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On solvable Z_3-graded alternative algebras

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Abstract. Let $A = A_0 \oplus A_1 \oplus A_2$ be an alternative Z_3-graded algebra. The main result of the paper is the following: if A_0 is solvable and the characteristic of the ground field not equal 2,3 and 5, then A is solvable.

1. Introduction

Let R be an algebra over a field F. Let G be a finite group of automorphisms of R, and $R^G = \{x \in R | \phi(x) = x \text{ for all } \phi \in G\}$ be a fixed points subalgebra of R.

For Lie algebras there is a classical Higman result: if a Lie algebra L has an automorphism ϕ of simple order p without fixed points ($L^\phi = 0$), then L is nilpotent [1]. Moreover, nil index $h(p)$ in this case depends only on the order p. The explicit estimate of the function $h(p)$ was found in the paper of Kreknin and Kostrikin [2]. At the same time Kreknin proved that a Lie ring with a regular automorphism of an arbitrary finite order is solvable [3]. It is also worth mentioning here a result of Makarenko [4] who proved that if a Lie algebra L admits an automorphism of a prime order p with a finite-dimensional fixed-point subalgebra of dimension t, then L has a nilpotent ideal of nilpotency class bounded in terms of p and of codimension bounded in terms of t and p.

If R is an associative algebra with a finite group of automorphisms G then classical Bergman-Isaacs theorem says that if the subalgebra of fixed

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points R^G is nilpotent and R has no $|G|$-torsion, then R is nilpotent [5]. Kharchenko proved that under the same conditions, if R^G is a PI-ring, then R is a PI-ring [6]. For Jordan algebras the analogue of Kharchenko's result was proved by Semenov [7].

The Bergman-Isaacs theorem was partially generalized by Martindale and Montgomery to the case when G is a finite group of so called Jordan automorphisms, that is a linear automorphisms that are automorphisms of the adjoint Jordan algebra $R^{(+)}$ (note that in this case R^G is not a subalgebra in R, but a subalgebra in $R^{(+)}$) [8].

Note, that in general for Jordan algebras Bergman-Isaacs theorem is false - there is an example of a solvable non-nilpotent Jordan algebra J with an automorphism of second order ϕ such that the ring of invariants J^ϕ is nilpotent. However, Zhelyabin in [11] proved, that if a Jordan algebra J over a field of characteristic not equal 2,3 admits an automorphism of second order ϕ such that the algebra of invariants J^ϕ is solvable, then J is solvable.

For alternative algebras in [12] it was proved that if A is an alternative algebra over a field of characteristic not equal 2 with an automorphism g of second order then the solvability of the algebra of fixed points A^g implies the solvability of A. On the other hand, if the characteristic of the ground field is zero and G is a finite group of automorphisms of an alternative algebra A, then again the solvability of the algebra of fixed points A^G implies the solvability of A [7]. At the same time it is not known if the similar result is true in positive characteristic.

In this work we study a special case of the problem for alternative algebras: we consider a Z_3-graded alternative algebra $A = A_0 \oplus A_1 \oplus A_2$ and prove, that if the characteristic of the ground field not equal 2,3 and 5 and A_0 is solvable, then A is solvable.

As a consequence we obtain the following result: if A is an alternative algebra with an automorphism ϕ of order $2^k 3^l$, then under the same conditions on the characteristic of the ground field, the solvability of the subalgebra of fixed points A^ϕ implies the solvability of A.

2. Definitions and preliminary results

Let F be a field of characteristic not equal 2,3,5, A be an algebra over F. If $x, y, z \in A$ then $(x, y, z) = (xy)z - x(yz)$ is the associator of elements x, y, z, $x \circ y = xy + xy$ is a Jordan product of elements x and y and $[x, y] = xy - yx$ is a commutator of the elements x and y.

Definition. An algebra A is called Z_3-graduated if A is a direct sum of subspaces $A_i,\ i \in Z_3 : A = A_0 \oplus A_1 \oplus A_2$ and $A_i A_j \subseteq A_{i+j}$.

If A is a Z_3-graded algebra, then for every $i \in Z_3$ and $x \in A$ by x_i we will denote the projection of the element x to the subspace A_i and if $M \subset A$ then $M_i = \{x_i |\ x \in M\}$. An ideal I of A is called *homogeneous* if $I_j \subset I, \quad j = 0, 1, 2$. If I is a homogeneous ideal of A, then the factor-algebra A/I is also a Z_3-graded algebra.

If ϕ is an automorphism of the algebra A, then by A^{ϕ} we denote the subalgebra of fixed points of ϕ, that is $A^g = \{x \in A |\ \phi(x) = x\}$.

Define subsets A^i, $A^{<i>}$ and $A^{(i)}$ as:

$$A^2 = A^{<2>} = A^{(1)} = AA, \quad A^n = \sum_{i=1}^{n-1} A^i A^{n-i}, \quad A^{<n>} = A^{<n-1>}A$$

$$A^{(1)} = A^2, \quad A^{(i)} = A^{(i-1)}A^{(i-1)}.$$

Definition. An algebra A is called *nilpotent* if $A^i = 0$ for some i. An algebra is called *solvable*, if $A^{(i)} = 0$ for some i.

It is clear that $A^{(i)} \subset A^{2^i}$, so every nilpotent algebra is solvable. If A is an associative algebra then the inverse is also true: every solvable associative algebra is nilpotent. But in general, a solvable algebra is not necessary nilpotent. An example of an alternative solvable non-nilpotent algebra was constructed by Dorofeev [15](can also be found in [13]).

Definition. An algebra A is called alternative, if for all $x, y \in A$:

$$(x, x, y) = (y, x, x) = 0. \tag{1}$$

Let A be an alternative algebra. We will need the following identities on A (that are the linearizations of the well-known Moufang identities):

$$(x_1, x_2 y, z) + (x_2, x_1 y, z) = (x_1, y, z)x_2 + (x_2, y, z)x_1. \tag{2}$$

$$(x_1, y x_2, z) + (x_2, y x_1, z) = x_1(x_2, y, z) + x_2(x_1, y, z). \tag{3}$$

$$(x_1 \circ x_2, y, z) = (x_1, x_2 y + y x_2, z) + (x_2, x_1 y + y x_1, z). \tag{4}$$

$$(x_1 \circ x_2, y, z) = (x_1, y, z) \circ x_2 + (x_2, y, z) \circ x_1. \tag{5}$$

Also, in A the following equalities hold([13]):

$$2[(a, b, c), d] = ([a, b], c, d) + ([b, c], a, d) + ([c, a], b, d), \tag{6}$$

$$(dx, y, z) + (d, x, [y, z]) = d(x, y, z) + (d, y, z)x. \quad (7)$$

Let $D(A)$ be the associator ideal of A, that is an ideal generated by all associators (x, y, z), $x, y, z \in A$. In [13] it was shown that

$$D(A) = (A, A, A) + (A, A, A)A = (A, A, A) + A(A, A, A), \quad (8)$$

where $(A, A, A) = \{\sum_i (x_i, y_i, z_i) | \ x_i, y_i, z_i \in A\}$.

Let $J_2(A) = \{\sum_i \alpha_i a_i^2 \,|\, \alpha_i \in F, a \in A\}$ and $J_6(A) = \{\sum_i \alpha_i a_i^6 \,|\, \alpha_i \in F, a \in A\}$. Suppose A is an alternative algebra, then if $\mathrm{char}(F) \neq 2$ then $J_2(A)$ is an ideal of A and if $\mathrm{char}(F) \neq 2, 3, 5$ then $J_6(A)$ is also an ideal in A(see, for example, [13]).

3. Properties of Z_3-graded alternative algebras

In this section we will get some technical results that we will need. Throughout this section $A = A_0 \oplus A_1 \oplus A_2$ is an arbitrary alternative Z_3-graded algebra.

Lemma 1.

1)
$$(A_0^2, A_1, A_2) \subset A_0^2 \quad (9)$$

2) *For every $x \in A_1$, $y \in A_2$, $a_1, a_2 \in A_0$:*

$$(x(a_1a_2))y = x((a_1a_2)y) + a', \quad (y(a_1a_2))x = y((a_1a_2)x) + a'' \quad (10)$$

for some $a', a'' \in A_0^2$.

3)
$$(A_0A_1)(A_0^2A_2) \subset A_0^2, \quad (A_0A_2)(A_0^2A_1) \subset A_0^2 \quad (11)$$
$$(A_1A_0^2)(A_2A_0) \subset A_0^2, (A_2A_0^2)(A_1A_0) \subset A_0^2. \quad (12)$$

Proof. Let $x \in A_1$, $y \in A_2$ and $a_1, a_2 \in A_0$. Then using (7) we get:

$$(a_1a_2, x, y) = -(a_1, a_2, [x, y]) + a_1(a_2, x, y) + (a_1, x, y)a_2 \subset A_0^2.$$

And (9) is proved. It is easy to see that (10) follows from (9).

Let us prove (11) and (12). It is easy to see that they are similar and it is enough to prove one of these inclusions. Using (7) and (9) we compute:

$$\begin{aligned}(A_0A_1)(A_0^2A_2) &\subset A_0(A_1(A_0^2A^2)) + (A_0, A_1, A_0^2A_2) \\ &\subset A_0^2 + (A_0, A_1, A_0^2)A_2 + A_0^2(A_0, A_1, A_2) + (A_1, A_0^2, A_2) \\ &\subset A_0^2 + ((A_0^2)A_1)A_2 \subset A_0^2.\end{aligned}$$

□

Remark. From (10) it is follows that $A_0^2 + (A_1A_0^2)A_2 = A_0^2 + A_1(A_0^2A_2)$ and $A_0^2 + (A_2A_0^2)A_1 = A_0^2 + A_2(A_0^2A_1)$. This allows us to omit brackets in such a sentences without ambiguity.

Lemma 2.

$$(D(A))_1 \subseteq A_0A_1 + (A_1, A_2, A_1) + (A_2, A_0, A_2). \tag{13}$$

$$(D(A))_2 \subseteq A_0A_2 + (A_2, A_1, A_2) + (A_1, A_0, A_1). \tag{14}$$

Proof. It is enough to prove one of these equations. Let us prove (13). Using (8) we have:

$$\begin{aligned}(D(A))_1 \subseteq\ & (A_1, A_1, A_1)A_1 + (A_0, A_0, A_0)A_1 + (A_2, A_2, A_2)A_1 \\ & + (A_1, A_0, A_0)A_0 + (A_1, A_2, A_1)A_0 + (A_2, A_2, A_0)A_0 \\ & + (A_1, A_1, A_0)A_2 + (A_2, A_2, A_1)A_2 + (A_0, A_2, A_0)A_2 \\ & + (A_0, A_0, A_1) + (A_1, A_2, A_1) + (A_2, A_0, A_2).\end{aligned}$$

Using (6) we get:

$$\begin{aligned}(A_1, A_0, A_0)A_0 &\subseteq A_0A_1 + [A_0, (A_1, A_0, A_0)] \\ &\subseteq A_0A_1 + (A_0, A_1, A_0) \subseteq A_0A_1.\end{aligned}$$

Similarly, we obtain that

$$(A_1, A_2, A_1)A_0 + (A_2, A_2, A_0)A_0 \subseteq A_0A_1 + (A_2, A_0, A_2).$$

By (2) we compute:

$$\begin{aligned}(A_1, A_1, A_0)A_2 &\subseteq (A_2, A_1, A_0)A_1 + (A_1, A_0, A_0) + (A_1, A_2, A_1) \\ &\subseteq A_0A_1 + (A_1, A_2, A_1), \\ (A_2, A_2, A_1)A_2 &\subseteq A_0A_1 + (A_2, A_0, A_2) + (A_1, A_2, A_1).\end{aligned}$$

And, finally, using (1) we obtain the following inclusion:

$$(A_0, A_2, A_0)A_2 \subseteq A_0A_1 + (A_2, A_0, A_2).$$

Summing up the obtained inclusions we finally have that:

$$(D(A))_1 \subseteq A_0A_1 + (A_1, A_2, A_1) + (A_2, A_0, A_2). \qquad \square$$

Lemma 3.

1)

$$(A_0^2A_1, A_0, A_2) \subset A_0^2, \quad (A_0^2A_2, A_0, A_1) \subset A_0^2. \tag{15}$$

2)

$$((A_1, A_2, A_1) + (A_2, A_0, A_2))A_0^2(A_1 \circ A_1) \subset A_0^2. \tag{16}$$

$$((A_2, A_1, A_2) + (A_1, A_0, A_1))A_0^2(A_2 \circ A_2) \subset A_0^2. \tag{17}$$

3) *For all* $n \geqslant 2$*:*

$$(A_1A_0^{<n>})(A_0A_2) \subset A_1(A_0^{<n+1>})A_2 + A_0^2, \tag{18}$$

$$(A_2A_0^{<n>})(A_0A_1) \subset A_2(A_0^{<n+1>})A_1 + A_0^2. \tag{19}$$

Proof. It is easy to see that it is enough to prove only one inclusion in every statement. We will prove the first inclusion in all cases.

By (7) we have:

$$\begin{aligned}(A_0^2A_1, A_0, A_2) &\subset A_0^2(A_1, A_0, A_2) + (A_0^2, A_0, A_2)A_1 + (A_0^2, A_1, A_2)\\ &\subset A_0^2 + (A_0^2A_2)A_1 \subset A_0^2.\end{aligned}$$

And (15) is proved.

Using (6), (9) and (15) we compute:

$$\begin{aligned}&((A_1, A_2, A_1) + (A_2, A_0, A_2))A_0^2(A_1 \circ A_1)\\ &\quad \subset A_0^2 + [(A_1, A_2, A_1) + (A_2, A_0, A_2), A_0^2(A_1 \circ A_1)]\\ &\quad \subset A_0^2 + (A_0^2(A_1 \circ A_1), A_0, A_1) + (A_0^2(A_1 \circ A_1), A_2, A_2)\\ &\quad \subset A_0^2 + (A_0^2(A_1 \circ A_1), A_2, A_2).\end{aligned}$$

Using (2) and (4) we have:

$$\begin{aligned}(A_0^2(A_1 \circ A_1), A_2, A_2) &\subset (A_1, A_0^2, A_2) + A_0^2 + (A_1 \circ A_1, A_0^2, A_2)A_2\\ &\subset A_0^2 + (A_1, A_0^2, A_0)A_2 \subset A_0^2.\end{aligned}$$

Thus, $((A_1, A_2, A_1) + (A_2, A_0, A_2))A_0^2(A_1 \circ A_1) \subset A_0^2$.

Let us prove (18). Using (9) and (15) we get:

$$\begin{aligned}(A_1A_0^{<n>})(A_0A_2) &\subset ((A_1A_0^{<n>})A_0)A_2 + (A_1A_0^{<n>}, A_0, A_2)\\ &\subset A_0^2 + (A_1A_0^{<n+1>})A_2 + (A_1, A_0^{<n>}, A_0)A_2\\ &\subset A_0^2 + A_1(A_0^{n+1}A_2).\end{aligned}$$

□

Lemma 4.

1) *Let* char$(F) \neq 2$. *Then A is solvable if and only if $J_2(A)$ is solvable.*
2) *Let* char$(F) \neq 2, 3, 5$. *Then A is solvable if and only if $J_6(A)$ is solvable.*

Proof. The proof is similar for both cases. Let us prove 2.

If A is solvable then clearly $J_6(A)$ is solvable.

Suppose $J_6(A)$ is solvable. Consider the factor algebra $\overline{A} = A/J_6(A)$. Then for every $\overline{x}$ in $\overline{A}$: $\overline{x}^6 = 0$, that is $\overline{A}$ is a nil algebra of nil-index 6. Since the characteristic of the ground field F not equal 2,3 or 5, then by Zhevlakov's theorem $\overline{A}$ is solvable ([14], the proof can also be found in [13]). Thus, A is solvable. □

Lemma 5. *Let A be a Z_3-graded alternative algebra over a field F of characteristic not equal 2,3,5. Then we have the following inclusions:*

1) $(J_2(A))_1 \subset A_0 \circ A_1 + A_2 \circ A_2$, $(J_2(A))_2 \subset A_0 \circ A_2 + A_1 \circ A_1$.
2) $(J_6(A))_0 \subset A_0^2 + A_1 A_0^2 A_2 + A_2 A_0^2 A_1$.

Proof. The first assertion is obvious.

Let us prove 2. We will use the following notation: if $u, v \in A$ then $u \equiv v$ means that $u - v \in A_0^2 + A_1 A_0^2 A_2 + A_2 A_0^2 A_1$

Let $x \in A_1, y \in A_2, a \in A_0$. It is sufficient to prove that $((x+y+a)^6)_0 \equiv 0$.

First we will proof the following inclusion:

$$x(y, x, a)x^2 + x^2(y, x, a)x \in A_0^2 \tag{20}$$

Indeed, using (5) and (2) we have

$$\begin{aligned} A_0^2 \ni 2(xy, x^3, a) &= (xy, x^2, a) \circ x + (xy, x, a) \circ x^2 \\ &= x(xy, x, a)x + (xy, x, a)x^2 + (xy, x, a) \circ x^2 \\ &= x(y, x, a)x^2 + 2(y, x, a)x^3 + x^2(y, x, a)x. \end{aligned}$$

Thus, $x(y, x, a)x^2 + x^2(y, x, a)x \in A_0^2$. Similarly, one can prove the following inclusion:

$$y(x, y, a)y^2 + y^2(x, y, a)y \in A_0^2. \tag{21}$$

Consider $p = (x + y + a)^3$. Then we have:

$$\begin{aligned} p_0 &= x^3 + y^3 + a^3 + (x \circ a)y + (y \circ a)x + (x \circ y)a, \\ p_1 &= x^2 y + (x \circ y)x + y^2 a + (a \circ y)y + a^2 x + (a \circ x)a, \\ p_2 &= y^2 x + (y \circ x)y + x^2 a + (a \circ x)x + a^2 y + (a \circ y)a. \end{aligned}$$

Since $((x+y+a)^6)_0 = p_0^2 + p_1 \circ p_2$, then it is enough to proof that:

$$(x^2y + (x\circ y)x)\circ(y^2x + (y\circ x)y) \equiv 0, \tag{22}$$

$$(y^2a + (a\circ y)y)\circ(y^2x + (y\circ x)y) \equiv 0, \tag{23}$$

$$(a^2x + (a\circ x)a)\circ(y^2x + (y\circ x)y) \equiv 0, \tag{24}$$

$$(x^2y + (x\circ y)x)\circ(x^2a + (a\circ x)x) \equiv 0, \tag{25}$$

$$(y^2a + (a\circ y)y)\circ(x^2a + (a\circ x)x) \equiv 0, \tag{26}$$

$$(a^2x + (a\circ x)a)\circ(x^2a + (a\circ x)x) \equiv 0, \tag{27}$$

$$(x^2y + (x\circ y)x)\circ(a^2y + (a\circ y)a) \equiv 0, \tag{28}$$

$$(y^2a + (a\circ y)y)\circ(a^2y + (a\circ y)a) \equiv 0, \tag{29}$$

$$(a^2x + (a\circ x)a)\circ(a^2y + (a\circ y)a) \equiv 0. \tag{30}$$

The equivalences (22),(27) and (29) are obvious. Let us prove (23). We have:

$$\begin{aligned}
&(y^2a)(y^2x) + (y^2x)(y^2a)\\
&\quad= (y^2ay)(yx) - (y^2a, y, yx) + ((y^2x)y^2)a - (y^2x, y^2, a)\\
&\quad\equiv -(a,y,x)y^3 - y(y^2x, y, a) - (y^2x, y, a)y \equiv y(y,x,a)y^2,\\
&(y^2a)((y\circ x)y) + (y\circ x)y)(y^2a)\\
&\quad= ((y^2a)(y\circ x)))y + (y^2a, y\circ x, y) + (y\circ x)(y^3a) + (y\circ x, y, y^2a)\\
&\quad\equiv (y^2(a(y\circ x)))y + (y^2, a, y\circ x)y + (y^2a, y\circ x, y) + (y\circ x, y, y^2a)\\
&\quad\equiv y(y,a,x)y^2 + 2(y\circ x, y, a)y^2 = 3y(y,a,x)y^2,\\
&((a\circ y)y)(y^2x) + (y^2x)((ay+ya)y)\\
&\quad= ((a\circ y)y^2)(yx) + ((a\circ y)y, y, yx) + y((yx)(ay^2)) + (y, yx, ay^2)\\
&\qquad+ (y^2xy)(ay) - (y^2x, y, ay)\\
&\quad\equiv y((yx)a)y^2 - y(yx, a, y^2) + y^2(y,x,a)y + ((y^2xy)a)y\\
&\qquad- (y^2xy, a, y) - y(x,y,a)y^2\\
&\quad\equiv y^2(y,x,a)y + y^2((xy)a)y + (y^2, xy, a)y - y(x,a,y)y^2 - y(x,y,a)y^2\\
&\equiv y^2(y,x,a)y,\\
&((a\circ y)y)((y\circ x)y)) + ((y\circ x)y))((a\circ y)y)\\
&\quad= (ay^2)((y\circ x)y) + (yay)(yxy + xy^2) + (y\circ x)(y(a\circ y)y)\\
&\qquad+ (y\circ x, y, (a\circ y)y)
\end{aligned}$$

$$
\begin{aligned}
&\equiv a(y^2((y\circ x)y)) + (a, y^2, (y\circ x)y) + (yay^2)(xy) - (yay, y, xy)\\
&\quad + ((yay)x)y^2 - (yay, x, y^2) + y^2(x, y, a)y\\
&\equiv 3y^2(x, y, a)y + y((ay)x)y^2 + (y, ay, x)y^2\\
&\equiv 3y^2(x, y, a)y + y(a(yx))y^2 + y(a, y, x)y^2 + y(y, a, x)y^2\\
&\equiv 3y^2(x, y, a)y.
\end{aligned}
$$

Summing up the obtained equations we have:

$$
\begin{aligned}
&(y^2a + (a\circ y)y)\circ(y^2x + (y\circ x)y)\\
&\qquad \equiv y(y, x, a)y^2 + 3y(y, a, x)y^2 + y^2(y, x, a)y + 3y^2(x, y, a)y \equiv 0.
\end{aligned}
$$

That proves (23). Using similar arguments one can obtain (25).

Let us prove (24):

$$
\begin{aligned}
&(a^2x)\circ(y^2x + (y\circ x)y)\\
&\qquad \equiv a^2(xy^2x + (y\circ x)y) + (a^2, x, y^2x + (y\circ x)y) \equiv 0,\\
&((a\circ x)a)\circ(y^2x + (y\circ x)y)\\
&\qquad = (xa^2)\circ(y^2x + (y\circ x)y) + (axa)(y^2x + (y\circ x)y)\\
&\qquad\quad + (y^2x + (y\circ x)y)(axa)\\
&\qquad \equiv a((xa)(y^2x + (y\circ x)y)) + (a, xa, (y^2x + (y\circ x)y))\\
&\qquad\quad + ((y^2x + (y\circ x)y)(ax))a - ((y^2x + (y\circ x)y), ax, a)\\
&\qquad \equiv a(a, x, (y^2x + (y\circ x)y)) - ((y^2x + (y\circ x)y), x, a)a \equiv 0.
\end{aligned}
$$

Thus, $(a^2x)\circ(y^2x + (y\circ x)y) + ((a\circ x)a)\circ(y^2x + (y\circ x)y) \equiv 0$ and (24) is proved. Similarly, one can prove (28).

Consider (26). We have:

$$
\begin{aligned}
(y^2a)\circ(x^2a) &= ((y^2a)x^2)a - (y^2a, x^2, a) + ((x^2a)y^2)a - (x^2a, y^2, a)\\
&\equiv -a(y^2a, x^2, a) - a(x^2, y^2, a) \equiv 0.
\end{aligned}
$$

Similarly, $(y^2a)\circ(ax^2) + (ay^2)\circ(ax^2) + (ay^2)\circ(x^2a) \equiv 0$. Further, we compute:

$$
\begin{aligned}
(y^2a)(xax) + (ay^2)(xax) &= y^2(axax) + (y^2, a, xax) + a(y^2(xax)) + (a, y^2, xax)\\
&\equiv y(((y(ax))a)x) - y(y(ax), a, x) \equiv -y(y(ax), a, x).
\end{aligned}
$$

Using (7) and (9) we get:

$$
\begin{aligned}
-y(y(ax), a, x) &= (y(ax), a, y)x - (y(ax), a, yx) - ([y(ax), a], y, x)\\
&\equiv (y(ax), a, y)x = ((ax, a, y)y)x = (((x, a, y)a)y)x \equiv 0.
\end{aligned}
$$

Using similar computations one can prove that $(xax)(y^2a)+(xax)(ay^2)\equiv 0$. Finally,

$$
\begin{aligned}
&(yay)(xax) + (xax)(yay)\\
&\quad = ((yay)x)(ax) - (yay, x, ax) + x((ax)(yay)) + (x, ax, yay)\\
&\quad \equiv ((ya)(yx))(ax) + (ya, y, x)(ax) + x(((ax)y)(ay)) - x(ax, y, ay)\\
&\quad \equiv (y(a(yx)))(ax) + (y, a, yx)(ax) + (ya, y, x)(ax) + x((((ax)y)a)y)\\
&\qquad - x((ax)y, a, y) - x(ax, y, ay)\\
&\quad \equiv ((y, a, x)y)(ax) + ((a, y, x)y)(ax) - x(y(ax, a, y))\\
&\qquad - x(y(ax, y, a)) = 0.
\end{aligned}
$$

And (26) is proved. The equality (30) can be proved in a similar way. □

4. The main part

Recall that if A is an algebra, then by $D(A)$ we denote the ideal generated by associators. Define subalgebras K_i and T_i as

$$K_1 := J_2(A),\ T_1 := D(K_1),\quad K_i := J_2(T_{i-1}),\ T_i := D(K_i).$$

It is easy to see that:

$$A \supseteq K_1 \supseteq T_1 \supseteq K_2 \supseteq ... \supseteq K_i \supseteq T_i \supseteq ...$$

Lemma 6. *If for some $i \geqslant 1$ T_i or K_i is solvable, then A is solvable.*

Proof. By lemma 4 A is solvable if and only if $J_2(A) = K_1$ is solvable. Since $D(K_1)$ is a homogeneous ideal, then $K_1/D(K_1)$ - is an associative Z_3-graded algebra with a solvable even part. Thus, by Bergman-Isaacs theorem $K_1/D(K_1)$ is nilpotent and if $D(K_1)$ is solvable, then A is solvable.

Similar arguments show that K_i and T_i are solvable if and only if T_{i-1} is solvable. □

Lemma 7. *Let A be a Z_3-graded algebra and $A_0 = 0$. If $\operatorname{char} F \neq 2, 3$, then A is solvable.*

Proof. Consider $J_3(A) = \{\sum_i x_i^3 | x_i \in A\}$. Using similar arguments as in lemma 4 we get, that A is solvable if and only if $J_3(A)$ is solvable. For all $x \in A_1$ and $y \in A_2$ we have:

$$(x + y)^3 = x^3 + y^3 + x^2y + yx^2 + xyx + y^2x + xy^2 + yxy.$$

But $x^3 \in A_0$, $y^3 \in A_0$ and $xy \in A_0$. Thus $(x+y)^3 = 0$ and $J_3(A) = 0$. □

Theorem 1. *Let A be a Z_3-graded alternative algebra over a field F. If A_0 is solvable and* $\operatorname{char} F \neq 2, 3, 5$, *then A is solvable.*

Proof. Let $A_0^{(m)} = 0$ and $n = 2^m$. Consider T_n and define $I = J_6(T_n)$. By lemmas 4 and 6 it is enough to prove that I is solvable. By lemma 5 we have $I_0 \subset A_0^2 + (T_n)_2(A_0^2)(T_n)_1 + (T_n)_1(A_0^2)(T_n)_2$.

Our aim now is to prove that

$$(T_n)_1(A_0^2)(T_n)_2 \subset A_0^2 + (T_{n-1})_1 A_0^{<3>} (T_{n-1})_2. \tag{31}$$

Indeed, since $T_n \subset K_n = J_2(T_{n-1})$ then by lemma 5:

$$(T_n)_2 \subset A_0 \circ (T_{n-1})_2 + (T_{n-1})_1 \circ (T_{n-1})_1.$$

By (12) and (18) we have that

$$(T_n)_1 A_0^2 (A_0 \circ (T_{n-1})_2) \subset (T_{n-1})_1 A_0^{<3>} (T_{n-1})_2 + A_0^2.$$

Using inclusion (13) we get:

$$\begin{aligned}(T_n)_1 &= (D(K_n))_1 \\ &\subseteq A_0(K_n)_1 + ((K_n)_1, (K_n)_2, (K_n)_1) + ((K_n)_2, A_0, (K_n)_2).\end{aligned}$$

And now it is left to use inclusions (11) and (16) to prove (31). Similar reasons shows us that $(T_n)_2(A_0^2)(T_n)_1 \subset A_0^2 + (T_{n-1})_2 A_0^{<3>} (T_{n-1})_1$ and we may conclude that

$$I_0 \subset A_0^2 + (T_{n-1})_1 A_0^{<3>} (T_{n-1})_2 + (T_{n-1})_2 A_0^{<3>} (T_{n-1})_1.$$

Now we can continue to use similar arguments and get that

$$I_0 \subset A_0^2 + (T_{n-2})_1 A_0^{<4>} (T_{n-2})^2 + (T_{n-2})_2 A_0^{<4>} (T_{n-2})_1.$$

And finally, we will get that

$$I_0 \subset A_0^2 + A_1 A_0^{<n>} A_2 + A_2 A_0^{<n>} A_1. \tag{32}$$

Let us prove that $A_1 A_0^{<n>} A_2 \subset A_0^2$. For this we will prove that for all $k \geqslant 2$:

$$A_1(A_0^k, A_0, A_0)A_2 \subset A_0^2. \tag{33}$$

Indeed, using (2) and (9) we have:

$$A_1(A_0^k, A_0, A_0)A_2 \subset A_1((A_2, A_0, A_0)A_0^k) + A_1(A_0^k, A_2, A_0) \subset A_0^2.$$

Moreover, from (18) and (33) we see that

$$A_1(((...((A_0^2, A_0, A_0)A_0)A_0)...A_0)A_2 \subset A_0^2. \tag{34}$$

Now we can use (34) to obtain the following inclusions:

$$\begin{aligned} A_1A_0^{<n>}A_2 &\subset A_1(((A_0^2A_0^2)A_0)...A_0)A_2 + A_0^2 \\ &\subset A_1(((A_0^2A_0^2)A_0^2)...A_0^2)A_2 + A_0^2 \\ &\subset A_1(((A_0^{(2)}A_0^{(2)})...)A_0^{(2)})A_2 + A_0^2 \\ &\subset ... \subset A_1A_0^{(m)}A_2 + A_0^2 = A_0^2. \end{aligned}$$

Similarly, $A_2A_0^{<n>}A_1 \subset A_0^2$. Thus, $I_0 \subset A_0^2 = A_0^{(1)}$.

Now we can start from the beginning with the ideal I and construct an ideal I' such that I (and, thus, A) is solvable if and only if I' is solvable and $I'_0 \subset I_0^2 \subset A_0^{(2)}$. Repeating this construction, in the end we will construct an sublagebra $\widetilde{I}$ such that A is solvable if and only if $\widetilde{I}$ is solvable and $\widetilde{I}_0 \subset A_0^{(m)} = 0$. But by lemma 7 $\widetilde{I}$ is solvable, so A is also solvable.

Corollary 1. *Let A be an alternative algebra with an automorphism ϕ of order 3. If* char $F \neq 2, 3, 5$ *and the subalgebra A^ϕ of fixed points with respect to ϕ is solvable, then A is solvable.*

Proof. If the ground field F is algebraically closed, then we can consider subspaces $A_\xi = \{x \in A| \ \ \phi(x) = \xi x\}$ and $A_{\xi^2} = \{x \in A| \ \ \phi(x) = \xi^2 x\}$, where ξ is a primitive cube root of unity. It is easy to see that $A = A_\xi \oplus A_{\xi^2} \oplus A^\phi$ and A is a Z_3-graded algebra. Since A^ϕ is solvable, then by theorem 1 A is solvable.

If F is not algebraically closed we can consider it's algebraic closure $\overline{F}$ and an algebra $\overline{A} = A \otimes_F \overline{F}$. Then $\overline{A}$ is an alternative algebra over $\overline{F}$ and A is solvable if and only if $\overline{A}$ is solvable. We can define an automorphisms $\overline{\phi}$ on $\overline{A}$ by putting: $\overline{\phi}(a \otimes \alpha) = \phi(a) \otimes \alpha$ for all $a \in A, \alpha \in \overline{F}$. Then $\overline{\phi}$ is an automorphism of order 3 and the subalgebra of fixed points $\overline{A}^{\overline{\phi}} = A^\phi \otimes \overline{F}$ is solvable. Thus, $\overline{A}$ is solvable and, finally, A is solvable. $\square$

Corollary 2. *Let $A = \sum\limits_{i=0}^{n-1} A_i$ be a Z_n-graded alternative algebra, where $n = 2^k 3^l$ and $k + l \geqslant 1$. If* char $F \neq 2, 3, 5$ *and the subalgebra A_0 is solvable, then A is solvable.*

Proof. If $k = 0$ then by corollary 1 A is solvable. Suppose $k \geqslant 1$. We will use an induction on l. If $l = 0$ then the result follows from the paper of Smirnov [12]. Let $l \geqslant 1$. Then we can consider subspaces $\widehat{A}_0 = \sum\limits_i A_{3i}$, $\widehat{A}_1 = \sum\limits_i A_{1+3i}$, $\widehat{A}_2 = \sum\limits_i A_{2+3i}$. Then $A = \widehat{A}_0 \oplus \widehat{A}_1 \oplus \widehat{A}_2$ - is a Z_3-gradation of A. By theorem 1 A is solvable if and only if $\widehat{A}_0$ is solvable. On the other hand it is easy to see that $\widehat{A}_0$ is a $Z_{n'}$-graded algebra, where $n' = 2^k 3^{l-1}$ and $(\widehat{A}_0)_0 = A_0$ is solvable. Now we may use the induction and get that $\widehat{A}_0$ is solvable. Hence, A is solvable. □

Corollary 3. *Let A be an alternative algebra with an automorphism ϕ of order $2^k 3^l$. If* $\operatorname{char} F \neq 2, 3, 5$ *and the subalgebra A^ϕ of fixed points with respect to ϕ is solvable, then A is solvable.*

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