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Generalizations of semicoprime preradicals

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Abstract. This article introduces the notions quasi-co-n-absorbing preradicals and semi-co-n-absorbing preradicals, generalizing the concept of semicoprime preradicals. We study the concepts quasi-co-n-absorbing submodules and semi-co-n-absorbing submodules and their relations with quasi-co-n-absorbing preradicals and semi-co-n-absorbing preradicals using the lattice structure of preradicals.

1. Introduction

The notion of 2-absorbing ideals of commutative rings was introduced by Badawi in [2], where a proper ideal I of a commutative ring R is called a *2-absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [1] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. In [24], the concept of 2-absorbing ideals was generalized to submodules of a module over a commutative ring. A proper submodule N of an R-module M is said to be a *2-absorbing submodule of* M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. For more studies concerning

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2-absorbing (submodules) ideals we refer to [3],[9],[24],[25]. In [13], Raggi et al. introduced the concepts of prime preradicals and prime submodules over noncommutative rings, and Raggi, Ríos and Wisbauer [18], studied the dual notions of these, coprime preradicals and coprime submodules. A generalization of prime preradicals and submodules, "2-absorbing preradicals and submodules" was investigated by Yousefian and Mostafanasab in [23]. In [14], Raggi et al. defined and investigated semiprime preradicals, and Mostafanasab and Yousefian [10], studied the concepts of quasi-n-absorbing and semi-n-absorbing preradicals. Raggi et al. [11] defined the notions of semicoprime preradicals and submodules. In this paper, we introduce the concepts of "quasi-co-n-absorbing preradicals" and "semi-co-n-absorbing preradicals". As well we investigate"quasi-co-n-absorbing submodules" and "semi-co-n-absorbing submodules" in this paper.

2. Preliminaries

Throughout this paper R is an associative ring with nonzero identity, and R-Mod denotes the category of all the unitary left R-modules. We denote by R-simp a complete set of representatives of isomorphism classes of simple left R-modules. For $M \in R$-Mod, we denote by $\mathrm{E}(M)$ the injective hull of M. Let $U, N \in R$-Mod, we say that N is *generated by* U (or N is U*-generated*) if there exists an epimorphism $U^{(\Lambda)} \to N$ for some index set Λ. Dually, we say that N is *cogenerated by* U (or N is U*-cogenerated*) if there exists a monomorphism $N \to U^{\Lambda}$ for some index set Λ. Also, we say that an R-module X is *subgenerated by* M (or X is M*-subgenerated*) if X is a submodule of an M-generated module. The category of M-subgenerated modules (the Wisbauer category) is denoted $\sigma[M]$ (see [21]). A *preradical* over the ring R is a subfunctor of the identity functor on R-Mod. Denote by R-pr the class of all preradicals over R. There is a natural partial ordering in R-pr given by $\sigma \preceq \tau$ if $\sigma(M) \leqslant \tau(M)$ for every $M \in R$-Mod. It is proved in [15] that with this partial ordering, R-pr is an atomic and co-atomic big lattice. The smallest and the largest elements of R-pr are denoted, respectively, 0 and 1.

Let $M \in R$-Mod. Recall ([5] or [15]) that a submodule N of M is called *fully invariant* if $f(N) \leqslant N$ for each R-homomorphism $f : M \to M$. In this paper, the notation $N \leqslant_{fi} M$ means that "N is a fully invariant submodule of M". Obviously the submodule K of M is fully invariant if and only if there exists a preradical τ of R-Mod such that $K = \tau(M)$. If $N \leqslant M$, then the preradicals α^M_N and ω^M_N are defined as follows: For $K \in R$-Mod,

1) $\alpha_N^M(K) = \sum\{f(N) | f \in \text{Hom}_R(M, K)\}$.
2) $\omega_N^M(K) = \bigcap\{f^{-1}(N) | f \in \text{Hom}_R(K, M)\}$.

Notice that for $\sigma \in R$-pr and $M, N \in R$-Mod we have that $\sigma(M) = N$ if and only if $N \leqslant_{fi} M$ and $\alpha_N^M \preceq \sigma \preceq \omega_N^M$. We have also that if $K \leqslant N \leqslant M$ with $K, N \leqslant_{fi} M$, then $\alpha_K^M \preceq \alpha_N^M$ and $\omega_K^M \preceq \omega_N^M$.

The atoms and coatoms of R-pr are, respectively, $\{\alpha_S^{E(S)} \mid S \in R\text{-simp}\}$ and $\{\omega_I^R \mid I$ is a maximal ideal of $R\}$ (See [15, Theorem 7]).

There are four classical operations in R-pr, namely, $\wedge, \vee, \cdot$ and $:$ which are defined as follows. For $\sigma, \tau \in R$-pr and $M \in R$-Mod:

1) $(\sigma \wedge \tau)(M) = \sigma M \cap \tau M$,
2) $(\sigma \vee \tau)(M) = \sigma M + \tau M$,
3) $(\sigma\tau)(M) = \sigma(\tau M)$ and
4) $(\sigma : \tau)(M)$ is determined by $(\sigma : \tau)(M)/\sigma M = \tau(M/\sigma M)$.

The meet $\wedge$ and join $\vee$ can be defined for arbitrary families of preradicals as in [15]. The operation defined in (3) is called *product*, and the operation defined in (4) is called *coproduct*. It is easy to show that for $\sigma, \tau \in R$-pr, $\sigma\tau \preceq \sigma \wedge \tau \preceq \sigma \vee \tau \preceq (\sigma : \tau)$. It is clear that in R-pr the operations (1)-(3) are associative, and in [22] it was shown that the coproduct " :″ is associative. Notice the fact that coproduct of preradicals preserves order on both sides, see [8, Remark 2.1]. We denote $\sigma\sigma\cdots\sigma$ (n times) by σ^n and $(\sigma : \sigma : \cdots : \sigma)$ (n times) by $\sigma_{[n]}$. Recall that $\sigma \in R$-pr is an *idempotent* if $\sigma^2 = \sigma$, while σ is a *radical* if $\sigma_{[2]} = \sigma$. Note that σ is a radical if and only if, $\sigma(M/\sigma(M)) = 0$ for each $M \in R$-Mod. We say that σ is *nilpotent* if $\sigma^n = 0$ for some $n \geqslant 1$, while σ is *unipotent* if $\sigma_{[n]} = 1$ for some $n \geqslant 1$.

Using the preradical ω_N^M, in the papers [6], [7] and [18], the following operation was introduced and studied:

$$\omega\text{-coproduct of submodules } K,\ N \leqslant M : (K :_M N) = (\omega_K^M : \omega_N^M)(M).$$

Henceforward, for brevity, $(K : N)$ is written instead of $(K :_M N)$. For any $\sigma \in R$-pr, we will use the following class of R-modules:

$$\mathbb{T}_\sigma = \{M \in R - \text{Mod} \mid \sigma(M) = M\}.$$

Let $\sigma \in R$-pr. By [18, Theorem 8.2], the following classes of modules are closed under taking arbitrary meets and arbitrary joins:

$$\mathcal{A}_e = \{\tau \in R\text{-pr} \mid \tau\sigma = \sigma\} \quad \text{and} \quad \mathcal{A}_t = \{\tau \in R\text{-pr} \mid (\sigma : \tau) = 1\}.$$

As in [16], we define, for $\sigma \in R$-pr, the following preradicals:

- $e(\sigma) = \bigwedge\{\tau \in \mathcal{A}_e\}$ the equalizer of σ;

- $t(\sigma) = \bigwedge\{\tau \in \mathcal{A}_t\}$ the totalizer of σ.

Clearly $e(\sigma)\sigma = \sigma$ and $(\sigma : t(\sigma)) = 1$. For undefined notions we refer the reader to [13, 15–17].

In [18], Raggi et al. defined the notions of coprime preradicals and coprime submodules as follows:

Let $\sigma \in R$-pr. σ is called *coprime in* R-pr if $\sigma \neq 0$ and for any $\tau,\ \eta \in R$-pr, $\sigma \preceq (\tau : \eta)$ implies that $\sigma \preceq \tau$ or $\sigma \preceq \eta$. Let $M \in R$-Mod and let $N \leqslant M$ be a nonzero fully invariant submodule of M. The submodule N is said to be *coprime in M* if whenever $K,\ L$ are fully invariant submodules of M with $N \leqslant (K : L)$, then $N \leqslant K$ or $N \leqslant L$. Also, Raggi et al. [11] defined a preradical σ *semicoprime in* R-pr if $\sigma \neq 0$ and for any $\tau \in R$-pr, $\sigma \preceq (\tau : \tau)$ implies that $\sigma \preceq \tau$. They said that a nonzero fully invariant submodule N of M is *semicoprime in M* if whenever K is a fully invariant submodule of M with $N \leqslant (K : K)$, then $N \leqslant K$. In special case, M is called a *coprime (resp. semicoprime) module* if M is a coprime (resp. semicoprime) submodule of itself.

Yousefian and Mostafanasab in [22] defined the notions of co-2-absorbing preradicals and co-2-absorbing submodules. The preradical $\sigma \in R$-pr is called *co-2-absorbing* if $\sigma \neq 0$ and, for each $\eta, \mu, \nu \in R$-pr, $\sigma \preceq (\eta : \mu : \nu)$ implies that $\sigma \preceq (\eta : \mu)$ or $\sigma \preceq (\eta : \nu)$ or $\sigma \preceq (\mu : \nu)$. More generally, a preradical $0 \neq \sigma$ in R-pr is said to be a *co-n-absorbing preradical* if whenever $\sigma \preceq (\eta_1 : \eta_2 : \cdots : \eta_{n+1})$ for $\eta_1, \eta_2, \ldots, \eta_{n+1} \in R$-pr, there are $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n+1\}$ such that $i_1 < i_2 < \cdots < i_n$ and $\sigma \preceq (\eta_{i_1} : \eta_{i_2} : \cdots : \eta_{i_n})$. They denoted by R-co-ass the class of all R-modules M that the operation ω-coproduct is associative over fully invariant submodules of M, i.e., for any fully invariant submodules K, N, L of M, $((K : N) : L) = (K : (N : L))$. Let $M \in R$-co-ass and K be a fully invariant submodule of M. Then $(K : K : \cdots : K)$ (n times) is simply denoted by $K_{[n]}$. By Proposition 5.4 of [7], we can see that if an R-module M is injective and artinian, then $M \in R$-co-ass. Let $M \in R$-co-ass and N a nonzero fully invariant submodule of M. The submodule N is said to be *co-2-absorbing in M* if whenever J, K, L are fully invariant submodules of M with $N \leqslant (J : K : L)$, then $N \leqslant (J : K)$ or $N \leqslant (J : L)$ or $N \leqslant (K : L)$. The generalization of co-2-absorbing submodules is that, the submodule N is said *co-n-absorbing in M* if whenever $N \leqslant (K_1 : K_2 : \cdots : K_{n+1})$ for fully invariant submodules $K_1, K_2, \ldots, K_{n+1}$ of M, there are $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n+1\}$ such that $i_1 < i_2 < \cdots < i_n$ and $N \leqslant (K_{i_1} : K_{i_2} : \cdots : K_{i_n})$. An R-module M is called a *co-n-absorbing module* if M is a co-n-absorbing submodule of itself.

We say that a preradical $0 \neq \sigma \in R$-pr is called a *quasi-co-n-absorbing preradical* if whenever $\sigma \preceq (\mu_{[n]} : \nu)$ for $\mu, \nu \in R$-pr, then $\sigma \preceq \mu_{[n]}$ or $\sigma \preceq (\mu_{[n-1]} : \nu)$. A preradical $0 \neq \sigma \in R$-pr is called a *semi-co-n-absorbing preradical* if whenever $\sigma \preceq \mu_{[n+1]}$ for $\mu \in R$-pr, then $\sigma \preceq \mu_{[n]}$. Let $M \in R$-co-ass. We say that a nonzero fully invariant submodule N of M is *quasi-co-n-absorbing in M* if for every fully invariant submodules $K,\ L$ of M, $N \leqslant (K_{[n]} : L)$ implies that $N \leqslant K_{[n]}$ or $N \leqslant (K_{[n-1]} : L)$. A nonzero fully invariant submodule N of M is called *semi-co-n-absorbing in M* if for every fully invariant submodule K of M, $N \leqslant K_{[n+1]}$ implies that $N \leqslant K_{[n]}$. An R-module M satisfies *the ω-property* if $(\tau(M) :_M \eta(M)) = (\tau : \eta)(M)$ for every $\tau,\ \eta \in$ *R-pr*, see [22].

We recall the definition of relative epi-projectivity (see [20]). Let M and N be modules. N is said to be *epi-M-projective* if, for any submodule K of M, any epimorphism $f : N \to \frac{M}{K}$ can be lifted to a homomorphism $g : N \to M$

Proposition 1 ([22, Proposition 2.9 (1)]). *Let $M \in$ R-Mod. If for any fully invariant submodule K of M, $\frac{M}{K}$ is epi-M-projective, then M has the ω-property.*

In the next sections we frequently use the following proposition.

Proposition 2 ([12, Proposition 1.2]). *Let $\{M_\gamma\}_{\gamma\in I}$ and $\{N_\gamma\}_{\gamma\in I}$ be families of modules in R-Mod such that for each $\gamma \in I$, $N_\gamma \leqslant M_\gamma$. Let $N = \bigoplus_{\gamma\in I} N_\gamma$, $M = \bigoplus_{\gamma\in I} M_\gamma$, $N' = \prod_{\gamma\in I} N_\gamma$ and $M' = \prod_{\gamma\in I} M_\gamma$.*
(1) *If $N \leqslant_{fi} M$, then for each $\gamma \in I$, $N_\gamma \leqslant_{fi} M_\gamma$ and $\alpha_N^M = \bigvee_{\gamma\in I} \alpha_{N_\gamma}^{M_\gamma}$.*
(2) *If $N' \leqslant_{fi} M'$, then for each $\gamma \in I$, $N_\gamma \leqslant_{fi} M_\gamma$ and $\omega_{N'}^{M'} = \bigwedge_{\gamma\in I} \omega_{N_\gamma}^{M_\gamma}$.*

3. Quasi-co-n-absorbing preradicals

Suppose that $m,\ n$ are positive integers with $n > m$. A preradical $\sigma \neq 0$ is called a *quasi-co-(n,m)-absorbing preradical* if whenever $\sigma \preceq (\mu_{[n-1]} : \nu)$ for $\mu, \nu \in R$-pr, then $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : \nu)$.

Proposition 3. *Let $\sigma \in$ R-pr and let $m > 0$. The following conditions are equivalent:*
(1) *σ is quasi-co-(n,m)-absorbing for every $n > m$;*
(2) *σ is quasi-co-(n,m)-absorbing for some $n > m$;*
(3) *σ is quasi-co-m-absorbing.*

Proof. (1)⇒(2) Is trivial.

(2)⇒(3) Assume that σ is quasi-co-(n,m)-absorbing for some $n>m$. Let $\sigma \preceq (\mu_{[m]} : \nu)$ for some $\mu,\nu \in R$-pr. Since $m \leqslant n-1$, then $(\mu_{[m]} : \nu) \preceq (\mu_{[n-1]} : \nu)$ and so $\sigma \preceq (\mu_{[n-1]} : \nu)$. Therefore $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : \nu)$. Consequently σ is quasi-co-m-absorbing.

(3)⇒(1) Suppose that σ is quasi-co-m-absorbing and get $n>m$. Let $\sigma \preceq (\mu_{[n-1]} : \nu)$ for some $\mu,\nu \in R$-pr. Therefore $\sigma \preceq (\mu_{[m]} : (\mu_{[n-1-m]} : \nu))$. Hence $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : (\mu_{[n-1-m]} : \nu)) = (\mu_{[n-2]} : \nu)$. Repeating this method implies that $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : \nu)$. Thus σ is quasi-co-(n,m)-absorbing. □

Remark 1. Let $\sigma \in R$-pr.

(1) σ is coprime if and only if σ is quasi-co-1-absorbing if and only if σ is co-1-absorbing.

(2) If σ is quasi-co-n-absorbing, then it is quasi-co-i-absorbing for all $i \geqslant n$.

(3) If σ is coprime, then it is quasi-co-n-absorbing for all $n \geqslant 1$.

(4) If σ is quasi-co-n-absorbing for some $n \geqslant 1$, then there exists the least $n_0 \geqslant 1$ such that σ is quasi-co-n_0-absorbing. In this case, σ is quasi-co-n-absorbing for all $n \geqslant n_0$ and it is not quasi-co-i-absorbing for $n_0 > i > 0$.

Proposition 4. *Let $\mathcal{C}$ be a family of coprime preradicals. Then $\bigvee_{\sigma\in\mathcal{C}} \sigma$ is a quasi-co-i-absorbing preradical for every $i \geqslant 2$.*

Proof. Let $\tau = \bigvee_{\sigma\in\mathcal{C}} \sigma$. By Remark 1(2), it is sufficient to show that τ is a quasi-co-2-absorbing preradical. Suppose that $\tau \preceq (\mu_{[2]} : \nu)$ for some $\mu,\nu \in R$-pr. Since every $\sigma \in \mathcal{C}$ is coprime and $\sigma \preceq (\mu_{[2]} : \nu)$, then $\sigma \preceq \mu$ or $\sigma \preceq \nu$. Hence $\tau \preceq (\mu : \nu)$, and so we conclude that τ is a quasi-co-2-absorbing preradical. □

Let $\zeta = \bigvee\{\alpha_S^S \mid S \in R\text{-simp}\}$. Note that for every R-module M, $\zeta(M) = \mathrm{Soc}(M)$. As in [14], ζ is called the socle preradical. Also, let $\kappa = \{\alpha_{R/I}^{R/I} \mid I$ a maximal ideal of $R\}$. We call κ the ultrasocle preradical, see [11].

As a direct consequence of Proposition 4 we have the following result.

Proposition 5. *ζ, κ are quasi-co–i-absorbing preradicals for every $i \geqslant 2$.*

Proof. By [18, p. 57], for each simple R-module S, α_S^S is coprime. Also, for every maximal ideal I of R, $\alpha_{R/I}^{R/I}$ is a coprime preradical, [11, Remark 6]. Then by Proposition 4, the claim holds. □

Proposition 6. *If R is a semisimple Artinian ring, then every nonzero preradical $\sigma \in R$-pr is a quasi-co-i-absorbing preradical for every $i \geqslant 2$.*

Proof. Suppose that R is a semisimple Artinian ring. According to [18, Proposition 3.2], every atom $\alpha_S^{E(S)}$ is a coprime preradical. On the other hand [15, Theorem 11] implies that $\sigma = \bigvee\{\alpha_S^{E(S)} \mid S \in R\text{-simp}, \alpha_S^{E(S)} \preceq \sigma\}$. Therefore every nonzero preradical σ in R-pr is quasi-co-i-absorbing for every $i \geqslant 2$, by Proposition 4. □

Remark 2. Let $S_1, S_2, \ldots, S_{n+1} \in R$-simp be distinct. Then by Proposition 4, $\alpha_{S_1}^{S_1} \vee \alpha_{S_2}^{S_2} \vee \cdots \vee \alpha_{S_{n+1}}^{S_{n+1}}$ is a quasi-co-i-absorbing preradical in R-pr for every $i \geqslant 2$. But, [22, Proposition 3.6] implies that $\alpha_{S_1}^{S_1} \vee \alpha_{S_2}^{S_2} \vee \cdots \vee \alpha_{S_{n+1}}^{S_{n+1}}$ is not a co-n-absorbing preradical. This remark shows that the two concepts of quasi-co-n-absorbing preradicals and of co-n-absorbing preradicals are different in general.

Corollary 1. *If R is a ring such that every quasi-co-n-absorbing preradical in R-pr is co-n-absorbing, then $|R\text{-simp}| \leqslant n$.*

Notice the fact that coproduct of preradicals preserves order on both sides.

Proposition 7. *Let R be a ring. The following statements are equivalent:*

(1) *for every $\mu, \nu \in R$-pr, $(\mu_{[n]} : \nu) = \mu_{[n]}$ or $(\mu_{[n]} : \nu) = (\mu_{[n-1]} : \nu)$;*

(2) *for every $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr,*

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]}$$

or

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n-1]} : \sigma_{n+1});$$

(3) *every preredical $0 \neq \sigma \in R$-pr is quasi-co-n-absorbing.*

Proof. (1)⇒(2) If $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, then by part (1) we have that,

$$\begin{aligned}(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) &\preceq ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]} : \sigma_{n+1}) \\ &= (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]},\end{aligned}$$

or

$$\begin{aligned}(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) &\preceq ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]} : \sigma_{n+1}) \\ &= ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n-1]} : \sigma_{n+1}).\end{aligned}$$

(2)⇒(1) For preradicals $\mu, \nu \in R$-pr, we have from (2),

$$(\mu_{[n]} : \nu) \preceq (\overbrace{\mu \vee \cdots \vee \mu}^{n \text{ times}})_{[n]} = \mu_{[n]}$$

or

$$(\mu_{[n]} : \nu) \preceq ((\overbrace{\mu \vee \cdots \vee \mu}^{n \text{ times}})_{[n-1]} : \nu) = (\mu_{[n-1]} : \nu).$$

Thus we have that $(\mu_{[n]} : \nu) = \mu_{[n]}$ or $(\mu_{[n]} : \nu) = (\mu_{[n-1]} : \nu)$.

(1)⇔(3) Is evident. □

In the next proposition we use $(\mu_1 : \cdots : \widehat{\mu_i} : \cdots : \mu_{n+1})$ when the *i-th* term is excluded from $(\mu_1 : \cdots : \mu_{n+1})$.

Proposition 8. *Let $0 \neq \sigma \in R$-pr be an idempotent radical.*

(1) *If σ is such that for any $\mu, \nu \in R$-pr, we have*

$$\mu \vee \nu \preceq \sigma \preceq (\mu_{[n]} : \nu) \Rightarrow [\sigma \preceq \mu_{[n]} \text{ or } \sigma \preceq (\mu_{[n-1]} : \nu)],$$

then σ is quasi-co-n-absorbing.

(2) *If σ is such that for any $\mu_1, \mu_2, \ldots, \mu_{n+1} \in R$-pr, we have*

$$\mu_1 \vee \mu_2 \vee \cdots \vee \mu_{n+1} \preceq \sigma \preceq (\mu_1 : \mu_2 : \cdots : \mu_{n+1}) \Rightarrow$$

$$[\sigma \preceq (\mu_1 : \cdots : \widehat{\mu_i} : \cdots : \mu_{n+1}), \quad \text{for some } 1 \leqslant i \leqslant n+1],$$

then σ is a co-n-absorbing preradical.

Proof. (1) Let $\sigma \neq 0$ be an idempotent radical that satisfies the hypothesis in part (1). Let $\sigma \preceq (\tau_{[n]} : \lambda)$ for some $\tau, \lambda \in R$-pr. Then, by [15, Theorem 8(3)] we have

$$\tau\sigma \vee \lambda\sigma \preceq \sigma = \sigma^2 \preceq (\tau_{[n]} : \lambda)\sigma = (\tau_{[n]}\sigma : \lambda\sigma) = ((\tau\sigma)_{[n]} : \lambda\sigma).$$

So, by hypothesis we have $\sigma \preceq (\tau\sigma)_{[n]} = \tau_{[n]}\sigma \preceq \tau_{[n]}$ or $\sigma \preceq ((\tau\sigma)_{[n-1]} : \lambda\sigma) = (\tau_{[n-1]} : \lambda)\sigma \preceq (\tau_{[n-1]} : \lambda)$. Therefore σ is quasi-co-n-absorbing.

(2) The proof is similar to that of (1). □

Proposition 9. *Let $\mathcal{C}$ be a chain of quasi-co-n-absorbing preradicals, that is, a subclass of quasi-co-n-absorbing preradicals which is linearly ordered. Then $\bigvee_{\sigma \in \mathcal{C}} \sigma$ is a quasi-co-n-absorbing preradical.*

Proof. Let $\tau = \bigvee_{\sigma \in \mathcal{C}} \sigma$ and assume that $\tau \preceq (\mu_{[n]} : \nu)$ for some $\mu, \nu \in R$-pr. If $\sigma \preceq \mu_{[n]}$ for each $\sigma \in \mathcal{C}$, then $\tau \preceq \mu_{[n]}$. If there exists $\sigma_0 \in \mathcal{C}$ such that $\sigma_0 \not\preceq \mu_{[n]}$, then $\sigma \not\preceq \mu_{[n]}$ for each $\sigma_0 \preceq \sigma$. Since all preradicals in $\mathcal{C}$ are quasi-co-n-absorbing, it follows that $\sigma \preceq (\mu_{[n-1]} : \nu)$ for each $\sigma_0 \preceq \sigma$. Thus $\sigma \preceq (\mu_{[n-1]} : \nu)$ for each $\sigma \in \mathcal{C}$, so that $\tau \preceq (\mu_{[n-1]} : \nu)$. Consequently, we deduce that τ is a quasi-co-n-absorbing preradical. □

Proposition 10. *If σ_i is a quasi-co-n_i-absorbing preradical in R-pr for every $1 \leqslant i \leqslant k$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a quasi-co-n-absorbing preradical for $n = n_1 + \cdots + n_k$.*

Proof. For $k = 1$ there is nothing to prove. Then, suppose that $k > 1$. Assume that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq (\mu_{[n]} : \nu)$ for some $\mu, \nu \in R$-pr. Notice that for every $1 \leqslant i \leqslant k$, $\sigma_i \preceq (\mu_{[n]} : \nu) = (\mu_{[n_i]} : \mu_{[n-n_i]} : \nu)$. Then, for every $1 \leqslant i \leqslant k$, either $\sigma_i \preceq \mu_{[n_i]}$ or $\sigma_i \preceq (\mu_{[n_i-1]} : \mu_{[n-n_i]} : \nu) = (\mu_{[n-1]} : \nu)$, because σ_i is quasi-co-n_i-absorbing. On the other hand, for every $1 \leqslant i \leqslant k$, $\mu_{[n_i]} \preceq \mu_{[n-1]}$ and so $\mu_{[n_i]} \preceq (\mu_{[n-1]} : \nu)$. Hence $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq (\mu_{[n-1]} : \nu)$ which shows that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a quasi-co-n-absorbing preradical. □

Proposition 11. *Let $\sigma_1, \sigma_2, \ldots, \sigma_t \in R$-pr.*

(1) *If σ_1 is a quasi-co-n-absorbing preradical and σ_2 is a quasi-co-m-absorbing preradical for $m \leqslant n$, then $\sigma_1 \vee \sigma_2$ is a quasi-co-$(n+1)$-absorbing preradical.*

(2) *If $\sigma_1, \sigma_2, \ldots, \sigma_t$ are quasi-co-n-absorbing preradicals, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is a quasi-co-$(n+t-1)$-absorbing preradical.*

(3) *If σ_i is a quasi-co-n_i-absorbing preradical for every $1 \leqslant i \leqslant t$ with $n_1 < n_2 < \cdots < n_t$ and $t > 2$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is a quasi-co-$(n_t + 1)$-absorbing preradical.*

Proof. (1) Let $\mu, \nu \in R$-pr be such that $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n+1]} : \nu)$. Since σ_1 is quasi-co-n-absorbing and $\sigma_1 \preceq (\mu_{[n]} : \mu : \nu)$, then either $\sigma_1 \preceq \mu_{[n]}$ or $\sigma_1 \preceq (\mu_{[n-1]} : \mu : \nu) = (\mu_{[n]} : \nu)$. Also, σ_2 is quasi-co-m-absorbing and $\sigma_2 \preceq (\mu_{[m]} : \mu_{[n+1-m]} : \nu)$, so either $\sigma_2 \preceq \mu_{[m]}$ or $\sigma_2 \preceq (\mu_{[m-1]} : \mu_{[n+1-m]} : \nu) = (\mu_{[n]} : \nu)$. There are four cases.

Case 1. Assume that $\sigma_1 \preceq \mu_{[n]}$ and $\sigma_2 \preceq \mu_{[m]}$. Then $\sigma_1 \vee \sigma_2 \preceq \mu_{[n]}$.

Case 2. Assume that $\sigma_1 \preceq \mu_{[n]}$ and $\sigma_2 \preceq (\mu_{[n]} : \nu)$. Then $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n]} : \nu)$.

Case 3. Assume that $\sigma_1 \preceq (\mu_{[n]} : \nu)$ and $\sigma_2 \preceq \mu_{[m]}$. Then $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n]} : \nu)$.

Case 4. Assume that $\sigma_1 \preceq (\mu_{[n]} : \nu)$ and $\sigma_2 \preceq (\mu_{[n]} : \nu)$. Then $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n]} : \nu)$. Hence $\sigma_1 \vee \sigma_2$ is quasi-co-$(n+1)$-absorbing.

(2) We use induction on t. For $t = 1$ there is nothing to prove. Let $t > 1$ and assume that for $t-1$ the claim holds. Then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{t-1}$ is quasi-co-$(n+t-2)$-absorbing. Since σ_t is quasi-co-n-absorbing, then it is quasi-co-$(n+t-2)$-absorbing, by Remark 1(2). Therefore $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is quasi-co-$(n+t-1)$-absorbing, by part (1).

(3) Induction on t: For $t = 3$ apply parts (1) and (2). Let $t > 3$ and suppose that for $t-1$ the claim holds. Hence $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{t-1}$ is quasi-co-$(n_{t-1}+1)$-absorbing. We consider the following cases:

Case 1. Let $n_{t-1}+1 < n_t$. In this case $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is quasi-co-(n_t+1)-absorbing, by part (1).

Case 2. Let $n_{t-1}+1 = n_t$. Thus $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is quasi-co-(n_t+1)-absorbing, by part (2).

Case 3. Let $n_{t-1}+1 > n_t$. Then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is quasi-co-$(n_{t-1}+2)$-absorbing, by part (1). Since $n_{t-1}+2 \leqslant n_t+1$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is quasi-co-(n_t+1)-absorbing. □

Proposition 12. *Let $\sigma \in R$-pr be a radical. If σ is quasi-co-n-absorbing, then $e(\sigma)$ is quasi-co-n-absorbing.*

Proof. Assume that σ is quasi-co-n-absorbing, and let $e(\sigma) \preceq (\mu_{[n]} : \nu)$ for some $\mu, \nu \in R$-pr. Then $\sigma = e(\sigma)\sigma \preceq (\mu_{[n]} : \nu)\sigma \preceq ((\mu\sigma)_{[n]} : \nu\sigma)$. Since σ is quasi-co-n-absorbing and radical, [15, Theorem 8(3)] implies that either $\sigma \preceq (\mu\sigma)_{[n]} = \mu_{[n]}\sigma \preceq \mu_{[n]}$ or $\sigma \preceq ((\mu\sigma)_{[n-1]} : \nu\sigma) = (\mu_{[n-1]} : \nu)\sigma \preceq (\mu_{[n-1]} : \nu)$. Consequently $e(\sigma)$ is quasi-co-n-absorbing. □

Definition 1. For $\tau,\ \rho \in R$-pr define the totalizer of ρ relative to τ as $t_\tau(\rho) = \bigwedge\{\eta \in R\text{-pr} |\ (\rho : \eta) \succeq \tau\}$. Note that $t_1(\rho) = t(\rho)$.

Proposition 13. *Let $\tau \in R$-pr. If τ is quasi-co-2-absorbing, then for each $\lambda \in R$-pr, either $\tau \preceq \lambda_{[n]}$ or $t_\tau(\lambda_{[n]}) = t_\tau(\lambda_{[n-1]})$. In particular, if 1 is a quasi-co-2-absorbing preradical, then for each $\lambda \in R$-pr, either $\lambda_{[n]} = 1$ or $t(\lambda_{[n]}) = t(\lambda_{[n-1]})$.*

Proof. Suppose that τ is quasi-co-2-absorbing and let $\lambda \in R$-pr such that $\tau \npreceq \lambda_{[n]}$. If $\nu \in R$-pr is such that $\tau \preceq (\lambda_{[n]} : \nu)$, then $\tau \preceq (\lambda_{[n-1]} : \nu)$, since σ is quasi-co-2-absorbing. Therefore $t_\tau(\lambda_{[n-1]}) \preceq t_\tau(\lambda_{[n]})$. On the other hand $\lambda_{[n-1]} \preceq \lambda_{[n]}$ and so $t_\tau(\lambda_{[n]}) \preceq t_\tau(\lambda_{[n-1]})$. Consequently $t_\tau(\lambda_{[n]}) = t_\tau(\lambda_{[n-1]})$. □

4. Semi-co-n-absorbing preradicals

Suppose that m, n are positive integers with $n > m$. A more general concept than semi-co-n-absorbing preradicals is the concept of semi-co-(n, m)-absorbing preradicals. A preradical $\sigma \neq 0$ is called a *semi-co-(n, m)-absorbing preradical* if whenever $\sigma \preceq \mu_{[n]}$ for $\mu \in R$-pr, then $\sigma \preceq \mu_{[m]}$.

Note that a semicoprime preradical is just a semi-co-1-absorbing preradical.

Theorem 1. *Let $\sigma \in R$-pr and m, n be positive integers with $n > m$.*

(1) *If σ is quasi-co-m-absorbing, then it is semi-co-(k, m)-absorbing for every $k > m$.*

(2) *If σ is semi-co-(n, m)-absorbing, then it is semi-co-(i, m)-absorbing for every $m < i < n$, in particular it is semi-co-m-absorbing.*

(3) *σ is semi-co-(n, m)-absorbing if and only if σ is semi-co-(n, k)-absorbing for each $n > k \geqslant m$ if and only if σ is semi-co-(i, j)-absorbing for each $n \geqslant i > j \geqslant m$.*

(4) *If σ is semi-co-(n, m)-absorbing, then it is semi-co-(nk, mk)-absorbing for every positive integer k.*

(5) *If σ is semi-co-(n, m)-absorbing and semi-co-(r, s)-absorbing for some positive integers $r > s$, then it is semi-co-(nr, ms)-absorbing.*

Proof. (1) Is trivial.

(2) Is easy.

(3) Straightforward.

(4) Suppose that σ is semi-co-(n, m)-absorbing. Let $\mu \in R$-pr and let k be a positive integer such that $\sigma \preceq \mu_{[nk]}$. Then $\sigma \preceq \left(\mu_{[k]}\right)_{[n]}$. Since σ is semi-co-(n, m)-absorbing, $\sigma \preceq \left(\mu_{[k]}\right)_{[m]} = \mu_{[mk]}$, and so σ is semi-co-(nk, mk)-absorbing.

(5) Assume that σ is semi-co-(n, m)-absorbing and semi-co-(r, s)-absorbing for some positive integers $r > s$. Let $\sigma \preceq \mu_{[nr]}$. Since σ is semi-co-(n, m)-absorbing, then $\sigma \preceq \mu_{[mr]}$; and since σ is semi-co-(r, s)-absorbing, $\sigma \preceq \mu_{[ms]}$. Hence σ is semi-co-(nr, ms)-absorbing. □

Corollary 2. *Let $\sigma \in R$-pr and n be a positive integer.*

(1) *If σ is quasi-co-n-absorbing, then it is semi-co-n-absorbing.*

(2) *Let $t \leqslant n$ be an integer. If σ is semi-co-$(n + 1, t)$-absorbing, then it is semi-co-$(nk + i, tk)$-absorbing for all $k \geqslant i \geqslant 1$.*

(3) *If σ is semi-co-n-absorbing, then it is semi-co-$(nk + i, nk)$-absorbing for all $k \geqslant i \geqslant 1$.*

(4) *If σ is semi-co-n-absorbing, then it is semi-co-$(nk+j)$-absorbing for all $k > j \geqslant 0$.*
(5) *If σ is semi-co-n-absorbing, then it is semi-co-(nk)-absorbing for every positive integer k.*
(6) *If σ is semicoprime, then it is semi-co-k-absorbing for every positive integer k.*
(7) *If σ is semicoprime, then for every $k \geqslant 1$ and every $\mu \in R$-pr, $\sigma \preceq \mu_{[k]}$ implies that $\sigma \preceq \mu$.*
(8) *If σ is semi-co-n-absorbing, then it is semi-co-$((n+1)^t, n^t)$-absorb -ing for all $t \geqslant 1$.*
(9) *If σ is semicoprime, then it is quasi-co-k-absorbing for every $k > 1$.*

Proof. (1) By parts (1), (2) of Theorem 1.

(2) Let σ be semi-co-$(n+1, t)$-absorbing. Then by Theorem 1(4), σ is semi-co-$(nk+k, tk)$-absorbing, for every positive integer k. Hence by Theorem 1(2), σ is semi-co-$(nk+i, tk)$-absorbing for every $k \geqslant i \geqslant 1$.

(3) In part (2) get $t = n$.

(4) By part (3).

(5) Is a special case of (4).

(6) Is a direct consequence of (5).

(7) By part (6).

(8) By Theorem 1(5).

(9) Assume that σ is semicoprime. Let $\sigma \preceq (\mu_{[k]} : \nu)$ for some $\mu, \nu \in R$-pr and some $k > 1$. Then $\sigma \preceq (\mu_{[k]} : \nu) \preceq (\mu : \nu)_{[k]}$. Therefore $\sigma \preceq (\mu : \nu)$, by part (7). So σ is quasi-co-k-absorbing. □

In the following remark we prove Proposition 4 in another way.

Remark 3. Clearly, an arbitrary join of a family of semicoprime (coprime) preradicals is semicoprime, and so it is quasi-co-k-absorbing for every $k > 1$, by Corollary 2(9).

Proposition 14. *Let $\sigma_1, \sigma_2, \ldots, \sigma_n \in R$-pr. If for every $1 \leqslant i \leqslant n$, σ_i is a semicoprime preradical, then $(\sigma_1 : \sigma_2 : \cdots : \sigma_n)$ is a semi-co-n-absorbing preradical. In particular, if σ is a semicoprime preradical, then $\sigma_{[n]}$ is a semi-co-n-absorbing preradical.*

Proof. Apply Corollary 2(7). □

Lemma 1. *Let $\sigma \in R$-pr. If $\sigma_{[n+1]}$ is a semi-co-n-absorbing preradical, then $\sigma_{[n+1]} = \sigma_{[n]}$. In particular, if $\sigma_{[2]}$ is a semicoprime preradical, then σ is radical.*

Proposition 15. *Let $\sigma \in R$-pr, $\sigma \neq 0$ be an idempotent radical. If σ is such that for any $\mu \in R$-pr, we have $\mu \preceq \sigma \preceq \mu_{[n+1]} \Rightarrow \sigma \preceq \mu_{[n]}$, then σ is semi-co-n-absorbing.*

Proof. The proof is similar to that of Proposition 8(1). □

Proposition 16. *Let $\sigma_1, \sigma_2, \ldots, \sigma_n \in R$-pr be semi-co-2-absorbing preradicals. Then $(\sigma_1 : \sigma_2 : \cdots : \sigma_n)$ is a semi-co-$(3^n - 1)$-absorbing preradical.*

Proof. Suppose that $(\sigma_1 : \sigma_2 : \cdots : \sigma_n) \preceq \mu_{[3^n]}$ for some $\mu \in R$-pr. For every $1 \leqslant i \leqslant n$, $\sigma_i \preceq \mu_{[3^n]} = \left(\mu_{[3^{n-1}]}\right)_{[3]}$ and σ_i is semi-co-2-absorbing, then $\sigma_i \preceq \left(\mu_{[3^{n-1}]}\right)_{[2]} = \mu_{[2\cdot 3^{n-1}]} = \left(\mu_{[2\cdot 3^{n-2}]}\right)_{[3]}$. Again, since σ_i is semi-co-2-absorbing, we conclude that $\sigma_i \preceq \mu_{[2^2 . 3^{n-2}]}$. Repeating this method implies that $\sigma_i \preceq \mu_{[2^n]}$. So $(\sigma_1 : \sigma_2 : \cdots : \sigma_n) \preceq \mu_{[n2^n]}$. On the other hand $n2^n \leqslant 3^n - 1$. So $(\sigma_1 : \sigma_2 : \cdots : \sigma_n) \preceq \mu_{[3^n - 1]}$ which shows that $(\sigma_1 : \sigma_2 : \cdots : \sigma_n)$ is semi-co-$(3^n - 1)$-absorbing. □

Proposition 17. *If σ_i is a semi-co-n_i-absorbing preradical in R-pr for every $1 \leqslant i \leqslant k$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a semi-co-$(n-1)$-absorbing preradical for $n = \prod_{i=1}^{k} (n_i + 1)$.*

Proof. Let $\mu \in R$-pr be such that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq \mu_{[n]}$. Thus for every $1 \leqslant i \leqslant k$, $\sigma_i \preceq \left(\mu_{[m]}\right)_{[n_i+1]}$, where $m = \prod_{j=1,\ j\neq i}^{k} (n_j + 1)$. Since σ_i's are semi-co-n_i-absorbing, then, for each $1 \leqslant i \leqslant k$, $\sigma_i \preceq \mu_{[n_i m]}$. Note that for every $1 \leqslant i \leqslant k$,

$$n_i m \leqslant \prod_{i=1}^{k} (n_i + 1) - 1 = n - 1.$$

So we have $\sigma_i \preceq \mu_{[n-1]}$ for every $1 \leqslant i \leqslant k$. Hence $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq \mu_{[n-1]}$ which implies that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a semi-co-$(n-1)$-absorbing preradical. □

Proposition 18. *Let $\sigma_1, \sigma_2 \in R$-pr and m, n be positive integers.*

(1) *If σ_1 is quasi-co-m-absorbing and σ_2 is semi-co-n-absorbing, then $(\sigma_1 : \sigma_2)$ is semi-co-$(n(m+1)+m)$-absorbing.*
(2) *If σ_1 is quasi-co-$(2m)$-absorbing and σ_2 is semi-co-m-absorbing, then $(\sigma_1 : \sigma_2)$ is semi-co-$(m^2 + 2m)$-absorbing.*

Proof. (1) Suppose that $(\sigma_1 : \sigma_2) \preceq \mu_{[(n+1)(m+1)]}$ for some $\mu \in R$-pr. Since σ_1 is quasi-co-m-absorbing and $\sigma_1 \preceq \mu_{[(n+1)(m+1)]}$, then $\sigma_1 \preceq \mu_{[m]}$.

On the other hand σ_2 is semi-co-n-absorbing and $\sigma_2 \preceq \mu_{[(n+1)(m+1)]}$, then $\sigma_2 \preceq \mu_{[n(m+1)]}$. Consequently $(\sigma_1 : \sigma_2) \preceq \mu_{[n(m+1)+m]}$, and so $(\sigma_1 : \sigma_2)$ is semi-co-$(n(m+1)+m)$-absorbing.

(2) Suppose that $(\sigma_1 : \sigma_2) \preceq \mu_{[(m+1)^2]}$ for some $\mu \in R$-pr. Since σ_1 is quasi-co-$(2m)$-absorbing and $\sigma_1 \preceq \mu_{[(m+1)^2]}$, then $\sigma_1 \preceq \mu_{[2m]}$. Since σ_2 is semi-co-m-absorbing and $\sigma_2 \preceq \mu_{[(m+1)^2]}$, then $\sigma_2 \preceq \mu_{[m^2]}$. Hence $(\sigma_1 : \sigma_2) \preceq \mu_{[m^2+2m]}$ which shows that $(\sigma_1 : \sigma_2)$ is semi-co-(m^2+2m)-absorbing. □

Proposition 19. *Let R be a ring. The following statements are equivalent:*

(1) *for every preradical $\sigma \in R$-pr, $\sigma_{[n+1]} = \sigma_{[n]}$;*

(2) *for all preradicals $\sigma_1\sigma_2, \ldots, \sigma_{n+1} \in R$-pr we have*

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n+1})_{[n]};$$

(3) *every preredical $0 \neq \sigma \in R$-pr is semi-co-n-absorbing.*

Proof. (1)⇒(2) If $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, then we get from (1),

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n+1})_{[n+1]} = (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n+1})_{[n]}.$$

(2)⇒(1) For a preradical $\sigma \in R$-pr, we have from (2),

$$\sigma_{[n+1]} \preceq (\overbrace{\sigma \vee \cdots \vee \sigma}^{n+1 \text{ times}})_{[n]} = \sigma_{[n]}.$$

So we have that $\sigma_{[n+1]} = \sigma_{[n]}$.

(1)⇔(3) Is clear. □

Remark 4. Let $\{\sigma_\alpha\}_{\alpha \in I} \subseteq R$-pr. If σ_α is semi-co-n-absorbing for every $\alpha \in I$, then $\bigvee_{\alpha \in I} \sigma_\alpha$ is semi-co-n-absorbing.

Proposition 20. *Let $\sigma \in R$-pr be radical. If σ is semi-co-n-absorbing, then $e(\sigma)$ is semi-co-n-absorbing.*

Proof. Is similar to the proof of Proposition 12. □

In Proposition 23 of [11], it was shown that $\sigma^0 := \bigvee\{\sigma \in R\text{-pr} \mid \sigma \text{ is semicoprime}\}$ is the unique greatest semicoprime preradical.

Proposition 21. *There exists in R-pr a unique greatest semi-co-n-absorbing preradical.*

Proof. Set $\sigma^0_{(n)} = \bigvee\{\sigma \in R\text{-pr} \mid \sigma \text{ is semi-co-}n\text{-absorbing}\}$. By Remark 4, $\sigma^0_{(n)}$ is the greatest semi-co-n-absorbing preradical. □

By notation in the the proof of the previous proposition we have that $\sigma^0_{(1)} = \sigma^0$.

Remark 5. As $\zeta \preceq \kappa \preceq \sigma^0$ are semicoprime preradicals, then $\zeta_{[n]}, \kappa_{[n]}$, $\sigma^0_{[n]}$ are semi-co-n-absorbing preradicals, by Proposition 14. Therefore $\zeta_{[n]} \preceq \kappa_{[n]} \preceq \sigma^0_{[n]} \preceq \sigma^0_{(n)}$.

Proposition 22. *The following statements hold:*

(1) $\sigma^0 = \bigwedge_{n\geqslant 1} \sigma^0_{(n)}$.

(2) $\sigma^0_{(n)} \preceq \sigma^0_{[nk]}$ *for every positive integer* k.

(3) $\sigma_{[n]} \preceq \sigma^0_{(n)}$ *for every semicoprime preradical* σ.

Proof. (1) By Corollary 2(6) every semicoprime preradical is semi-co-n-absorbing for every $n \geqslant 1$. Then $\sigma^0 \preceq \sigma^0_{(n)}$ for every $n \geqslant 1$.

(2) By Corollary 2(5).

(3) By Proposition 14. □

In Proposition 26 of [11] it was shown that $\sigma^0 \preceq \nu_0$, where $\nu_0 = \bigwedge\{\tau \mid \tau \in R\text{-pr},\ \tau \text{ is unipotent}\}$.

The following proposition is straightforward.

Proposition 23. *Suppose that* $\nu_0^{(n)} := \bigwedge\{\tau_{[n]} \mid \tau \in R\text{-}pr,\ \tau_{[n+1]} = 1\}$. *Then:*

(1) $\sigma^0_{(n)} \preceq \nu_0^{(n)}$;

(2) $\nu_0 \preceq \nu_0^{(1)}$.

Corollary 3. *The following statements hold:*

(1) *If* $\zeta_{[n+1]} = 1$, *then* $\zeta_{[n]} = \kappa_{[n]} = \sigma^0_{[n]} = \sigma^0_{(n)} = \nu_0^{(n)}$;

(2) *If* $\zeta_{[2]} = 1$, *then* $\zeta = \kappa = \sigma^0 = \nu_0 = \nu_0^{(1)}$.

Proof. (1) By Remark 5 and Proposition 23 we have that $\zeta_{[n]} \preceq \kappa_{[n]} \preceq \sigma^0_{[n]} \preceq \sigma^0_{(n)} \preceq \nu_0^{(n)}$. If $\zeta_{[n+1]} = 1$, then $\nu_0^{(n)} \preceq \zeta_{[n]}$, and so $\zeta_{[n]} = \kappa_{[n]} = \sigma^0_{[n]} = \sigma^0_{(n)} = \nu_0^{(n)}$.

(2) By part (1) and [11, Corollary 27]. □

Proposition 24. *For a ring* R *the following statements are equivalent:*

(1) *For every* $\mu \in R\text{-}pr$, $\mu_{[n+1]} = 1$ *implies that* $\mu_{[n]} = 1$;

(2) 1 *is a semi-co-n-absorbing preradical;*
(3) $\sigma^0_{(n)} = 1$*;*
(4) $\nu^{(n)}_0 = 1$.

Proof. Is easy. □

For $\tau \in R$-pr define

$$C^{(n)}(\tau) = \bigvee\{\sigma \in R\text{-pr} \mid \sigma \preceq \tau,\ \sigma \text{ semi-co-}n\text{-absorbing}\},$$

which is the unique greatest semi-co-n-absorbing preradical less than or equal to τ. Notice that in [11], $C^{(1)}$ is denoted by C.

Proposition 25. *Let R be a ring.*
(1) $\sigma^0_{(n)} = C^{(n)}(1) = \bigvee_{\tau\in R\text{-}pr} C^{(n)}(\tau)$.
(2) *For each* $\tau \in R$*-pr,* $C^{(n)}(\tau) \preceq \tau$.
(3) *For each* $\tau, \sigma \in R$*-pr we have* $\tau \preceq \sigma \Rightarrow C^{(n)}(\tau) \preceq C^{(n)}(\sigma)$.
(4) *For each* $\tau \in R$*-pr,* $C^{(n)}(\tau_{[n+1]}) = C^{(n)}(\tau_{[n]})$.
(5) *For each* $\tau \in R$*-pr,* τ *is semi-co-n-absorbing if and only if* $\tau = C^{(n)}(\tau)$.
(6) $\{\tau \in R\text{-}pr \mid \tau \text{ is semi-co-}n\text{-absorbing}\} = Im\, C^{(n)} = \{C^{(n)}(\sigma) \mid \sigma \in R\text{-}pr\}$.
(7) $\left[C^{(n)}\right]^2 = C^{(n)}$*. Thus,* $C^{(n)}$ *is a closure operator on R-pr.*
(8) *For each family* $\{\tau_\alpha\}_{\alpha\in I} \subseteq R$*-pr, we have*

$$C^{(n)}(\bigwedge_{\alpha\in I} \tau_\alpha) = C^{(n)}(\bigwedge_{\alpha\in I} C^{(n)}(\tau_\alpha)).$$

(9) $C^{(n)} = \bigwedge_{k\geqslant 1} C^{(nk)}$*, in particular* $C = \bigwedge_{k\geqslant 1} C^{(k)}$.
(10) $C^{(n)}(\sigma_{[n+1]}) = C^{(n)}(\sigma_{[n]}) = \sigma_{[n]}$ *for any semicoprime preradical* σ.

Proof. The proofs of (1), (2), (3), (5) and (6) is easy.

(4) For any $\tau \in R$-pr, part (3) implies that $C^{(n)}(\tau_{[n]}) \preceq C^{(n)}(\tau_{[n+1]})$. Since $C^{(n)}(\tau_{[n+1]})$ is semi-co-n-absorbing (by Remark 4) and $C^{(n)}(\tau_{[n+1]}) \preceq \tau_{[n+1]}$, then $C^{(n)}(\tau_{[n+1]}) \preceq \tau_{[n]}$. Hence $C^{(n)}(\tau_{[n+1]}) \preceq C^{(n)}(\tau_{[n]})$. So the equality holds.

(7) Is a direct consequence of part (5).

(8) The proof is similar to that of [11, Proposition 31](5).

(9) By Corollary 2(5).

(10) Apply Proposition 14 and parts (4), (5). □

Now consider the operator $\overline{(\_)}$ in R-pr that assigns to each preradical σ the least radical over σ (see [19, p. 137]).

Lemma 2. *Let $\sigma, \tau \in R$-pr be such that σ is radical and τ is semi-co-n-absorbing. Then:*

(1) $C^{(n)}(\sigma) \preceq \overline{C^{(n)}(\sigma)} \preceq \sigma$.
(2) $C^{(n)}(\sigma) = C^{(n)}(\overline{C^{(n)}(\sigma)})$.
(3) $\tau \preceq C^{(n)}(\overline{\tau}) \preceq \overline{\tau}$.
(4) $\overline{\tau} = \overline{C^{(n)}(\overline{\tau})}$.

Proof. Similar to the proof of [11, Lemma 32]. □

Proposition 26. *Let R be a ring.*

(1) *The operator $\overline{C^{(n)}(\_)}$ defines an interior operator on the ordered class of radicals.*
(2) *The operator $C^{(n)}(\overline{(\_)})$ defines a closure operator on the ordered class of semi-co-n-absorbing preradicals.*

Notice that the "open" radicals associated with the interior operator $\overline{C^{(n)}(\_)}$ are

$$\mathcal{O}^{(n)}_{rad} = \{\sigma \text{ radical} \mid \sigma = \overline{\tau} \text{ for some semi-co-}n\text{-absorbing } \tau\}.$$

The "closed" semi-co-n-absorbing preradicals associated with the closure operator $C^{(n)}(\overline{(\_)})$ are

$$\mathcal{C}^{(n)}_{sca} = \{\tau \text{ semi-co-}n\text{-absorbing} \mid \tau = C^{(n)}(\sigma) \text{ for some radical } \sigma\}.$$

The following result is immediate.

Corollary 4. *For a ring R the operators $C^{(n)}(\_)$ and $\overline{(\_)}$ restrict to mutually inverse maps between $\mathcal{O}^{(n)}_{rad}$ and $\mathcal{C}^{(n)}_{sca}$.*

Definition 2. Let $\tau \in R$-pr. Define

$$C^{(n)}_1(\tau) = \bigwedge\{\sigma_{[n]} \mid \sigma \in R\text{-pr}, \tau \preceq \sigma_{[n+1]}\}.$$

Proposition 27. *For a ring R the following conditions hold:*

(1) *For each $\tau \in R$-pr, $C^{(n)}_1(\tau) \preceq \tau_{[n]}$.*
(2) *For each $\tau \in R$-pr, τ is semi-co-n-absorbing if and only if $\tau \preceq C^{(n)}_1(\tau)$.*
(3) *1 is a semi-co-n-absorbing preradical if and only if $C^{(n)}_1(1) = 1$.*

(4) *Let* $\tau,\ \sigma \in R$*-pr. If* $\tau \preceq \sigma$*, then* $C_1^{(n)}(\tau) \preceq C_1^{(n)}(\sigma)$.

(5) *For each family* $\{\tau_\alpha\}_{\alpha\in I} \subseteq R$*-pr,* $C_1^{(n)}(\bigwedge_{\alpha\in I}\tau_\alpha) \preceq \bigwedge_{\alpha\in I} C_1^{(n)}(\tau_\alpha)$ *and* $\bigvee_{\alpha\in I} C_1^{(n)}(\tau_\alpha) \preceq C_1^{(n)}(\bigvee_{\alpha\in I}\tau_\alpha)$.

Proof. The assertions have straightforward verifications. □

We apply an "Amitsur construction" to $C_1^{(n)}$ as follows:

Definition 3. Let $\tau \in R$-pr. We define recursively the preradical $C_\lambda^{(n)}(\tau)$ for each ordinal λ as follows:

(1) $C_0^{(n)}(\tau) = \tau$.

(2) $C_{\lambda+1}^{(n)}(\tau) = C_1^{(n)}(C_\lambda^{(n)}(\tau))$.

(3) If λ is a limit ordinal, then $C_\lambda^{(n)}(\tau) = \bigwedge_{\beta<\lambda} C_\beta^{(n)}(\tau)$.

(4) $C_\Omega^{(n)}(\tau) = \bigwedge_{\lambda\ ordinal} C_\lambda^{(n)}(\tau)$.

Proposition 28. *Let* $\tau \in R$*-pr. Then the following statements are equivalent:*

(1) τ *is semi-co-*n*-absorbing;*

(2) *For each ordinal* λ*,* $\tau \preceq C_\lambda^{(n)}(\tau)$*;*

(3) $C_\Omega^{(n)}(\tau) = \tau$.

Proof. By Proposition 27 and using transfinite induction we have the claim. □

As is the case with $C_1^{(n)}$, all of the operators $C_\lambda^{(n)}$ preserve order between preradicals.

Proposition 29. *Let* $\tau,\ \sigma \in R$*-pr be such that* $\tau \preceq \sigma$*. Then:*

(1) *For each ordinal* λ*,* $C_\lambda^{(n)}(\tau) \preceq C_\lambda^{(n)}(\sigma)$.

(2) $C_\Omega^{(n)}(\tau) \preceq C_\Omega^{(n)}(\sigma)$.

Proposition 30. *For each* $\tau \in R$*-pr,* $C^{(n)}(\tau) \preceq C_\Omega^{(n)}(\tau)$.

Proof. Let $\tau \in R$-pr. We use transfinite induction. First, note that $C^{(n)}(\tau) \preceq \tau = C_0^{(n)}(\tau)$. Assume that λ is an ordinal such that $C^{(n)}(\tau) \preceq C_\lambda^{(n)}(\tau)$. Since $C^{(n)}(\tau)$ is semi-co-n-absorbing, $C^{(n)}(\tau) \preceq C_1^{(n)}(C^{(n)}(\tau)) \preceq C_1^{(n)}(C_\lambda^{(n)}(\tau)) = C_{\lambda+1}^{(n)}(\tau)$, by parts (2) and (4) of Proposition 27. If λ is a limit ordinal and $C^{(n)}(\tau) \preceq C_\beta^{(n)}(\tau)$ for each $\beta < \lambda$, then $C^{(n)}(\tau) \preceq \bigwedge_{\beta<\lambda} C_\beta^{(n)}(\tau) = C_\lambda^{(n)}(\tau)$. □

In the following result we give equivalent conditions for the equality $C_\Omega^{(n)}(\tau) = C^{(n)}(\tau)$.

Proposition 31. *For each $\tau \in R$-pr the following statements are equivalent:*

(1) $C_\Omega^{(n)}(\tau)$ *is semi-co-n-absorbing;*

(2) $C_\Omega^{(n)}(\tau) \preceq C_1^{(n)}(C_\Omega^{(n)}(\tau))$*;*

(3) *For each ordinal λ we have $C_\Omega^{(n)}(\tau) \preceq C_\lambda^{(n)}(C_\Omega^{(n)}(\tau))$;*

(4) $C_\Omega^{(n)}(C_\Omega^{(n)}(\tau)) = C_\Omega^{(n)}(\tau)$*;*

(5) $C_\Omega^{(n)}(\tau) = C^{(n)}(\tau)$.

Proof. (1)⇒(2) By Proposition 27(2).

(2)⇒(3) It follows by using transfinite induction on λ.

(3)⇒(4) Is easy.

(4)⇒(1) By Proposition 28.

(1)⇒(5) Assume that $C_\Omega^{(n)}(\tau)$ is semi-co-n-absorbing. Since $C_\Omega^{(n)}(\tau) \preceq \tau$, the definition of $C^{(n)}(\tau)$ implies that $C_\Omega^{(n)}(\tau) \preceq C^{(n)}(\tau)$. On the other hand $C^{(n)}(\tau) \preceq C_\Omega^{(n)}(\tau)$, by Proposition 30. So the equality holds.

(5)⇒(1) Is straightforward. □

5. Quasi-co-n-absorbing and semi-co-n-absorbing submodules

Remark 6. Let $M \in R$-co-ass and N be a nonzero fully invariant submodule of M. Then we have:

(1) N is co-n-absorbing in $M \Rightarrow N$ is quasi-co-n-absorbing in $M \Rightarrow N$ is semi-co-n-absorbing in M.

(2) N is a quasi-co-1-absorbing submodule of M if and only if N is a coprime submodule of M.

(3) N is a semi-co-1-absorbing submodule of M if and only if N is a semicoprime submodule of M.

Proposition 32. *Let $\sigma \in R$-pr. If for every $M \in R$-Mod, $\sigma(M)$ is a semicoprime submodule of M, then σ is a semicoprime preradical.*

Proof. By hypothesis, [11, Proposition 19] implies that $\alpha^M_{\sigma(M)}$ is a semicoprime preradical. So $\sigma = \bigvee\{\alpha^M_{\sigma(M)} \mid M \in R\text{-Mod}\}$ (see [17, Remark 1]) is a semicoprime preradical. □

Corollary 5. *Let R be a ring. If every nonzero R-module is semicoprime, then* 1 *is a semicoprime preradical in R-pr.*

Lemma 3 ([7, Lemma 2.5])**.** *Let $M \in R$-Mod. Then for any submodules $N,\ K$ of M, $\alpha^M_{N+K} = \alpha^M_N \vee \alpha^M_K$.*

Proposition 33. *Let $M \in R$-Mod. Suppose that $\{N_i\}_{i\in I}$ is a family of semicoprime submodules of M. Then $N = \sum\limits_{i\in I} N_i$ is a semicoprime submodule of M.*

Proof. Let $\{N_i\}_{i\in I}$ be a family of semicoprime submodules of M. Then, by [11, Proposition 19], $\alpha^M_{N_j}$'s are semicoprime preradicals. Thus $\alpha^M_N = \bigvee_{i\in I} \alpha^M_{N_i}$ is a semicoprime preradical. Again by [11, Proposition 19], $N = \sum\limits_{i\in I} N_i$ is a semicoprime submodule of M. □

Proposition 34. *Let R be a ring and $\{M_i\}_{i\in I}$ be a family of semicoprime R-modules. Then $M = \bigoplus\limits_{i\in I} M_i$ is a semicoprime R-module.*

Proof. Since for every $i \in I$, M_i is a semicoprime R-module, then for every $i \in I$, $\alpha^{M_i}_{M_i}$ is a semicoprime preradical, by [11, Proposition 19]. Therefore $\bigvee\limits_{i\in I} \alpha^{M_i}_{M_i} = \alpha^M_M$ is a semicoprime preradical, and so again by [11, Proposition 19], $M = \bigoplus\limits_{i\in I} M_i$ is a semicoprime R-module. □

Proposition 35. *For a ring R the following statements are equivalent:*

(1) *R is a finite product of simple rings;*
(2) *$\kappa = 1$;*
(3) *1 is a semicoprime preradical;*
(4) *${}_RR$ is a semicoprime R-module;*
(5) *There exists a semicoprime R-module that is a generator in R-Mod.*

Proof. (1)⇔(2) By [11, Theorem 10].

(1)⇔(3) By [11, Theorem 29].

(3)⇔(4) Notice the fact that an R-module G is a generator in R-Mod if and only if $\alpha^G_G = 1$. Since R is a generator in R-Mod, then $\alpha^R_R = 1$. Now, use [11, Proposition 19].

(4)⇒(5) Is trivial.

(5)⇒(3) See the proof of (3)⇔(4). □

Theorem 2. *Let $M \in R$-co-ass and N a fully invariant submodule of M. Consider the following statements.*

(a) N is co-n-absorbing in M.

(b) α^M_N is a co-n-absorbing preradical.

Then (b) ⇒ (a), and if M satisfies the ω-property, then (a) ⇒ (b).

Proof. The proof is similar to that of [22, Theorem 4.2]. □

Theorem 3. *Let $M \in R$-co-ass and N a fully invariant submodule of M. Consider the following statements:*

(1) *N is quasi-co-n-absorbing (resp. semi-co-n-absorbing) in M.*

(2) *α_N^M is a quasi-co-n-absorbing (resp. semi-co-n-absorbing) preradical.*

Then (2) $\Rightarrow$ (1), and if M satisfies the ω-property, then (1) $\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2) Assume that N is quasi-co-n-absorbing in M and that $(\eta(M) : \mu(M)) = (\eta : \mu)(M)$ for every $\eta,\ \mu \in R$-pr. Since $N \neq 0$ we have $\alpha_N^M \neq 0$. Now let $\eta, \mu \in R$-pr be such that $\alpha_N^M \preceq (\eta_{[n]} : \mu)$. In this case we have

$$N = \alpha_N^M(M) \leqslant (\eta_{[n]} : \mu)(M) = (\eta(M)_{[n]} : \mu(M)).$$

Since N is quasi-co-n-absorbing in M, by hypothesis we have that $N \leqslant \eta(M)_{[n]} = \eta_{[n]}(M)$ or $N \leqslant (\eta(M)_{[n-1]} : \mu(M)) = (\eta_{[n-1]} : \mu)(M)$. It follows from [15, Proposition 5] that $\alpha_N^M \preceq \alpha^M_{\eta_{[n]}(M)} \preceq \eta_{[n]}$ or $\alpha_N^M \preceq \alpha^M_{(\eta_{[n-1]}:\mu)(M)} \preceq (\eta_{[n-1]} : \mu)$, and so α_N^M is quasi-co-n-absorbing.

(2) $\Rightarrow$ (1) Assume that α_N^M is a quasi-co-n-absorbing preradical. Since $\alpha_N^M \neq 0$, we have $N \neq 0$. Suppose that $J,\ K$ are fully invariant submodules of M such that $N \leqslant (J_{[n]} : K)$. Then we have $N \leqslant \left((\omega_J^M)_{[n]} : \omega_K^M\right)(M)$. By [15, Proposition 5], we get

$$\alpha_N^M \preceq \alpha^M_{\left((\omega_J^M)_{[n]}:\omega_K^M\right)(M)} \preceq \left((\omega_J^M)_{[n]} : \omega_K^M\right).$$

Since α_N^M is quasi-co-n-absorbing, we have $\alpha_N^M \preceq (\omega_J^M)_{[n]}$ or $\alpha_N^M \preceq \left((\omega_J^M)_{[n]} : \omega_K^M\right)$. Therefore $N = \alpha_N^M(M) \preceq (\omega_J^M)_{[n]}(M) = J_{[n]}$ or $N = \alpha_N^M(M) \preceq \left((\omega_J^M)_{[n]} : \omega_K^M\right)(M) = (J_{[n-1]} : K)$. Hence N is a quasi-co-n-absorbing submodule. A similar proof can be stated for semi-co-n-absorbing preradicals. □

Remark 7. Note that in Theorem 3, for the case $n = 2$ we can omit the condition $M \in R$-co-ass, by the definition of quasi-co-2-absorbing (semi-co-2-absorbing) submodules.

Definition 4. Let $M \in R$-co-ass. We say that M is a quasi-co-n-absorbing (resp. semi-co-n-absorbing) module if M is a quasi-co-n-absorbing (resp. semi-co-n-absorbing) submodule of itself.

Corollary 6. *Let $M_1, M_2, \ldots, M_t$ be injective Artinian R-modules. Suppose that M_i's are quasi-co-n-absorbing modules that satisfy the ω-property. Then $M = \bigoplus_{i=1}^{t} M_i$ is a quasi-co-$(n+t-1)$-absorbing R-module.*

Proof. Let $M_1, M_2, \ldots, M_t$ be quasi-co-n-absorbing R-modules. Then, by Theorem 3, $\alpha_{M_1}^{M_1}, \alpha_{M_2}^{M_2}, \ldots, \alpha_{M_t}^{M_t}$ are quasi-co-n-absorbing preradicals, and so $\alpha_M^M = \alpha_{M_1}^{M_1} \vee \alpha_{M_2}^{M_2} \vee \cdots \vee \alpha_{M_t}^{M_t}$ is a quasi-co-$(n+t-1)$-absorbing preradical, by Proposition 11(2). Again by Theorem 3, $M = \bigoplus_{i=1}^{t} M_i$ is a quasi-co-$(n+t-1)$-absorbing R-module. □

Corollary 7. *Let R be a ring. The following statements hold:*

(1) *If the preradical 1 is quasi-co-2-absorbing (resp. semi-co-2-absorbing), then every generator R-module is a quasi-co-2-absorbing (resp. semi-co-2-absorbing) R-module.*

(2) *If R is a semisimple Artinian ring, then every R-module is quasi-co-i-absorbing for every $i \geqslant 2$.*

Proof. (1) Suppose that 1 is a quasi-co-2-absorbing (resp. semi-co-2-absorbing) preradical and G is a generator R-module. Since $\alpha_G^G = 1$, the result follows from Theorem 3.

(2) By Proposition 6 and Theorem 3. □

Example 1. Let R be a semisimple Artinian ring and $S_1, S_2, \ldots, S_{n+1} \in R$-simp be distinct. Then the injective Artinian R-module $\bigoplus_{i=1}^{n+1} S_i$ is quasi-co-n-absorbing, by Corollary 7(2). But note that, by [22, Proposition 3.6] and Theorem 2, $\bigoplus_{i=1}^{n+1} S_i$ is not co-n-absorbing. This example shows that the two concepts of quasi-co-n-absorbing modules and of co-n-absorbing modules are different in general.

The following two propositions can be proved similar to [22, Proposition 4.10] and [22, Theorem 4.11], respectively.

Proposition 36. *Let N, $H \in R$-co-ass such that H be a fully invariant submodule of N and N be self-injective. For a fully invariant submodule K of H,*

(1) *If K is quasi-co-n-absorbing in N, then K is quasi-co-n-absorbing in H.*

(2) *If K is quasi-co-n-absorbing in N and $K \in R$-co-ass, then K is a quasi-co-n-absorbing module.*

(3) *If α_K^N is a quasi-co-n-absorbing preradical and N satisfies the ω-property, then α_K^H is a quasi-co-n-absorbing preradical.*

Proposition 37. *Let N, $Q \in R$-co-ass such that N be a fully invariant submodule of Q and Q be self-injective. Then N is a quasi-co-n-absorbing module if and only if N is quasi-co-n-absorbing in Q.*

Theorem 4. *Let $M \in R$-co-ass that satisfies the ω-property. The following statements are equivalent:*

(1) *M is quasi-co-n-absorbing;*
(2) *α_M^M is quasi-co-n-absorbing;*
(3) *For each $\tau, \eta \in R$-pr, $M \in \mathbb{T}_{(\tau_{[n]}:\eta)} \Rightarrow M \in \mathbb{T}_{\tau_{[n]}}$ or $M \in \mathbb{T}_{(\tau_{[n-1]}:\eta)}$.*

Proof. (1) $\Leftrightarrow$ (2) Is clear by Theorem 3.

(2) $\Rightarrow$ (3) Suppose that α_M^M is quasi-co-n-absorbing. Let $\tau, \eta \in R$-pr such that $M \in \mathbb{T}_{(\tau_{[n]}:\eta)}$. Then $(\tau_{[n]} : \eta)(M) = M$, and so $\alpha_M^M \preceq (\tau_{[n]} : \eta)$. Therefore $\alpha_M^M \preceq \tau_{[n]}$ or $\alpha_M^M \preceq (\tau_{[n-1]} : \eta)$. Hence $\tau_{[n]}(M) = M$ or $(\tau_{[n-1]} : \eta)(M) = M$. Consequently $M \in \mathbb{T}_{\tau_{[n]}}$ or $M \in \mathbb{T}_{(\tau_{[n-1]}:\eta)}$.

(3) $\Rightarrow$ (2) has a routine verification. □

Similarly we can prove the following theorem.

Theorem 5. *Let $M \in R$-co-ass that satisfies the ω-property. The following statements are equivalent:*

(1) *M is semi-co-n-absorbing;*
(2) *α_M^M is semi-co-n-absorbing;*
(3) *For each $\tau \in R$-pr, $M \in \mathbb{T}_{\tau_{[n+1]}} \Rightarrow M \in \mathbb{T}_{\tau_{[n]}}$.*

Theorem 6. *Let $M \in R$-Mod be such that, for each pair K, L of fully invariant submodules of M, we have $\left(\omega_K^M : \omega_L^M\right) = \omega_{(K:L)}^M$. Then, for each quasi-co-n-absorbing (resp. semi-co-n-absorbing) preradical σ such that $\sigma(M) \neq 0$, we have that $\sigma(M)$ is quasi-co-n-absorbing (resp. semi-co-n-absorbing) in M.*

Proof. By hypothesis $M \in R$-co-ass, [22, Lemma 4.12]. Let σ be a quasi-co-n-absorbing preradical such that $\sigma(M) \neq 0$. If K, L are fully invariant submodules of M such that $\sigma(M) \leqslant (K_{[n]} : L)$, then

$$\sigma \preceq \omega_{\sigma(M)}^M \preceq \omega_{(K_{[n]}:L)}^M = \left((\omega_K^M)_{[n]} : \omega_L^M\right).$$

Since σ is quasi-co-n-absorbing, then

$$\sigma \preceq (\omega_K^M)_{[n]} \text{ or } \sigma \preceq \left((\omega_K^M)_{[n-1]} : \omega_L^M\right).$$

In the first case we have $\sigma(M) \leqslant (\omega_K^M)_{[n]}(M) = K_{[n]}$; in the second case we have $\sigma(M) \leqslant \left((\omega_K^M)_{[n-1]} : \omega_L^M\right)(M) = (K_{[n-1]} : L)$. Consequently $\sigma(M)$ is quasi-co-n-absorbing. □

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