ON UNIVERSALITY OF COUNTABLE POWERS OF ABSOLUTE RETRACTS

We construct an absolute retract $X$ of arbitrary high Borel complexity, such that the countable power $X^\omega$ is not universal for the Borelian class $\mathcal{A}_1$ of sigma-compact spaces, and the product $X^\omega \times \Sigma$, where $\Sigma$ is the radial interior of the Hilbert cube, is not universal for the Borelian class $\mathcal{A}_2$ of absolute $G_{\delta_0}$-spaces.

By $\mathcal{A}_1$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{A}_2$ we denote respectively the class of all sigma-compact spaces, the class of all Polish spaces, the class of all absolute $F_{\sigma\delta}$-spaces, and the collection of all absolute $G_{\delta_0}$-spaces; $Q = [-1, 1]^\omega$ is the Hilbert cube, and $\Sigma = \{ (t_i)_{i \in \mathbb{N}} \in Q : \sup_{i \in \mathbb{N}} |t_i| < 1 \}$ is its radial interior. A closed set $A \subset X$ in an absolute retract $X$ is called a $Z$-set, provided every map $f : Q \to X$ can be uniformly approximated by maps into $X \setminus A$ [1]. An absolute retract $X$ is called a $Z_\sigma$-space, provided $X$ is a countable union of its $Z$-sets.

All spaces considered are metrizable and separable, all maps are continuous.

Let $C$ be a collection of spaces. We say that a space $X$ is $C$-universal, provided for every space $C \in C$ there is a closed embedding $f : C \to X$.

In [2] (Corollary 2.5) T. Dobrowolski and J. Mogilski proved that, if an absolute retract $X$ is a $Z_\sigma$-space, then the countable power $X^\omega$ is $\mathcal{M}_2$-universal. In this note we show that in the above result, the condition on $X$ to be a $Z_\sigma$-space can not be replaced by the conditions on Borelian complexity of $X$ (for example, $X \notin \mathcal{M}_1$ or $X \notin \mathcal{A}_2$).

By $\overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and $D = \{ z \in \mathbb{C} : |z| < 1 \}$ we denote respectively the closed and the open disks in the complex plane $\mathbb{C}$, and by $P = \{ z \in \mathbb{C} : |z| = 1, \ \arg(z)/\pi \text{ is irrational} \}$ the set of irrationals in the circle $S^1 = \overline{D} \setminus D$. It is obvious

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that, for every dense \( A \subset P \), both \( A \) and \( S^1 \setminus D \) are zero-dimensional. Moreover, the set \( D \cup A \) is convex, and consequently, is an absolute retract (see Theorem 3.1 [1] (II, § 3)).

**Theorem 1.** For every dense set \( A \subset P \) the space \((D \cup A)^\circ\) is not \( A_1\)-universal.

**Proof.** Assume, on the contrary, that the space \((D \cup A)^\circ\) is \( A_1\)-universal. Then there exists a closed embedding \( f: \Sigma \to (D \cup A)^\circ \). We consider the space \((D \cup A)^\circ\) to be a subset in the compactum \( \overline{D}^\circ \). According to Lavrentiev Theorem [3] (Theorem 4.3.21), there exists an embedding \( \tilde{f}: G \to \overline{D}^\circ \) of some \( G_\sigma\)-set \( G \), \( \Sigma \subset G \subset Q \), extending the embedding \( f \). Since \( \Sigma \) is dense in \( G \) and \( f(\Sigma) \) is closed in \((D \cup A)^\circ\), \( \tilde{f}(G \setminus \Sigma) \subset \overline{D}^\circ \setminus (D \cup A)^\circ \). Now notice that, \( \overline{D}^\circ \setminus (D \cup A)^\circ = \bigcup_{n=1}^\infty X_n \), where \( X_n = \{ (t_i)_{i=1}^m \in \overline{D}^\circ \mid t_n \in S^1 \setminus A \} \). Since \( G \setminus \Sigma \) is a \( G_\sigma\)-set in \( Q \), by the Baire Theorem [3] (Theorem 3.9.3), there is an open set \( U \subset G \setminus \Sigma \) such that the set \( \tilde{f}(U) \cap X_n \) is dense in \( \tilde{f}(U) \) for some \( n \in \mathbb{N} \). Let \( V = Q \setminus cl_Q((G \setminus \Sigma) \cup U) \). Obviously, \( V \) is an open set in \( Q \) such that \( \tilde{f}(G \setminus \Sigma) \cap V = U \). Let \( V' \subset V \) be an open set of the form \( V' = \{ (t_i)_{i=1}^m \in Q \mid a_i < t_i < b_i, 1 \leq i \leq m \} \), where \( m \in \mathbb{N} \), and \( a_i < b_i, 1 \leq i \leq m \), are reals. Put finally, \( W = V' \cap G \). One can verify that \( W \cap \Sigma = V' \cap \Sigma \) is a connected (even convex) dense set in \( W \) and the set \( W \setminus \Sigma \) is dense in \( W \). Since \( \tilde{f} \) is an embedding, \( \tilde{f}(W) \cap X_n \) is dense in \( \tilde{f}(W) \). Denote by \( pr_n : \overline{D}^\circ \to \overline{D} \) the projection onto the \( n \)-th factor. Note that \( pr_n^{-1}(S^1 \setminus A) = X_n \). Since the set \( \tilde{f}(W \setminus \Sigma) \cap X_n \) is dense in \( \tilde{f}(W \setminus \Sigma) \) (remark that \( W \setminus \Sigma \) is an open set in \( U \)) and \( W \setminus \Sigma \) is dense in \( W \), \( pr_n(\tilde{f}(W)) \subset S^1 \) and \( pr_n(\tilde{f}(W)) \cap (S^1 \setminus A) \neq \emptyset \). Recalling that \( \tilde{f}(\Sigma) \subset (D \cup A)^\circ \), we obtain that \( pr_n(\tilde{f}(W \cap \Sigma)) \subset S^1 \cap (D \cup A)^\circ = A \). Since the set \( W \cap \Sigma \) is connected, and \( A \) is zero-dimensional, the image \( pr_n(\tilde{f}(W \cap \Sigma)) \) consists of only the point \( a \in A \). Since \( W \cap \Sigma \) is dense in \( W \), we obtain \( pr_n(\tilde{f}(W)) = \{ a \} \). But this contradicts to \( pr_n(\tilde{f}(W)) \cap (S^1 \setminus A) \neq \emptyset \). Theorem is proved.

In connection with [4] (Question 6.3), the following problem seems to be interesting.

**Question.** Let \( A \subset P \) be a dense set. Can the space \((D \cup A)^\circ \times \Sigma^\circ\) be \( A_2\)-universal?

**Theorem 2.** For every dense set \( A \subset P \) the space \((D \cup A)^\circ \times \Sigma^\circ\) is not \( A_2\)-universal.

**Proof.** We will slightly modify the proof of Theorem 1. Let \( s = Q \setminus \Sigma \). Assume that the space \((D \cup A)^\circ \times \Sigma \) is \( A_2\)-universal. Then, since \( Q^\circ \setminus \Sigma^\circ \in A_2 \), there is a closed embedding \( f: Q^\circ \setminus \Sigma^\circ \to (D \cup A)^\circ \times \Sigma \). We consider the space \((D \cup A)^\circ \times \Sigma \) to be a subset of the compactum \( \overline{D}^\circ \times Q \). According to Lavrentiev Theorem, there exists an embedding \( \tilde{f} : G \to \overline{D}^\circ \times Q \) of some \( G_\sigma\)-set \( G \), \( Q^\circ \setminus \Sigma^\circ \subset G \subset Q^\circ \), extending the embedding \( f \). Since \( Q^\circ \setminus \Sigma^\circ \) is dense in \( G \) and \( f(Q^\circ \setminus \Sigma^\circ) \) is closed in \((D \cup A)^\circ \times \Sigma \), we have \( \tilde{f}(G \setminus (Q^\circ \setminus \Sigma^\circ)) \subset (\overline{D}^\circ \times Q) \setminus ((D \cup A)^\circ \times \Sigma) = (\overline{D}^\circ \setminus (D \cup A)^\circ) \times Q \cup (\overline{D}^\circ \times (Q \setminus \Sigma)) \). Notice that

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$\overline{D}^\omega \times (Q \setminus \Sigma) = \overline{D}^\omega \times s$ is an absolute $G_\delta$-set. Consequently, the intersection $\overline{f}(G) \cap (\overline{D}^\omega \times s)$ is also an absolute $G_\delta$-set. Moreover, since $\overline{f}(Q^\omega \setminus \Sigma^\omega) \subseteq \overline{D}^\omega \times \Sigma$, we have $\overline{f}(G) \cap (\overline{D}^\omega \times s) \subseteq \overline{f}(G \cap \Sigma^\omega)$. Let us show that the space $G \cap \Sigma^\omega$ is of the first Baire category. Indeed, since the complement $\Sigma^\omega \setminus G = Q^\omega \setminus G$ is sigma-compact and the space $\Sigma^\omega$ is nowhere sigma-compact, the set $\Sigma^\omega \cap G = \Sigma^\omega \setminus (Q^\omega \setminus G)$ is dense in $\Sigma^\omega$, and consequently, in $Q^\omega$. Now, since the space $\Sigma^\omega$ is of the first Baire category [5] (§ 10, IV, 2) implies that the intersection $G \cap \Sigma^\omega$ is also of the first Baire category. By the Baire Theorem [3] (Theorem 3.9.3) and [5] (§ 10, IV, 3), the absolute $G_\delta$-set $\overline{f}(G) \cap (\overline{D}^\omega \times s)$ is nowhere dense-in $\overline{f}(G \cap \Sigma^\omega)$.

Then the set $F = \text{cl}(\overline{f}^{-1}(\overline{D} \times s)) Q^\omega$ is nowhere dense in $Q^\omega$. Using known universal properties of the couple $(Q^\omega, \Sigma^\omega)$ (see e.g. [6]), one can find a compactum $K \subseteq Q^\omega \setminus F$ such that the pair $(K, K \cap \Sigma^\omega)$ is homeomorphic to $(Q, s)$. Then $K \setminus \Sigma^\omega$ is homeomorphic to $Q \setminus s = \Sigma$. Let $(X, Y) = (K \cap G, (K \cap G) \setminus \Sigma^\omega) = (K \cap G, K \setminus \Sigma^\omega)$ (recall that $Q^\omega \setminus \Sigma^\omega \subseteq G$). Considering the restriction $g = \overline{f} | K \cap G$ we obtain the embedding $g : X \to \overline{D}^\omega \times Q$ of absolute $G_\delta$-set such that $g(Y) \subseteq (D \cup \bigcup A)^\omega \setminus \Sigma$, $g(X \setminus Y) \subseteq (\overline{D}^\omega \setminus (D \cup A)^\omega) \times \Sigma$, and the space $Y = K \setminus \Sigma^\omega$ is homeomorphic to $\Sigma$. Proceeding by analogy with the proof of Theorem 1 we obtain a contradiction.


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