Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **21** (2016). Number 2, pp. 202–213

Generalization of primal superideals

Ameer Jaber

Communicated by Z. Marciniak

Abstract. Let R be a commutative super-ring with unity $1 \neq 0$. A proper superideal of R is a superideal I of R such that $I \neq R$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function, where $\mathfrak{I}(R)$ denotes the set of all proper superideals of R. A homogeneous element $a \in R$ is *ϕ-prime* to I if $ra \in I - \phi(I)$ where r is a homogeneous element in R, then $r \in I$. We denote by $\nu_\phi(I)$ the set of all homogeneous elements in R that are not ϕ-prime to I. We define I to be *ϕ-primal* if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

forms a superideal of R. For example if we take $\phi_\varnothing(I) = \varnothing$ (resp. $\phi_0(I) = 0$), a *ϕ-primal* superideal is a primal superideal (resp., a weakly primal superideal). In this paper we study several generalizations of primal superideals of R and their properties.

1. Introduction

A supercase on a ring is a $\mathbb{Z}_2$-grading on that ring. In general the grading on a ring, or a module, usually leads computation by allowing one to focus on the homogeneous elements, which are simpler and easier than random elements. However, to do this work you need to know that the constructions being studied are graded. One approach to this issue is to

2010 MSC: 13A02, 16D25, 16W50.

Key words and phrases: primal superideal, ϕ-P-primal superideal, ϕ-prime superideal.

redefine the constructions entirely in terms of graded modules and avoid any consideration of non-graded modules or non-homogeneous elements. Unfortunately, while such an approach helps to understand the graded modules, it will only help to understand the original construction, where the graded version of the concept coincide with original one. Therefore, notably, the studying of the graded rings (or modules) is very important.

Because of the importance of the grading, the author made many researches in different subjects in mathematics in super-rings and graded rings few years ago. For example in [1,2,4], the author studied existence of superinvolutions and pseudo superinvolutions of kinds one and two, also in [3, 5] he studied Division $\mathbb{Z}_3$-Algebra, and primitive $\mathbb{Z}_3$-algebra with $\mathbb{Z}_3$-involution. Moreover, in [7] he studied Δ-supergraded submodules and in [6] he studied product of graded submodules. Finally, in [8] the author studied weakly primal graded superideals.

A few years ago Y. A. Bahturin and A. Giambruno in [12] studied Group Gradings on associative algebras with involution.

Let R be any ring with unity, then R is called a super-ring if R is a $\mathbb{Z}_2$-graded ring such that if $a, b \in \mathbb{Z}_2$ then $R_a R_b \subseteq R_{a+b}$ where the subscripts are taken modulo 2. Let $h(R) = R_0 \cup R_1$. Then $h(R)$ is the set of homogeneous elements in R and $1 \in R_0$.

Throughout, R will be a commutative super-ring with unity. By a proper superideal of R we mean a superideal I of R such that $I \neq R$. We will denote the set of all proper superideals of R by $\mathfrak{I}(R)$. If I and J are in $\mathfrak{I}(R)$, then the superideal $\{r \in R \ : \ rJ \subseteq I\}$ is denoted by $(I : J)$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function and let $I \in \mathfrak{I}(R)$, we say that I is a *ϕ-prime* if whenever $x, y \in h(R)$ with $xy \in I - \phi(I)$, then $x \in I$ or $y \in I$. Since $I - \phi(I) = I - (\phi(I) \cap I)$, there is no loss of generality to assume that $\phi(I) \subseteq I$ for every proper superideal I of R.

Given two functions $\psi_1, \psi_2 : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$, we define $\psi_1 \leqslant \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathfrak{I}(R)$.

Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function, then an element $a \in h(R)$ is *ϕ-prime* to I, if whenever $ra \in I - \phi(I)$, where $r \in h(R)$, then $r \in I$. That is $a \in h(R)$ is ϕ-prime to I, if

$$h((I : a)) - h((\phi(I) : a)) \subseteq h(I).$$

Let $\nu_\phi(I)$ be the set of all homogeneous elements in R that are not ϕ-prime to I. We define I to be *ϕ-primal* if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

forms a superideal in R. In this case we say that I is a ϕ-P-primal superideal of R, and P is the adjoint superideal of I.

In the next example we give some famous functions $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ and their corresponding ϕ-primal superideals.

Example 1.1.

- $\phi_\varnothing$, $\phi_\varnothing(I) = \varnothing \forall I \in \mathfrak{I}(R)$ — primal superideal.
- ϕ_0, $\phi_0(I) = \{0\} \forall I \in \mathfrak{I}(R)$ — weakly primal superideal.
- ϕ_2, $\phi_2(I) = I^2 \forall I \in \mathfrak{I}(R)$ — almost primal superideal.
- ϕ_n, $\phi_n(I) = I^n \forall I \in \mathfrak{I}(R)$ — n-almost primal superideal.
- ϕ_ω, $\phi_\omega(I) = \cap_{n=1}^{\infty} I^n \forall I \in \mathfrak{I}(R)$ — ω-primal superideal.

Observe that $\phi_\varnothing \leqslant \phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2$.

For the nongraded case one can easily check that if I is a ϕ-P-primal ideal of R, with $\phi \neq \phi_\varnothing$, then $P = (\nu_\phi(I) \cup \{0\}) + \phi(I)$ if and only if $P = \nu_\phi(I) \cup \phi(I)$. But if $\phi = \phi_\varnothing$ then $P = \nu_\phi(I)$.

Y. Darani in [13] defined that for a commutative ring R with unity and for a function $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ a proper ideal I of R is a ϕ-P-primal ideal of R if $P = \phi(I) \cup \nu_\phi(I)$ is an ideal in R, where $\nu_\phi(I)$ is the set of all elements in R that are not ϕ-prime to I.

By comparing the two definitions (in the trivial case and in the supercase), we can see that the definition of ϕ-primal superideals is a generalization of the definition of the ϕ-primal ideals to the supercase.

In section 2, we give some examples and properties of ϕ-primal superideals of R. Also, we prove that if R is ϕ-torsion free super-ring, then every ϕ-primary superideal of R is ϕ-primal and hence if R is torsion free super-ring then every weakly primary (i.e., ϕ_0-primary) superideal of R is weakly primal.

In section 3, we introduce some conditions under which ϕ-primal superideals are primal.

2. ϕ-Primal superideals

Let R be a commutative super-ring with unity $1 \neq 0 \in R_0$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function and let I be a proper superideal of R. Suppose that $\nu_\phi(I)$ is the set of all homogeneous elements in R that are not ϕ-prime to I, we recall that I is a ϕ-primal superideal of R if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

forms a superideal in R. In this case P is called the adjoint superideal of I.

In the next examples we show that the concepts "primal superideals" and "ϕ-primal superideals" are different.

Example 2.1. Let $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$, where $u^2 = 0$, be a commutative super-ring and assume that $\phi = \phi_0$. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$.

(1) Since $0 \neq \bar{2} \cdot \bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get that $\bar{2}$ and $\bar{4}$ are not ϕ-prime to I. Easy computations imply that $\bar{2} + \bar{4} = \bar{6}$ is ϕ-prime to I. Thus we obtain that I is not a ϕ-primal superideal of R.

(2) Set $P = 2\mathbb{Z}_{24} + u\mathbb{Z}_{24}$. We show that I is a primal superideal of R. It is easy to check that every element of $h(P)$ is not prime to I. Conversely, assume that $\bar{a} \in h(R) - h(P)$, then $\bar{a} \in \mathbb{Z}_{24}$ with $gcd(a, 8) = 1$. If $\bar{a} \cdot \bar{n} \in I$ for some $\bar{n} \in \mathbb{Z}_{24}$, then 8 divides n; hence $\bar{n} \in I$. Therefore, $h(P)$ is exactly the set of elements in $h(R)$ which are not prime to I. Thus I is a primal superideal of R.

Example 2.2. Let $\phi = \phi_0$, and let $T(R)$ be the collection of all homogeneous zero divisors of R. If R is not a superdomain such that $Z(R) = T_0(R) + T_1(R)$ is not a superideal of R, then the trivial superideal of R is a ϕ-primal superideal which is not primal.

According to Examples 2.1 and 2.2 a primal superideal of R need not to be ϕ-primal and a ϕ-primal superideal of R need not to be primal.

In the next lemma we show that if I is a ϕ-primal superideal in R, then $I \subseteq P$. The same result for the non graded case has been proved in [13].

Lemma 2.3. *Let I be a superideal of R, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. Suppose that I is ϕ-primal superideal of R with the adjoint superideal P. Then*

(1) $I \subseteq P$.

(2) $h(P) = h(\phi(I)) \cup \nu_\phi(I)$.

Proof. (1) Let r be any homogeneous element in I, if $r \in \phi(I)$, then $r \in P$. If $r \in h(I) - h(\phi(I))$, then $1.r \in I - \phi(I)$ with $1 \notin I$, hence $r \in P$. Thus, $I \subseteq P$.

(2) It is trivial that $\nu_\phi(I) \subseteq h(P) - h(\phi(I))$. For the reverse inclusion, let $x \in h(P) - h(\phi(I))$ then $x = x_\alpha + y_\alpha$, where $x_\alpha \neq 0 \in \nu_\phi(I)$ and $y_\alpha \in (\phi(I))_\alpha$, for some α in $\mathbb{Z}_2$. Since $x_\alpha \neq 0 \in \nu_\phi(I)$, there exists $r \in h(R) - h(I)$ with $rx_\alpha \in I - \phi(I)$. Thus, $rx = rx_\alpha + ry_\alpha \in I - \phi(I)$ since $ry_\alpha \in \phi(I)$. Hence $x \in \nu_\phi(I)$. $\square$

Proposition 2.4. *Let I, P be proper superideals of R. Then the following statements are equivalent.*

(1) I is a ϕ-primal superideal of R with the adjoint superideal P.

(2) For $x \in h(R)$ with $x \notin h(P) - h(\phi(I))$ we have $h((I : x)) = h(I) \cup h((\phi(I) : x))$. If $x \in h(P) - h(\phi(I))$ then $h((I : x)) \supsetneqq h(I) \cup h((\phi(I) : x))$.

Proof. (1) $\Rightarrow$ (2) If $x \in h(P) - h(\phi(I))$, then $x \in \nu_\phi(I)$, so there exists $r \in h(R) - h(I)$ with $rx \in I - \phi(I)$. Thus $r \in h((I : x))$ and $r \notin h(I) \cup h((\phi(I) : x))$. Since it is easy to see that $h((I : x)) \supseteq h(I) \cup h((\phi(I) : x))$, we have that $h((I : x)) \supsetneqq h(I) \cup h((\phi(I) : x))$.

Now let $x \notin h(P) - h(\phi(I))$, where $x \in h(R)$, then $x \notin \nu_\phi(I)$ hence x is ϕ-prime to I. Let $r \in h((I : x))$, if $rx \notin \phi(I)$ then $r \in h(I)$. If $rx \in \phi(I)$ then $r \in h((\phi(I) : x))$. Hence

$$h((I : x)) \subseteq h(I) \cup h((\phi(I) : x)) \subseteq h((I : x)).$$

(2) $\Rightarrow$ (1) From part (2) we have $h(P) - h(\phi(I)) = \nu_\phi(I)$. Thus I is a ϕ-primal superideal of R. $\square$

Theorem 2.5. *If I is a ϕ-primal superideal of R, then*

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

is a ϕ-prime superideal of R.

Proof. Suppose that $a, b \in h(R) - h(P)$ we show that $ab \in \phi(P)$ or $ab \notin P$. Assume that $ab \notin \phi(P)$, then $ab \notin \phi(I)$, since $\phi(I) \subseteq \phi(P)$. Let $rab \in I - \phi(I)$ for some $r \in h(R)$. Then by Proposition 2.4, we have $ra \in h((I : b)) = h(I) \cup h((\phi(I) : b))$, but $ra \notin (\phi(I) : b)$; hence $ra \in h(I)$. Moreover $ra \notin h(\phi(I))$, for if $ra \in h(\phi(I))$, then $rab \in h(\phi(I))$, which is a contradiction. Therefore, $ra \in h(I) - h(\phi(I))$ and again by Proposition 2.4, $r \in h((I : a)) = h(I) \cup h((\phi(I) : a))$. Since $ra \notin \phi(I)$, we have $r \notin h((\phi(I) : a))$, so $r \in h(I)$. Hence ab is ϕ-prime to I which implies that $ab \notin P$. $\square$

Remark 2.6. Let I is a ϕ-primal superideal of R then by Theorem 2.5,

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

is a ϕ-prime superideal of R. In this case P is called the ϕ-prime adjoint superideal (simply adjoint superideal) of I, and I is called a ϕ-P-primal superideal of R.

The next result shows that every ϕ-prime superideal of R is ϕ-primal.

Theorem 2.7. *Every ϕ-prime superideal of R is ϕ-primal.*

Proof. Let P be a ϕ-prime superideal of R, we show that P is a ϕ-P-primal superideal of R. Thus we must prove that

$$P = \begin{cases} [(\nu_\phi(P))_0 + (\nu_\phi(P))_1 \cup \{0\}] + \phi(P) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(P))_0 + (\nu_\phi(P))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

Case 1. Suppose that $P \neq \phi(P)$. We show that $h(P) - h(\phi(P)) = \nu_\phi(P)$. Let $a \in h(P) - h(\phi(P))$. Then $a.1 \in P - \phi(P)$ with $1 \notin P$, so $a \in \nu_\phi(P)$. On the other hand let $a \notin h(P) - h(\phi(P))$. If $a \in h(\phi(P))$, then $ra \in \phi(P)$ for all $r \in h(R)$, so a is ϕ-prime to P and hence $a \notin \nu_\phi(P)$. If $a \notin h(\phi(P))$, then $a \notin P$, so for any $r_\alpha \in R_\alpha$ with $r_\alpha a \in P - \phi(P)$ we have $r_\alpha \in P_\alpha$, since P is ϕ-prime. Thus a is ϕ-prime to P, hence $a \notin \nu_\phi(P)$. Therefore, $h(P) - h(\phi(P)) = \nu_\phi(P)$ which implies that P is a ϕ-P-primal superideal of R.

Case 2. Suppose that $P = \phi(P)$ then it is easy to check that $\nu_\phi(P) = \varnothing$, hence P is a ϕ-P-primal superideal of R. □

In the next example we introduce a ϕ-P-primal superideal I of R such that I itself is not ϕ-prime.

Example 2.8. Let $\phi = \phi_0$ and let $R = \mathbb{Z}_8 + u\mathbb{Z}_8$ where $u^2 = 0$. Then R is a commutative super-ring with unity. If $I = 4\mathbb{Z}_8 + u\mathbb{Z}_8$, then I is not a ϕ-prime superideal of R, since $\bar{2} \cdot \bar{2} \neq 0 \in I$, but $\bar{2} \notin I$. Let $P = 2\mathbb{Z}_8 + u\mathbb{Z}_8$, we show that I is a ϕ-P-primal superideal of R. It is enough to show that $\nu(I) = h(P) - \{0\}$. Let $0 \neq \bar{a} \in h(P)$, if $\bar{a} \in 2\mathbb{Z}_8$ then $\bar{a} = 2k \in \mathbb{Z}_8$. If k is an odd number, then $0 \neq \bar{2}\bar{a} \in I$, but $\bar{2} \notin I$, and if k is an even number $0 \neq \bar{1}\bar{a} \in I$ with $\bar{1} \notin I$; hence $\bar{a} \in \nu(I)$. If $\bar{a} \in u\mathbb{Z}_8$ then $\bar{a} \in I \subseteq \nu(I)$. On the other hand, if $\bar{a} \in h(R) - h(P)$, then $\bar{a}$ is an odd number in $\mathbb{Z}_8$. If $0 \neq \bar{a}\bar{m} \in I$ for some $\bar{m} \in \mathbb{Z}_8$ then 4 divides am and so, 4 divides m since $(4, a) = 1$; hence $\bar{m} \in I$. Thus I is a ϕ-P-primal superideal of R.

Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. We assume that for any $I, J \in \mathfrak{I}(R)$, $\phi(J) \subseteq \phi(I)$ if $J \subseteq I$. We produced in Example 2.2 a ψ_2-primal which is not ψ_1-primal, where $\psi_1 \leqslant \psi_2$. In the next theorem we give the condition on ψ_2-P-primal superideal to be ψ_1-P-primal.

Theorem 2.9. *Suppose that $\psi_1 \leqslant \psi_2$, where ψ_1 and ψ_2 are maps from $\mathfrak{I}(R)$ into $\mathfrak{I}(R) \cup \{\varnothing\}$, and let I be a ψ_2-P-primal superideal of R, with $I_0 I_\alpha \neq \psi_2(I)_\alpha$ for all $\alpha \in \mathbb{Z}_2$. If P is a prime superideal of R, then I is ψ_1-P-primal.*

Proof. Since I is a ψ_2-P-primal superideal of R, then

$$P = \begin{cases} [(\nu_{\psi_2}(I))_0 + (\nu_{\psi_2}(I))_1 \cup \{0\}] + \psi_2(I) & : \quad \text{if } \psi_2 \neq \phi_\varnothing \\ (\nu_{\psi_2}(I))_0 + (\nu_{\psi_2}(I))_1 & : \quad \text{if } \psi_2 = \phi_\varnothing \end{cases}$$

To show that I is a ψ_1-P-primal superideal of R we must prove that

$$P = \begin{cases} [(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing \end{cases}$$

If $\psi_2 = \phi_\varnothing$, then $\psi_1 = \psi_2$ and hence we have that $P = \nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1$ which implies that I is a ψ_1-P-primal superideal of R. Now we may assume that $\psi_2 \neq \phi_\varnothing$, so we need to prove that

$$P = \begin{cases} [(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing \end{cases}.$$

Let $a \in \nu_{\psi_2}(I)$, then there exists $r \in h(R) - h(I)$ with $rs \in I - \psi_2(I) \subseteq I - \psi_1(I)$, so $a \in \nu_{\psi_1}(I)$ which implies that

$$(\nu_{\psi_2}(I))_0 + (\nu_{\psi_2}(I))_1 \subseteq (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \tag{1}$$

Now, let $a \in h(\psi_2(I))$, if $a \notin \psi_1(I)$ then $a \in I - \psi_1(I)$, so $1.a \in I - \psi_1(I)$ with $a \notin I$, hence $a \in \nu_{\psi_1}(I)$. Therefore,

$$\psi_2(I) \subseteq \begin{cases} [(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing \end{cases}. \tag{2}$$

From (1) and (2) we have that

$$P \subseteq \begin{cases} [(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing \end{cases}. \tag{3}$$

Since $\psi_1(I) \subseteq \psi_2(I) \subseteq P$, by (3)

$$P = \begin{cases} [(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing \end{cases}$$

if $\nu_{\psi_1}(I) \subseteq P$.

Let $a \in (\nu_{\psi_1}(I))_\alpha$. Then there exists $r_\beta \in R_\beta - I_\beta$ with $ar_\beta \in I - \psi_1(I)$. If $ar_\beta \in I - \psi_2(I)$, then $a \in \nu_{\psi_2}(I) \subseteq P$. So we may assume that $ar_\beta \notin$

$I - \psi_2(I)$, hence $ar_\beta \in \psi_2(I)$. First suppose that $aI_\beta \not\subseteq (\psi_2(I))_{\alpha\beta}$, say $as_\beta \in I_{\alpha\beta} - (\psi_2(I))_{\alpha\beta}$ with $s_\beta \in I_\beta$. Then $a(r_\beta + s_\beta) = ar_\beta + as_\beta \notin \psi_2(I)$ with $r_\beta + s_\beta \in R_\beta - I_\beta$, hence $a \in \nu_{\psi_2}(I) \subseteq P$. Therefore, we may assume that $aI_\beta \subseteq (\psi_2(I))_{\alpha\beta}$.

Now suppose that $r_\beta I_0 \not\subseteq (\psi_2(I))_\beta$, then there exists $c \in I_0$ with $r_\beta c \in I_\beta - (\psi_2(I))_\beta$. Since $a^2 \in R_0$, we have that $(a^2+c)r_\beta \in I_\beta - (\psi_2(I))_\beta$ with $r_\beta \notin I_\beta$, so $a^2 + c \in P_0$, but $c \in I_0 \subseteq P_0$, therefore $a^2 \in P$ and hence $a \in P$, since P is a prime superideal. So we may assume that $r_\beta I_0 \subseteq (\psi_2(I))_\beta$. Since $(I_0 I_\beta) \neq (\psi_2(I))_\beta$ there exists $\overline{a} \in I_0$ and $\overline{b} \in I_\beta$ with $\overline{a}\overline{b} \notin (\psi_2(I))_\beta$. Thus, $(a^2+\overline{a})(r_\beta+\overline{b}) = a^2 r_\beta + a^2\overline{b} + \overline{a}r_\beta + \overline{a}\overline{b} \notin \psi_2(I)$, so $(a^2+\overline{a})(r_\beta+\overline{b}) \in I - \psi_2(I)$ with $r_\beta + \overline{b} \in R_\beta - I_\beta$ which implies that $a^2 + \overline{a} \in (\nu_{\psi_2}(I))_0 \subseteq P_0$, hence $a^2 \in P_0 \subseteq P$ and then $a \in P$, since P is a prime superideal of R. Therefore, $\nu_{\psi_1}(I) \subseteq P$, so

$$P = \begin{cases} [(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing \end{cases}$$

and hence I is a ψ_1-P-primal superideal of R. $\square$

We end the section by proving the following results about the relationship between ϕ-primary and ϕ-primal superideals. For more properties about primary and primal superideals see [8, section 4].

Definition 2.10. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function such that $\phi \neq \phi_\varnothing$, then R is a ϕ-torsion free if $ab \in \phi(P)$ where $P \in \mathfrak{I}(R)$, then $a \in \phi(P)$ or $b \in \phi(P)$.

For example if $\phi = \phi_0$, then a ϕ-torsion free super-ring is just a torsion free super-ring.

Theorem 2.11. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function, where $\phi \neq \phi_\varnothing$, and let R be a ϕ-torsion free. Then every ϕ-primary superideal of R is ϕ-primal.*

Proof. Let I be a ϕ-primary superideal of R. We show that

$$\sqrt{I} = [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I).$$

($\supseteq$) Let $r \in \nu_\phi(I)$, then there exists $a \in h(R) - h(I)$ with $ra \in I - \phi(I)$ which implies that $r \in \sqrt{I}$, since I is ϕ-primary. Moreover, $\phi(I) \subseteq I \subseteq \sqrt{I}$.

($\subseteq$) Let $b \in h(\sqrt{I})$. If $b \in \phi(I)$, then done. So, we may assume that $b \notin \phi(I)$. Let n be the smallest positive integer such that $b^n \in I$. Suppose

$n = 1$. If $b \in \phi(I)$, then done. If $b \notin \phi(I)$, then $1.b \in I - \phi(I)$ and $1 \notin I$ so $b \in \nu_\phi(I)$. Therefore we may assume that $n > 1$. If $b^n \in \phi(I)$, then $b^n = b^{n-1}b \in \phi(I)$ and $b^{n-1} \notin \phi(I)$, since $b^{n-1} \notin I$ and $\phi(I) \subseteq I$, which is a contradiction since R is ϕ-torsion free. So, $b^n = b^{n-1}b \in I - \phi(I)$ and $b^{n-1} \notin I$, hence $b \in \nu_\phi(I)$. □

Corollary 2.12. *If R is a torsion free, then every weakly primary superideal of R is weakly primal.* □

3. Conditions on ϕ-primal superideals

In this section, we introduce some conditions under which ϕ-primal superideals are primal.

Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. We have to remind that if I is a ϕ-P-primal superideal of R, then

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

is a ϕ-prime superideal of R.

Definition 3.1. Let r be a homogeneous element in R, then $|r| = \alpha$ if $r \in R_\alpha$ for some $\alpha \in \mathbb{Z}_2$.

In the next theorem we provide some conditions under which a ϕ-primal superideal is primal.

Theorem 3.2. *Let R be a commutative super-ring with unity and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. Suppose that I is a ϕ-P-primal superideal of R with $I_\gamma I_\delta \nsubseteq \phi(I)$ for each $\gamma, \delta \in \mathbb{Z}_2$. If P is a prime superideal of R, then I is P-primal.*

Proof. Assume that a is a homogeneous element in P. Then $a \in \phi(I)$ or $a \in (\nu_\phi(I))_\alpha$ for some $\alpha \in \mathbb{Z}_2$ or $a = b_\beta + c_\beta$ where $b_\beta \in (\nu_\phi(I))_\beta$ and $c_\beta \in \phi(I)$ for some $\beta \in \mathbb{Z}_2$. If the first two cases hold, then a is not prime to I, since it is not ϕ-prime to I. In the last case, let d be a homogeneous element in R such that $d \notin I$ with $b_\beta d \in I - \phi(I)$. Then $ad = b_\beta d + c_\beta d \in I - \phi(I)$, because $ad \in \phi(I)$ implies that $b_\beta d \in \phi(I)$, since $c_\beta d \in \phi(I)$ which is a contradiction. Thus a is not ϕ-prime to I and hence a is not prime to I. Now assume that $b \in h(R)$ is not prime to I, so $rb \in I$ for some homogeneous element $r \in R - I$. If $rb \notin \phi(I)$, then b is not ϕ-prime to I, so $b \in P$. Thus assume that $rb \in \phi(I)$. Suppose that

$|r| = \alpha$. First suppose that $bI_\alpha \nsubseteq \phi(I)$. Then, there exists $r' \in I_\alpha$ such that $br' \notin \phi(I)$. So $b(r + r') \in I - \phi(I)$, where $r + r'$ is a homogeneous element in $R - I$, implies that b is not ϕ-prime to I, that is $b \in P$. Therefore, we may assume that $bI_\alpha \subseteq \phi(I)$. Let $|b| = \beta$. If $rI_\beta \nsubseteq \phi(I)$, then $rc \notin \phi(I)$ for some $c \in I_\beta$. In this case $r(b + c) \in I - \phi(I)$ with $r \in R - I$, that is $b + c \in P$ and hence $b \in P$, since $c \in I \subseteq P$. So we may assume that $rI_\beta \subseteq \phi(I)$. Since $I_\alpha I_\beta \nsubseteq \phi(I)$, there are $b' \in I_\alpha$ and $a' \in I_\beta$ with $b'a' \notin \phi(I)$. Then $(b + a')(r + b') \in I - \phi(I)$, where $r + b'$ is a homogeneous element in $R - I$, implies that $b + a'$ is a homogeneous element in P. On the other hand $a' \in I \subseteq P$, so that $b \in P$. We have already shown that P is exactly the set of all elements of R that are not prime to I. Hence I is P-primal. $\square$

Let R and S be commutative super-rings. It is easy to prove that the prime superideals of $R \times S$ have the forms $P \times S$ or $R \times Q$ where P is a prime superideal of R and Q is a prime superideal of S. Also we have the following two propositions about primal superideals of $R \times S$. We leave the easy proof for the next two results to the reader. For the trivial case (i.e., $R_1 = \{0\}$) they have proved in [10, Lemma 13] and [9, Theorem 16].

Proposition 3.3. *Let R and S be commutative super-rings. If P is a primal superideal of R and Q is a primal superideal of S, then $P \times S$ and $R \times Q$ are primal superideals of $R \times S$.* $\square$

Proposition 3.4. *Let R_1 and R_2 be commutative super-rings with unities and let $\psi_i : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be functions. Let $\phi = \psi_1 \times \psi_2$. Then ϕ-primes of $R_1 \times R_2$ have exactly one of the following three types:*

(1) *$I_1 \times I_2$ where I_i is a proper superideal of R_i with $\psi_i(I_i) = I_i$;*

(2) *$I_1 \times R_2$ where I_1 is a ψ_1-prime of R_1 which must be prime if $\psi_2(R_2) \neq R_2$;*

(3) *$R_1 \times I_2$ where I_2 is a ψ_2-prime of R_2 which must be prime if $\psi_1(R_1) \neq R_1$.* $\square$

Now let R_1, R_2 be commutative super-rings with unities and let $R = R_1 \times R_2$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be a function. In the next theorem, we provide some conditions under which a ϕ-primal superideal of R is primal, but first we start with the following remark.

Remark 3.5. Let I be a proper superideal of a commutative super-ring R and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be a function. If a homogeneous element a is not ϕ-prime to I, then there is a homogeneous element r in $R - I$ such that $ar \in I - \phi(I) \subseteq I$ so a is not prime to I.

Theorem 3.6. *Let R_1, R_2 be commutative super-rings with unities and let $R = R_1 \times R_2$. Let $\psi_i : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be functions with $\psi_i(R_i) \neq R_i$ for $i = 1, 2$. Let $\phi = \psi_1 \times \psi_2$. Assume that P is a superideal of R with $\phi(P) \neq P$. If I is a ϕ-P-primal superideal of R, then either $I = \phi(I)$ or I is primal.*

Proof. Suppose $\phi(I) \neq I$. By Theorem 2.5, P is a ϕ-prime superideal of R. Therefore, by Proposition 3.4, P has one of the following three cases.

Case 1. $P = P_1 \times P_2$ where P_i is a proper superideal of R_i with $\psi_i(P_i) = P_i$ for $i = 1, 2$. In this case $\phi(P) = P$, a contradiction.

Case 2. $P = P_1 \times R_2$ where P_1 is a ψ_1-prime superideal of R_1. Since $\psi_2(R_2) \neq R_2$, by Proposition 3.4(2), P_1 is a prime superideal of R_1 and so P is a prime superideal of R.

We will show that $I_2 = R_2$. Since $I \neq \phi(I)$, there exists a homogeneous element (a, b) in $I - \phi(I)$. So $(a, 1)(1, b) = (a, b) \in I - \phi(I)$. If $(a, 1) \notin I$, then $(1, b)$ is not ϕ-prime to I, hence $(1, b) \in P = P_1 \times P_2$, so $1 \in P_1$ a contradiction. Thus $(a, 1) \in I = I_1 \times I_2$ i.e., $1 \in I_2$ that is $I_2 = R_2$.

Now we prove that I_1 is a P_1-primal superideal of R_1. Let a_1 be a homogeneous element in P_1. Then $(a_1, 0) \in P_1 \times R_2 = P$. If $(a_1, 0) \in \phi(I) = \psi_1(I_1) \times \psi_2(R_2)$, then $a_1 \in \psi_1(I_1) \subseteq I_1$ so a_1 is not prime to I_1. Therefore, we may assume that $(a_1, 0) \in \nu_\phi(I)$. In this case there exists a homogeneous element $(r_1, r_2) \in R - I$ such that $(a_1, 0)(r_1, r_2) \in I - \phi(I)$ so $a_1 r_1 \in I_1 - \psi_1(I_1)$ with $r_1 \in R_1 - I_1$, since $R - I = (R_1 - I_1) \times R_2$, implies that a_1 is not ψ_1-prime to I_1, hence by Remark 3.5, a_1 is not prime to I_1. Conversely, let b_1 be a homogeneous element in R_1 such that b_1 is not prime to I_1. Then there exists a homogeneous element c_1 in $R_1 - I_1$ with $b_1 c_1 \in I_1$. Since $\psi_2(R_2) \neq R_2$, $(b_1, 1)(c_1, 1) = (b_1 c_1, 1) \in I_1 \times R_2 - (I_1 \times \psi_2(R_2)) \subseteq I - \phi(I)$ with $(c_1, 1) \in R - I$. Hence $(b_1, 1)$ is not ϕ-prime to I which implies that $(b_1, 1) \in P = P_1 \times R_2$ and so $b_1 \in P_1$.

We have already shown that the set of homogeneous elements in P_1 consists exactly of the homogeneous elements of R_1 that are not prime to I_1. Hence I_1 is P_1-primal superideal of R_1 so by Proposition 3.3, I is a P-primal superideal of R.

Case 3. $P = R_1 \times P_2$ where P_2 is a ψ_2-primal superideal of R_2. The proof of case(3) is similar to that of case(2). □

References

[1] *A. Jaber*, Central simple superalgebras with anti-automorphisms of order two of the first kind, J. Algebra, 323, **7** (2010) 1849–1859.

[2] *A. Jaber*, Central simple superalgebras with superantiautomorphism of order two of the second kind, Turkish Journal of Mathematics, **35** (2011), 11-21.

[3] *A. Jaber*, Division Z3-Algebras, International Electronic Journal of Algebra, **7** (2010), 1-11.

[4] *A. Jaber*, Existence of Pseudo-Superinvolutions of the First Kind, International Journal of Mathematics and Mathematical Sciences, Article ID 386468, 12 pages doi:10.1155/2008/386468.

[5] *A. Jaber*, Primitive $\mathbb{Z}_3$-algebras with $\mathbb{Z}_3$-involution, Far East Journal of Mathematical Sciences, **48** (2011), no. 2, 225-244.

[6] *A. Jaber*, Product of graded submodules, Turkish Journal of Mathematics, **35** (2011) , $1-12$.

[7] *A. Jaber*, Δ-supergraded Submodules, International Mathematical Forum, **5** (2010), 22, 1091-1104.

[8] *A. Jaber*, Weakly Primal Graded Superideals, Tamkang Journal of Mathematics, **43** (2012), 1, 123-135.

[9] *D. Anderson, M. Bataineh*, Generalizations of prime ideals, Comm. in Algebra **36** (2008), 686-696.

[10] *E. Atani, Y. Darani*, On Weakly Primal Ideals(I), Demonstratio Math. **40** (2007), 23-32.

[11] *L. Fuchs*, On Primal ideals, Amer. Math. Soc. **1** (1950), 1-6.

[12] *Y. A. Bahturin, A. Giambruno*, Group Gradings on associative algebras with involution, DOI:10.4153/CMB-2008-020-7, Canad. Math. Bull., **51** (2008), 182-194.

[13] *Y. Darani*, Generalizations of primal ideals in commutative rings, Matematiqki Vesnik, **64** (2012), 1, 25–31.

Contact information

A. Jaber Department of Mathematics, The Hashemite University, Zarqa 13115, Jordan
E-Mail(s): `ameerj@hu.edu.jo`

Received by the editors: 21.09.2015
and in final form 14.02.2016.