mathbb{R}^{\infty} \,\middle|\, \sum_{j=1}^{\infty} x_j^2 q_j < \infty \right\} \ (q_j > 0).$$

The space $H_{-\tau'}$, as a weighted space l_2, belongs to the σ-algebra $\mathfrak{C}_\sigma(\mathbb{R}^\infty)$ generated by the cylinder sets in $\mathbb{R}^\infty$, and so, for an arbitrary set α from $\mathfrak{C}_\sigma(H_{-\tau'})$, we have $\alpha \in \mathfrak{C}_\sigma(\mathbb{R}^\infty)$ [5]. Hence, we can define the measure $\tilde{\mu}(\cdot)$ on $\mathfrak{C}_\sigma(\mathbb{R}^\infty)$ by

$$\mathfrak{C}_\sigma(\mathbb{R}^\infty) \ni \alpha \mapsto \tilde{\mu}(\alpha) = \mu'(\alpha \cap H_{-\tau'}).$$

The function

$$\tilde{\Omega}(x) = \exp\{-\| x \|_{H_{-\tau'}}^2\} = \exp\left\{ -\sum_{j=1}^{\infty} x_j^2 \frac{1}{p_j} \right\}$$

is defined for $\tilde{\mu}$-almost all $x = (x_j)_{j=1}^{\infty} \in \mathbb{R}^\infty$ and belongs to $L_2(\mathbb{R}^\infty, d\tilde{\mu}(\cdot))$. It satisfies the condition of Lemma 4 ([3], Ch. 1, § 2.5, p. 87). Therefore, the linear span of the functions $x^\beta \tilde{\Omega}(x) = (x_1^{\beta_1} x_2^{\beta_2} \ldots)\tilde{\Omega}(x)$ $(\beta = (\beta_n)_{n=1}^{\infty} \in \mathbb{Z}_{+,0}^{\infty}$ is a set of finite sequences from $\mathbb{Z}_{+,0}^{\infty})$ is a dense set on $L_2(\mathbb{R}^\infty, d\tilde{\mu}(\cdot))$ and the linear span of the functions $((x, e_1)_{H_0}^{\beta_1} (x, e_2)_{H_0}^{\beta_2} \ldots)\Omega(x)$ $(\beta \in \mathbb{Z}_{+,0}^{\infty})$, where $\Omega(\cdot) = \tilde{\Omega} \restriction H_{-\tau'}$, is dense in $L_2(H_{-\tau'}, d\mu'(\cdot))$. By using this and the estimate of the type

$$\| (x, f_1)_{H_0} \cdots (x, f_n)_{H_0} \exp\{-\| x \|_{H_{-\tau'}}^2\} \|_{L_2(H_{-\tau'}, d\mu'(\cdot))}^2 \le$$

$$\le \| f_1 \|_{H_{\tau'}}^2 \cdots \| f_n \|_{H_{\tau'}}^2 \int_{H_{-\tau'}} \| x \|_{H_{-\tau'}}^{2n} \exp\{-2 \| x \|_{H_{-\tau'}}^2\} d\mu'(x) =$$

$$= \| f_1 \|_{H_{\tau'}}^2 \cdots \| f_n \|_{H_{\tau'}}^2 C_n \quad (f_1, \ldots, f_n \in H_{\tau'},\ 0 \le C_n < \infty,\ n \in \mathbb{N}),$$

we obtain that, for arbitrary $\varphi_1, \ldots, \varphi_n \in \Phi$, $\Omega \in \mathcal{D}(A'_{\varphi_1} \ldots A'_{\varphi_n})$, and every function of the form $((x, e_1)_{H_0}^{m_1} \ldots (x, e_n)_{H_0}^{m_n})\Omega(x)$ $(m_1, \ldots, m_n \in \mathbb{Z}_+,\ n \in \mathbb{N})$ can be approximated with an arbitrary accuracy by a function of the form $((x, \varphi_1)_{H_0}^{m_1} \ldots$ $\ldots (x, \varphi_n)_{H_0}^{m_n})\Omega(x)(\varphi_1, \ldots, \varphi_n \in \Phi)$. Therefore $\Omega(\cdot)$ is a cyclic vector of the family $(A'_\varphi)_{\varphi \in \Phi}$.

"Only if" part. Let Ω be a cyclic vector of the family A. Without loss of generality, we suppose that $\Omega(x) \equiv 1$ $(x \in \Phi')$. Indeed, if it is not so, by taking into account that $\Omega(x) \ne 0$ (mod $\mu(\cdot)$), we can construct the measure $d\mu_1(x) = |\Omega(x)|^2 d\mu(x)$, which is equivalent to μ, and proceed, using the unitary transformation

$$L_2(\Phi', d\mu(\cdot)) \ni f(x) \mapsto (Kf)(x) = \frac{f(x)}{\Omega(x)} \in L_2(\Phi', d\mu_1(\cdot)),$$

to considering the family $B = (B_\varphi)_{\varphi \in \Phi}$ of the operators B_φ acting in the space $L_2(\Phi', d\mu_1(\cdot))$ as operators of multiplication by $(x, \varphi)_{H_0}$ $(x \in \Phi')$ $(B_\varphi = KA_\varphi K^{-1})$, for which the function $\hat{\Omega}(x) \equiv 1$ $(x \in \Phi')$ will be a cyclic vector.

The idea of the proof is to apply the projective spectral theorem [5] to the family A, and to show then that the generalized spectrum of the family A, which is a set of full measure $\mu(\cdot)$, is embedded into some Hilbert space $H_{-\tau'}$. So we have to construct a rigging of $L_2(\Phi', d\mu(\cdot))$ that is connected with A in a standard way. Since this task is quite complicated, we, first, construct a Hilbert space $\hat{H}$, proceed, by using a unitary isomorphism $\mathcal{F}: L_2(\Phi', d\mu(\cdot)) \to \hat{H}$, to considering the family $\mathcal{A} = (\mathcal{A}_\varphi)_{\varphi \in \Phi}$ of operators $\mathcal{A}_\varphi = \mathcal{F} A_\varphi \mathcal{F}^{-1}$ acting in the Hilbert space $\hat{H}$, and then construct a rigging of $\hat{H}$ connected with $\mathcal{A}$ in a standard way.

Let us consider the following mapping:

$$\Phi^{\otimes n} \ni (\varphi^{(1)}, \ldots, \varphi^{(n)}) \mapsto (\varphi^{(1)}, x)_{H_0} \cdot \ldots \cdot (\varphi^{(n)}, x)_{H_0} \equiv$$

$$\equiv (\varphi^{(1)} \otimes \ldots \otimes \varphi^{(n)}, x^{\otimes n})_{H_0^{\otimes n}} \in L_2(\Phi', d\mu(\cdot)) \quad (n \in \mathbb{N}). \tag{1}$$

Since the function $\Omega(x) \equiv 1$ $(x \in \Phi')$ is a cyclic vector of the family A, the functions $(\varphi^{(1)} \otimes \ldots \otimes \varphi^{(n)}, x^{\otimes n})_{H_0^{\otimes n}} \in L_2(\Phi', d\mu(\cdot))$ belong to $L_p(\Phi', d\mu(\cdot))$ for every $p \geq 1$ and, therefore, for every $n \in \mathbb{N}$, function (1) is continuous [4]. This yields that the operator

$$\Phi^{\otimes n} \ni \varphi_n \mapsto (\varphi_n, x^{\otimes n})_{H_0^{\otimes n}} \in L_2(\Phi', d\mu(\cdot)) \quad (n \in \mathbb{N})$$

is continuous.

Let Φ_c be the complexification of Φ, which is also a nuclear space. The set $\mathcal{P}(\Phi')$ of polynomials of the form

$$\varphi_0 + (\varphi_1, x)_{H_0} + \ldots + (\varphi_n, x^{\otimes n})_{H_0^{\otimes n}} \quad (n \in \mathbb{Z}_+),$$

where $x \in \Phi'$ and $\varphi_i \in \Phi_c^{\otimes i}$, is dense in $L_2(\Phi', d\mu(\cdot))$ by virtue of the definition of a cyclic vector.

Let us consider the mapping

$$\mathcal{P}(\Phi') \ni \varphi_0 + (\varphi_1, x)_{H_0} + \ldots + (\varphi_n, x^{\otimes n})_{H_0^{\otimes n}} \mapsto (\varphi_0, \varphi_1, \ldots, \varphi_n, 0, 0, \ldots) \in D_0,$$

where D_0 is the set of finite sequences, in which the i-th position is occupied by an element from $\Phi_c^{\otimes i}$. Define a quasiscalar product on D_0 by

$$\langle (\varphi_0, \ldots, \varphi_n, 0, 0, \ldots), (\psi_0, \ldots, \psi_m, 0, 0, \ldots) \rangle =$$

$$= (\varphi_0 + \ldots + (\varphi_n, x^{\otimes n})_{H_0^{\otimes n}}, \psi_0 + \ldots + (\psi_m, x^{\otimes m})_{H_0^{\otimes m}})_{L_2(\Phi', d\mu(\cdot))} \quad (n, m \in \mathbb{Z}_+). \tag{2}$$

By the factorization $D_0 \ni f \mapsto \hat{f} \in \hat{D}_0$ w. r. t. the set $\{f \in D_0 \mid \langle f, f \rangle = 0\}$ and by completion, we obtain the Hilbert space $\hat{H}$, which is unitary isomorphic to the space $L_2(\Phi', d\mu(\cdot))$ by construction. Let $\mathcal{F}: L_2(\Phi', d\mu(\cdot)) \to \hat{H}$ be this isomorphism. Then, for the operators $\mathcal{A}_\varphi = \mathcal{F} A_\varphi \mathcal{F}^{-1}$ $(\varphi \in \Phi)$, we get $\hat{D}_0 \subset \mathcal{D}(A\varphi)$ and, for any $\hat{f} \in \hat{D}_0$, $\hat{f} = (\varphi_0, \ldots, \varphi_n, 0, 0, \ldots)$, we have

$$\mathcal{A}_\varphi \hat{f} = (0, \varphi \otimes \varphi_0, \ldots, \varphi \otimes \varphi_n, 0, 0, \ldots) \in \hat{D}_0. \tag{3}$$

Define moment forms $s_n(\cdot)$ by

$$s_n(\varphi_n) = \int_{\Phi'} (\varphi_n, x^{\otimes n})_{H_0^{\otimes n}} \, d\mu(x) \quad (\varphi_n \in \Phi_c^{\otimes n}),$$

which are continuous functionals on $\Phi_c^{\otimes n}$ $(n \in \mathbb{N})$. Since $(\Phi_c^{\otimes n})' = \operatorname{ind\,lim}_{\tau \in \mathcal{T}} H_{-\tau,c}^{\otimes n}$ ($H_{-\tau,c}$ is the complexification of $H_{-\tau}$), for every $n \in \mathbb{N}$, there exists $\tau(n) \in \mathcal{T}$ such that $s_n(\cdot) \in H_{-\tau(n),c}^{\otimes n}$. Fix the sequence $(\tau(n))_{n=1}^{\infty}$.

Henceforth, we follow the proof of the theorem on the generalized power moment problem [5, 6] to construct the Fourier transform that corresponds to the family $\mathcal{A} = (\mathcal{A}_\varphi)_{\varphi\in\Phi}$. To do this, we, first, construct a rigging of the space $\hat{H}$

$$\hat{H}_- \supset \hat{H} \supset \hat{H}_+ \supset \hat{D}, \tag{4}$$

where $\hat{D}$ is a separable projective limit of Hilbert spaces, $\hat{H}_+$ is a Hilbert space embedded into $\hat{H}$ quasinuclearly, and $\hat{H}_-$ is the dual of $\hat{H}_+$ w. r. t. $\hat{H}$.

We equip the set D_0 with the topology of the topological direct sum of the nuclear spaces $\Phi_c^{\otimes n}$ $(n\in\mathbb{Z}_+)$, i.e., $D_0 = \bigoplus_{n=0}^{\infty}\Phi_c^{\otimes n}$ (if $n=0$, we have $\Phi_c^{\otimes n} = \mathbb{C}$). So, D_0 is a nuclear space that can be defined as the projective limit of all the spaces of the form

$$\mathcal{H}((\tau_n)_{n=0}^{\infty}, (p_n)_{n=0}^{\infty}) = \bigoplus_{n=0}^{\infty} H_{\tau_n,c}^{\otimes n} p_n\, (\tau_n \in \mathcal{T},\ p_n > 0),$$

which consist of sequences $(f_0, f_1, \dots)$ such that $f_n \in H_{\tau_n,c}^{\otimes n}$ and

$$\|(f_0, f_1, \dots)\|^2_{\mathcal{H}((\tau_n),(p_n))} = \sum_{n=0}^{\infty} \|f_n\|^2_{H_{\tau_n}^{\otimes n}} p_n < \infty.$$

For an arbitrary $f = (\varphi_0, \dots, \varphi_n, 0, 0, \dots) \in D_0$, by virtue of (3) and the Cauchy inequality, we get

$$\langle f, f\rangle = \sum_{j,k=0}^{\infty} \int_{\Phi'} (\varphi_j, x^{\otimes j})_{H_0^{\otimes j}} \overline{(\varphi_k, x^{\otimes k})}_{H_0^{\otimes k}}\, d\mu(x) \le$$

$$\le \sum_{j,k=0}^{\infty} \left(\int_{\Phi'} \left| \left(\varphi_j,\ x^{\otimes j}\right)_{H_0^{\otimes j}} \right|^2 d\mu(x) \int_{\Phi'} \left| \left(\varphi_k,\ x^{\otimes k}\right)_{H_0^{\otimes k}} \right|^2 d\mu(x) \right)^{1/2} =$$

$$= \left(\sum_{j=0}^{\infty} \left(\int_{\Phi'} \left(\varphi_j \otimes \overline{\varphi_j},\ x^{\otimes 2j}\right)_{H_0^{\otimes 2j}} d\mu(x) \right)^{1/2} \right)^2 =$$

$$= \left(|\varphi_0| + \sum_{j=1}^{\infty} \left(\left(\varphi_j \otimes \overline{\varphi_j},\ s_{2j}\right)_{H_{0,c}^{\otimes 2j}} \right)^{1/2} \right)^2 \le$$

$$\le |\varphi_0|^2 p_0 + \sum_{j=1}^{\infty} \|\varphi_j\|^2_{H_{\tau(2j),c}^{\otimes j}} \|s_{2j}\|_{H_{-\tau(2j),c}^{\otimes 2j}} p_j = \|f\|^2_{G_+}, \tag{5}$$

where the sequence $(p_j)_{j=0}^{\infty}$ satisfies the inequality $\sum_{j=0}^{\infty} \frac{1}{p_j} \le 1$,

$$G_+ = \bigoplus_{j=0}^{\infty} H_{\tau(2j),c}^{\otimes j} q_j, \quad q_j = \|s_{2j}\|_{H_{-\tau(2j),c}^{\otimes 2j}} p_j,$$

and $\overline{\varphi}_j$ denotes the element of $\Phi_c^{\otimes j}$ complex conjugate to φ_j. It follows from (5) that the bilinear form $D_0 \times D_0 \ni (f, g) \mapsto \langle f, g\rangle \in \mathbb{C}$ can be extended to the space $G_+ \otimes G_+$ by continuity and the set $\{f \in G_+ \mid \langle f, f\rangle = 0\}$ is closed in G_+. Therefore, we can factorize the space G_+ w. r. t. this set and get the Hilbert space $\hat{G}_+$ embedded into $\hat{H}$ topologically. Further, it follows from (5) and the definition of the nuclear space D_0 that it is possible to factorize D_0 w. r. t. the set $\{f \in D_+ \mid \langle f, f\rangle = 0\}$, giving rise to a nuclear space $\hat{D}_0$ ([5], Ch. 5, § 5.1).

Let $(\tau'(n))_{n=1}^{\infty}$ be a sequence of indices from $\mathcal{T}$ such that all the inclusions $H_{\tau'(n)} \subset H_{\tau(n)}$ are quasinuclear. Construct the Hilbert space $H_+ = \bigoplus_{n=0}^{\infty} H_{\tau'(2n),c}^{\otimes n} q'_n$, where the sequence $(q'_n)_{n=0}^{\infty}$ $(q'_n > 0)$ is chosen so that the imbedding H_+ into G_+ is quasinuclear, i. e.

$$|O_{H_+, G_+}|^2 = \sum_{n=0}^{\infty} |O_{H_{\tau'(2n)}^{\otimes n}, H_{\tau(2n)}^{\otimes n}}|^2 \frac{q_n}{q'_n} = \sum_{n=0}^{\infty} |O_{\tau'(2n), \tau(2n)}|^{2n} \frac{q_n}{q'_n},$$

where $O_{X,Y}$ denotes the inclusion operator $X \to Y$, $|\cdot|$ is the Hilbert–Schmidt norm. By factorization of H_+ w. r. t. the set $\{f \in H_+ \mid \langle f, f\rangle = 0\}$, we get the Hilbert space $\hat{H}_+$ embedded quasinuclearly into $\hat{G}_+$ and, thus, into $\hat{H}$ as well. Hence, the space $\hat{D}_0$ is embedded into $\hat{H}_+$ topologically. So, we obtain chain (4) with $\hat{D} = \hat{D}_0$.

Since the set $\hat{D}$ is dense in $\hat{H}_+$, we have that $\omega = \mathcal{F}\Omega(\cdot) = (1, 0, 0, \dots) \in \hat{D}$ is a strong cyclic vector of the family $\mathcal{A}$, i. e. the linear span of all the vectors of the form $\mathcal{A}_{\varphi_1}^{m_1} \dots \mathcal{A}_{\varphi_p}^{m_p} \omega$ is dense not only in $\hat{H}$ but also in $\hat{H}_+$. Besides, $\hat{D} \subset \mathcal{D}(\mathcal{A}_\varphi)$ $(\varphi \in \Phi)$ and, by virtue of (3), we have $\mathcal{A}_\varphi \in \mathcal{L}(\hat{D})$ and, therefore, $\mathcal{A}_\varphi \in \mathcal{L}(\hat{D}, \hat{H}_+)$ $(\varphi \in \Phi)$, where $\mathcal{L}(Y, Z)$ denotes the set of all bounded operators from Y into Z. So, the family $\mathcal{A}$ is connected with the chain (4) in a standard way. According to the theory of the expansion in generalized joint eigenvectors ([5], Ch. 3), we get the following results.

1) The joint r. i. $\mathcal{E}$ of the family $\mathcal{A}$ and the probability measure

$$\rho(\cdot) = (\mathcal{E}(\cdot)\omega, \omega)_{\hat{H}} \tag{6}$$

defined on $(\mathbb{R}^\Phi, \mathfrak{C}_\sigma(\mathbb{R}^\Phi))$ (here $\mathfrak{C}_\sigma(\mathbb{R}^\Phi)$ is the σ-algebra generated by cylinder sets in $\mathbb{R}^\Phi$) are equivalent and we have

$$O^+\mathcal{E}(\alpha)O = \int_\alpha P(\lambda(\cdot))\, d\rho(\lambda(\cdot)) \quad (\alpha \in \mathfrak{C}_\sigma(\mathbb{R}^\Phi)), \tag{7}$$

where $O\colon \hat{H}_+ \to \hat{H}$, $O^+\colon \hat{H} \to \hat{H}_-$ are inclusion operators, $P(\cdot)\colon \mathbb{R}^\Phi \ni \lambda(\cdot) \mapsto P(\lambda(\cdot)) \in \mathcal{L}(\hat{H}_+, \hat{H}_-)$ is a weak-measurable function w. r. t. $\mathfrak{C}_\sigma(\mathbb{R}^\Phi)$, defined for $\rho(\cdot)$-almost all $\lambda(\cdot) \in \mathbb{R}^\Phi$ and such that $P(\lambda(\cdot)) \geq 0$, $|P(\lambda(\cdot))| < \infty$ $(\lambda(\cdot) \in \mathbb{R}^\Phi)$.

2) There exists a set $\pi \subset \mathbb{R}^\Phi$ of full measure $\rho(\cdot)$ such that, for every $\lambda(\cdot) \in \pi$, the range $\mathfrak{R}(P(\lambda(\cdot)))$ is a one-dimensional subspace of $\hat{H}_-$, which consists of the generalized joint eigenvectors of the family $\mathcal{A}$, i.e.,

$$(P(\lambda(\cdot))\hat{f}, \mathcal{A}_\varphi \hat{g})_{\hat{H}} = \lambda(\varphi)(P(\lambda(\cdot))\hat{f}, \hat{g})_{\hat{H}} \quad (\varphi \in \Phi,\ \hat{f} \in \hat{H}_+,\ \hat{g} \in \hat{D}). \tag{8}$$

3) For every $\hat{f} \in \hat{H}_+$, we can define the Fourier transform

$$\hat{H}_+ \ni \hat{f} \mapsto \tilde{\hat{f}}(\lambda(\cdot)) = (\hat{f}, \hat{\xi}(\lambda(\cdot)))_{\hat{H}} = (F\hat{f})(\lambda(\cdot)) \in \mathbb{C}^1,$$

for ρ_π-almost all $\lambda(\cdot) \in \pi$, where $\rho_\pi(\cdot)$ is the modification of $\rho(\cdot)$ by π: $\rho_\pi(\alpha \cap \pi) = \rho(\alpha)$ $(\alpha \in \mathfrak{C}_\sigma(\mathbb{R}^\Phi))$, and $\hat{\xi}(\lambda(\cdot))$ are nonzero vectors from $\mathfrak{R}(P(\lambda(\cdot))) \subset \hat{H}_-$, fixed for every $\lambda(\cdot) \in \pi$; furthermore, the Parseval equality holds:

$$(\hat{f}, \hat{g})_{\hat{H}} = \int_\pi \tilde{\hat{f}}(\lambda(\cdot))\overline{\tilde{\hat{g}}(\lambda(\cdot))}\, d\rho_\pi(\lambda(\cdot)) \quad (\hat{f}, \hat{g} \in \hat{H}_+). \tag{9}$$

4) By virtue of (9), the operator F is extended by continuity to an isometric operator from $\hat{H}$ into $L_2(\pi, d\rho_\pi(\cdot))$.

Let $\lambda(\cdot) \in \pi$, $\hat{f}, \hat{g} \in \hat{H}_+$, $I_1: H_- \to H_+$ be the isometry corresponding to the rigging (4), and let Q be the orthogonal projector $\varphi \mapsto \hat{\varphi}$ in H_+ on its subspace $\hat{H}_+$ (here, we understand $\hat{H}_+$ as the orthogonal complement to the subspace $\{f \in H_+ \mid \langle f, f\rangle = 0\}$). Then

$$(P(\lambda(\cdot))\hat{f}, \hat{g})_{\hat{H}} = (I_1 P(\lambda(\cdot))\hat{f}, \hat{g})_{\hat{H}_+} = (QI_1 P(\lambda(\cdot))Qf, g)_{H_+}, \tag{10}$$

where f, g are elements of H_+ corresponding to the elements $\hat{f}, \hat{g} \in \hat{H}_+$. Since $|P(\lambda(\cdot))| < \infty$ $(\lambda(\cdot) \in \pi)$, $QI_1P(\lambda(\cdot))Q: H_+ \to H_+$ is a Hilbert – Schmidt operator, and, by the theorem on the kernel of a quasinuclear operator, there exists $S(\lambda(\cdot)) \in$ $\in H_+ \otimes H_+$ such that

$$(QI_1P(\lambda(\cdot))Qf, g)_{H_+} = (S(\lambda(\cdot)), g \otimes \bar{f})_{H_+ \otimes H_+} \quad (f, g \in H_+). \tag{11}$$

Consider the chain $H_- \supset G \supset H_+$, where $G = \bigoplus_{n=0}^{\infty} H_{0,c}^{\otimes n}$, H_+ is the above-defined space and H_- is the dual of H_+ w. r. t. G. The tensor square of this chain is

$$H_-^{\otimes 2} \supset G^{\otimes 2} \supset H_+^{\otimes 2}. \tag{12}$$

Let I_2: $H_-^{\otimes 2} \to H_+^{\otimes 2}$ be the isometry corresponding to (12). Then

$$(S(\lambda(\cdot)), g \otimes \bar{f})_{H_+^{\otimes 2}} = (I_2^{-1}S(\lambda(\cdot)), g \otimes \bar{f})_{G^{\otimes 2}} = (T(\lambda(\cdot)), g \otimes \bar{f})_{G^{\otimes 2}}, \tag{13}$$

where $(T(\lambda(\cdot)) = I_2^{-1}S(\lambda(\cdot)) \in H_-^{\otimes 2}$. By using (10), (11), and (13), we get

$$(P(\lambda(\cdot))\hat{f}, \hat{g})_{\hat{H}} = (T(\lambda(\cdot)), g \otimes \bar{f})_{G^{\otimes 2}}. \tag{14}$$

This equality, together with (8), yields

$$(T(\lambda(\cdot)), (\mathcal{A}_\varphi g) \otimes \bar{f})_{G^{\otimes 2}} = \lambda(\varphi)(T(\lambda(\cdot)), g \otimes \bar{f})_{G^{\otimes 2}} \tag{15}$$

$$(\varphi \in \Phi, g \in D_0, f \in H_+).$$

Note that $T(\lambda(\cdot))$, as an element of $H_-^{\otimes 2}$, can be represented as an infinite matrix $(T_{j,k}(\lambda(\cdot)))_{j,k=0}^{\infty}$, where

$$T_{j,k}(\lambda(\cdot)) \in H_{-\tau'(2j),\,c}^{\otimes j} \otimes H_{-\tau'(2k),\,c}^{\otimes k}, \ \| T(\lambda(\cdot)) \|_{H_-^{\otimes 2}}^2 =$$

$$= \sum_{j,k=0}^{\infty} \| T_{j,k}(\lambda(\cdot)) \|_{H_{-\tau'(2j),c}^{\otimes j} \otimes H_{-\tau'(2k),c}^{\otimes k}}^2 (q'_j q'_k)^{-1}. \tag{16}$$

Since the operators $\mathcal{A}_\varphi$ $(\varphi \in \Phi)$ are real w. r. t. the complex conjugation in $\hat{H}$, so are the operators $P(\lambda(\cdot))(\lambda(\cdot) \in \pi)$; therefore, owing to (14), we find that the elements of the matrix $T_{j,k}(\lambda(\cdot)))_{j,k=0}^{\infty}$ are real $(\lambda(\cdot) \in \pi)$. This implies that (16) holds without all the indices c of complexification.

By substituting $f = (0, \ldots, 0, \varphi_k, 0, \ldots)$ and $g = (0, \ldots, 0, \psi_j, 0, \ldots)$ $(\varphi_k \in \Phi^{\otimes k}$, $\psi_j \in \Phi^{\otimes j}$, $k, j \in \mathbb{Z}_+)$ in (15), we get:

$$(T_{j+1,k}(\lambda(\cdot)), \varphi \otimes \psi_j \otimes \varphi_k)_{H_0^{\otimes(j+k+1)}} = \lambda(\varphi)(T_{j,k}(\lambda(\cdot)), \psi_j \otimes \varphi_k)_{H_0^{\otimes(j+k)}}. \tag{17}$$

For every $f, g \in H_+$, by virtue of (14) and (7), we have

$$(T(\lambda(\cdot)),\ g \otimes \bar{f})_{G^{\otimes 2}} = (P(\lambda(\cdot))\hat{f},\ \hat{g})_{\hat{H}} =$$
$$= (\hat{f},\ P(\lambda(\cdot))\hat{g})_{\hat{H}} = (T(\lambda(\cdot)),\ f \otimes \bar{g})_{G^{\otimes 2}}$$

and, therefore,

$$(T_{j,k}(\lambda(\cdot)),\ \psi_j \otimes \varphi_k)_{H_0^{\otimes(j+k)}} = (T_{k,j}(\lambda(\cdot)),\ \varphi_k \otimes \psi_j)_{H_0^{\otimes(j+k)}} \quad (\varphi_k \in \Phi^{\bullet k},\ \psi_j \in \Phi^{\bullet j}). \tag{18}$$

Owing to (16) and (18), we get

$$(T_{k,j+1}(\lambda(\cdot)),\ \varphi_k \otimes \varphi \otimes \psi_j)_{H_0^{\otimes(j+k+1)}} = \lambda(\varphi)(T_{k,j}(\lambda(\cdot)),\ \varphi_k \otimes \psi_j)_{H_0^{\otimes(j+k)}}. \tag{19}$$

By using formulas (17) and (19), it is easy to check by induction that

$$(T_{j,k}(\lambda(\cdot)),\ \varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_{j+k})_{H_0^{\otimes(j+k)}} = \lambda(\varphi_1)\lambda(\varphi_2)\ldots\lambda(\varphi_{j+k})T_{0,0}(\lambda(\cdot))$$
$$(\varphi_1, \varphi_2, \ldots, \varphi_{j+k} \in \Phi,\ j, k \in \mathbb{Z}_+,\ \lambda(\cdot) \in \pi).$$

This implies

$$T_{0,0}(\lambda(\cdot)) \neq 0,\ \lambda(\cdot) = T_{0,1}(\lambda(\cdot)) = T_{1,0}(\lambda(\cdot)) \in H_{-\tau'(2)},$$
$$T_{j,k)}(\lambda(\cdot)) = (\lambda(\cdot))^{\bullet(j+k)}T_{0,0}(\lambda(\cdot)) \in H_{-\tau'(2)}^{\otimes(j+k)}.$$

Hence, we have the inclusion $\pi \subset H_{-\tau'(2)}$ and, therefore, $H_{-\tau'(2)}$ is a set of full measure $\rho(\cdot)$.

Let $\rho_{\Phi'}$ be the modification of the measure $\rho(\cdot)$ by Φ': $\rho_{\Phi'}(\alpha \cap \Phi') = \rho(\alpha)$ ($\alpha \in \mathfrak{C}_\sigma(\mathbb{R}^\Phi)$). It is easy to see that

$$\rho_{\Phi'}(\cdot) = \mu(\cdot). \tag{20}$$

To prove this it suffices to show that these measures are equal on the sets of the form

$$C(\varphi_1, \ldots, \varphi_n, \Delta_1 \times \ldots \times \Delta_n) = \{\lambda \in \Phi' \mid \lambda(\varphi_1) \in \Delta_1, \ldots, \lambda(\varphi_n) \in \Delta_n\}$$
$$(\varphi_1, \ldots, \varphi_n \in \Phi,\ \Delta_1, \ldots, \Delta_n \in \mathcal{B}(\mathbb{R}),\ n \in \mathbb{N}).$$

We prove this for $n = 1$; the proof for $n > 1$ can be carried out by analogy. By (6), we get

$$\rho(C(\varphi, \Delta)) = (\mathcal{E}(C(\varphi, \Delta))\omega, \omega)_{\hat{H}} = (E(C(\varphi, \Delta))\Omega(x), \Omega(x))_{L_2(\Phi', d\mu(\cdot))} =$$
$$= (E_\varphi(\Delta)\,\Omega(x), \Omega(x))_{L_2(\Phi', d\mu(\cdot))} = (\chi_\Delta(A_\varphi)\,\Omega(x), \Omega(x))_{L_2(\Phi', d\mu(\cdot))} =$$
$$= \int_{\Phi'} \chi_\Delta((\varphi, x)_{H_0})\, d\mu(x) = \mu(C(\varphi, \Delta)) \ \ (\varphi \in \Phi, \Delta \in \mathcal{B}(\mathbb{R})),$$

where E is the joint r. i. of A, E_φ is the r. i. of A_φ, and $\chi_\Delta(\cdot)\colon \mathbb{R} \to \mathbb{R}$ is the characteristic function of the set Δ. So, equality (20) is proved, and $H_{-\tau'(2)}$ is a set of full measure $\mu(\cdot)$.

3. Under some restrictions, a family $B = (B_\varphi)_{\varphi \in \Phi}$ of self-adjoint commuting operators B_φ in a Hilbert space H is unitary equivalent to the family $A = (A_\varphi)_{\varphi \in \Phi}$ in $L_2(\Phi', d\mu(\cdot))$ and, therefore, *B has a cyclic vector in H iff $\exists \tau' \in \mathfrak{T}$ such that $H_{-\tau'}$ is a set of full operator-valued measure R, where R is the joint r. i. of the* *family* B. To find this, it suffices to note that the resolutions of the identity of unitary equivalent families of operators are equivalent. We give two types of such restrictions on the family B.

a) Let

$$H_- \supset H \supset H_+ \supset D \tag{21}$$

be a rigging of H analogous to (4). The restrictions on B are the following:

a1) The space D forms a core for every operator B_φ, i. e. $(B_\varphi \upharpoonright D)^\sim = B_\varphi$ $(\varphi \in \Phi)$.

a2) The family B is connected with the chain (21) in a standard way, i.e., $D \in \in \mathfrak{D}(B_\varphi)$, $B_\varphi \in \mathfrak{L}(D, H_+)$ $(\varphi \in \Phi)$.

a3) $\forall f \in D$ the mapping $\Phi \ni \varphi \mapsto B_\varphi f \in H_+$ is linear and weakly continuous.

a4) $\exists\, \Omega' \in H$, and there exist a collection of distinct points $x_1, \dots, x_p \in X$ and numbers $m_1, \dots, m_p \in \mathbb{Z}_+$ $(p \in \mathbb{N})$ such that $\Omega' \in \mathfrak{D}(B_{x_1}^{m_1} \dots B_{x_p}^{m_p})$, $B_{x_1}^{m_1} \dots B_{x_p}^{m_p} \Omega' \in \in H_+$, and the linear span of all such vectors $B_{x_1}^{m_1} \dots B_{x_p}^{m_p} \Omega'$ is dense in H_+. (Note that we demanded in the definition of a cyclic vector that it should belong to the domains of all the operators of the form $B_{x_1}^{m_1} \dots B_{x_p}^{m_p}$.)

Then, according to the theory of the expansion in generalized joint eigenvectors, the family B is unitary equivalent to the family A in $L_2(\Phi', d\mu(\cdot))$, where $\mu(\cdot)$ is the spectral measure of B.

b) Let $U = (U_\varphi)_{\varphi \in \Phi}$ be the family of unitary operators such that $U_\varphi = \exp\{iB_\varphi\}$ $(\varphi \in \Phi)$. We demand that the family U be a unitary cyclic representation of the space Φ, i.e.,

b1) $U_{\varphi_1 + \varphi_2} = U_{\varphi_1} U_{\varphi_2} (\varphi_1, \varphi_2 \in \Phi)$;

b2) $\forall f \in H$ the function $\Phi \ni \varphi \mapsto U_\varphi f \in H$ is weakly continuous;

b3) The family U has a cyclic vector, i. e. $\exists\, h \in H$ such that the linear span of all the vectors $U(\varphi)h(\varphi \in \Phi)$ is a set dense in H_0.

Then the family B is unitary equivalent to A in $L_2(\Phi', d\mu(\cdot))$ [7], where $\mu(\cdot)$ is defined by the equality

$$(U(\varphi)h, h)_H = \int_{\Phi'} \exp\{i\,(x, \varphi)_{H_0}\}\, d\mu(x) \quad (\varphi \in \Phi).$$

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