UDC 512.54

B. AMBERG, Prof. (Univ. Mainz, Germany),
S. FRANCIOSI, Prof. (Univ. Salermo, Italy),
F. DE GIOVANNI, Prof. (Univ. Napoli, Italy)

Rank formulae for faktorized groups

Формулы ранга для факторизуемых групп

The following inequalities for the torsion-free rank $r_0(G)$ of the group $G = AB$ and for the p^∞-rank $r_p(G)$ of the soluble-by-finite group $G = AB$ are stated:

$$r_0(G) \leqslant r_0(A) + r_0(B) - r_0(A \cap B),$$
$$r_p(G) \leqslant r_p(A) + r_p(B) - r_p(A \cap B).$$

Для свободного ранга $r_0(G)$ группы $G = AB$ и для p^∞-ранга $r_p(G)$ почти разрешимой группы $G = AB$ установлены следующие неравенства:

$$r_0(G) \leqslant r_0(A) + r_0(B) - r_0(A \cap B),$$
$$r_p(G) \leqslant r_p(A) + r_p(B) - r_p(A \cap B).$$

Для вільного ранга $r_0(G)$ групи $G = AB$ і для p^∞-ранга $r_p(G)$ майже розв'язної групи $G = AB$ встановлені такі нерівності:

$$r_0(G) \leqslant r_0(A) + r_0(B) - r_0(A \cap B),$$
$$r_p(G) \leqslant r_p(A) + r_p(B) - r_p(A \cap B).$$

1. Introduction. A group G has finite torsion-free rank if it has a series of finite length whose factors are either periodic or infinite cyclic. The number $r_0(G)$ of infinite cyclic factors in such a series is an invariant of G, called the torsion-free rank of G. Thus the function r_0 is constant on isomorphism classes, satisfies $r_0(H) \leqslant r_0(G)$ for every subgroup H of the group G and is additive on extensions, i. e. $r_0(G) = r_0(N) + r_0(G/N)$ for each normal subgroup N of G.

Let the group $G = AB$ with finite torsion-free rank be the product of two subgroups A and B. If one of the factors A and B is normal in G, it is clear that

$$r_0(G) = r_0(A) + r_0(B) - r_0(A \cap B).$$

Therefore it is natural to investigate the relations between the numbers $r_0(G)$, $r_0(A)$ and $r_0(B)$, when A and B are arbitrary subgroups of G. Results of this type can for instance be found in [1—5]. Our first theorem on this subject is the following.

Theorem A. *Let the group $G = AB$ be the product of two subgroups A and B. If G has finite torsion-free rank, then*

$$r_0(G) \leqslant r_0(A) + r_0(B) - r_0(A \cap B).$$

It seems to be unknown whether the inequality in Theorem A is actually an equality. This was shown to be true by Wilson [5] for a soluble-by-finite group with finite abelian section rank. Here a group is said to have finite abe-

© B. AMBERG, S. FRANCIOSI, F. DE CIOVANNI, 1991

lian section rank if it has no infinite abelian sections of prime exponent.

Of course, the same type of problems can be considered for other rank functions on a factorized group. If p is a prime, a group G has finite p^∞-rank if it has a series of finite length whose factors either are of type p^∞ or have no sections of type p^∞. The number $r_p(G)$ of factors of type p^∞ in such a series is an invariant of G, called the p^∞-rank of G. Clearly a soluble-by-finite group with finite abelian section rank has finite p^∞-rank for every prime p.

Theorem B. *Let the soluble-by-finite group $G = AB$ be the product of two subgroups A and B. If G has finite p^∞—rank, then*

$$r_p(G) \leqslant r_p(A) + r_p(B) - r_p(A \cap B).$$

Again, it is unknown whether the inequality in Theorem B is actually an equality. It was shown by Wilson [5] that this is the case for a soluble-by-finite minimax group. Our next result extends Wilson's theorem to a wider class of groups. Reccal that a soluble-by-finite group G is an S_1-group if it has finite abelian section rank and the set $\pi(G)$ of prime divisors of orders of elements of G is finite.

Theorem C. *Let the soluble-by-finite group $G = AB$ with finite abelian section rank be the product of two subgroups A and B. If at least one of the sets $\pi(A)$ and $\pi(B)$ is finite, then*

$$r_p(G) = r_p(A) + r_p(B) - r_p(A \cap B)$$

for every prime p. In particular, the p^∞-rank equality holds if $G = AB$ is an S_1-group.

Most of our notation is standard and can be found in [6].

2. Rank inequalities. A map μ assigning to each group G either a non-negative integer $\mu(G)$ or ∞ is called an additive function if it is constant on isomorphism classes, satisfies $\mu(H) \leqslant \mu(G)$ for every subgroup H of the group G and $\mu(G) = \mu(N) + \mu(G/N)$ whenever N is a normal subgroup of G. The additive function μ is of infinite type if $\mu(E) = 0$ for every finite group E, but there exists a countable abelian group U such that $\mu(U) \neq 0$. If μ is an additive function of infinite type, it is clear that $\mu(G) = \infty$ for some countable abelian group G. Examples of additive functions of infinite type on groups are given by the rank functions r_0 and r_p for every prime p. Thus Theorems A and B will be obtained as special cases of a result concerning additive functions of infinite type on factorized groups.

Let μ be an additive function. We shall say that the μ-inequality holds for the factorized group $G = AB$ if

$$\mu(G) \leqslant \mu(A) + \mu(B) - \mu(A \cap B).$$

Similarly, the factorized group $G = AB$ satisfies the μ-*equality* if

$$\mu(G) = \mu(A) + \mu(B) - \mu(A \cap B).$$

Our first two lemmas were already proved in [1] for the torsion-free rank.

Lemma 1. *Let group $G = AB$ be the product of two subgroups A and B. If μ is an additive function and A contains a normal subgroup N of G such that the factor-group $G/N = (A/N)(BN/N)$ satisfies the μ-equality (respectively: the μ-inequality), then also $G = AB$ satisfies the μ-equality (respectively: the μ-inequality).*

Proof. Suppose that the μ-equality holds for the factor-group $G/N = (A/N)(BN/N)$. Since $(A \cap BN)/N \simeq (A \cap B)/(N \cap B)$, it follows that

$$\mu(G) = \mu(N) + \mu(G/N) = \mu(N) + \mu(A/N) + \mu(BN/N) - \mu((A \cap BN)/N) =$$
$$= \mu(A) + \mu(B) - \mu(B \cap N) - \mu(A \cap B) + \mu(N \cap B) =$$
$$= \mu(A) + \mu(B) - \mu(A \cap B).$$

The proof for the μ-inequality is similar.

The following lemma will be used to reduce the proofs of our theorems to triply factorized groups. Recall that if N is a normal subgroup of a factorized group $G = AB$, the factorizer $X(N)$ of N in G is the subgroup $AN \cap BN$. It is well-known that $X(N)$ has the triple factorization

$$X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN).$$

Lemma 2. *Let the group $G = AB$ be the product of two subgroups A and B. If μ is an additive function and N is a normal subgroup of G such that the μ-equalities (respectively: the μ-inequalities) hold for the factorizer $X(N) = (A \cap BN)(B \cap AN)$ and for the factor-group $G/N = (AN/N)(BN/N)$, then also $G = AB$ satisfies the μ-equality (respectively: the μ-inequality).*

Proof. Suppose that the μ-equalities hold for the factorizer $X = X(N)$ and for the factor-group G/N. Since $X/N \simeq (A \cap BN)/(A \cap N) \simeq (B \cap AN)/(B \cap N)$, it follows that

$$\mu(G) = \mu(G/N) + \mu(N) = \mu(AN/N) + \mu(BN/N) - \mu(X/N) + \mu(X) - \mu(X/N) =$$
$$= \mu(A) - \mu(A \cap N) + \mu(B) - \mu(B \cap N) - 2\mu(X/N) + \mu(A \cap BN) +$$
$$+ \mu(B \cap AN) - \mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

The proof for the μ-inequality is similar.

The next lemma on triply factorized groups was proved by Wilson in [5]. We give here a shorter proof.

Lemma 3. *Let the soluble-by-finite group $G = AB = AK = BK$ be the product of two subgroups A and B and a torsion-free abelian normal subgroup $K \neq 1$ such that $A \cap K = B \cap K = 1$ and $C_G(K) = K$. If G has finite torsion-free rank; then it cannot act rationally irreducibly on K.*

Proof. Since A is a linear group over the field of rational numbers, its periodic subgroups are finite (see [6, p. 85], Pt. 1), so that the set of primes $\pi(G)$ is finite and G has finite Prüfer rank (see [6], Pt. 2, Lemma 9.34). Then G is an S_1-group and has no non-trivial periodic normal subgroups since $C_G(K) = K$. Therefore the Fitting subgroup F of G is nilpotent and G/F is a finitely generated abelian-by-finite group (see [6, p. 169], Pt. 2]).

Assume that G acts rationally irreducibly on K. If $[K, F] \neq 1$, then the factor group $K/[K, F]$ is periodic. Since F is nilpotent, there exists a positive integer i such that $[K, \underbrace{F, \ldots, F}_{i}] = 1$, so that K is periodic. This contradiction shows that $[K, F] = 1$, so that $F \leqslant C_G(K) = K$ and $K = F$. Therefore A and B are finitely generated abelian-by-finite groups. In particular G is finitely generated, and hence nilpotent-by-finite by a theorem of Zaĭcev (see [7], Theorem 2). Thus G/K if finite, so that A and B are finite. It follows that $G = AB$ is finite, and this contradiction proves the lemma.

It is well-known that a group with finite torsion-free rank has a normal series of finite length whose factors are either periodic or torsion-free abelian groups of finite rank. Therefore Theorem A is special case of the following result.

Theorem 1. *Let the group $G = AB$ be the product of two subgroups A and B, and let μ be an additive function of infinite type such that $\mu(G)$ is finite. If G has a normal series of finite length whose factors are eithers torsion-free abelian groups or abelian groups with the minimal condition or groups on which μ is zero, then the μ-inequality holds for $G = AB$.*

Proof. Since the additive function μ is of infinite type and $\mu(G)$ is finite, G has no free abelian sections of infinite rank, and in particular every torsion-free abelian section of G has finite rank. If Σ is a normal series of finite length of G whose factors are either torsion-free abelian groups or abelian groups with the minimal condition or groups on which μ is zero, $\mu_0(\Sigma)$ denotes the sum of the ranks of the torsion-free abelian factors of Σ on which μ is not zero. We shall denote by $\mu_0(G)$ the minimum of all $\mu_0(\Sigma)$'s. The length of a shortest normal series Σ of G for which $\mu_0(E) = \mu_0(G)$ will be denoted by $\mu_1(G)$. It is clear that $\mu_0(H) \leqslant \mu_0(G)$ and $\mu_1(\Sigma) \leqslant \mu_1(G)$ for every subgroup H of G. Moreover, if N is a normal subgroup and U/V is a torsion-free abelian normal section of G, the torsion subgroup of UN/VN is the direct product of a G-invariant subgroup satisfying the minimal condition and a G-invariant subgroup on which μ is zero. Hence $\mu_0(G/N) \leqslant \mu_0(G)$, and if $\mu_0(G/N) = \mu_0(G)$, then $\mu_1(G/N) \leqslant \mu_1(G)$.

Assume that Theorem 1 is false, and among all the counterexamples for which $\mu_0(G)$ is minimal choose one $G = AB$ such that also $\mu_1(G)$ is minimal. Let Σ be a normal series of G of length $\mu_1(G)$ for which $\mu_0(\Sigma) = \mu_0(G)$. If K is the smallest non-trivial term of Σ, the μ-inequality holds for the factor group $G/K = (AK/K)(BK/K)$. Hence the factorizer $X(K)$ of K is also a counterexample by Lemma 2, and so we may suppose that G has a triple factorization

$$G = AB = AK = BK,$$

where K is a normal subgroup of G.

Assume first that $\mu(K) = 0$. Then

$$\mu(G) = \mu(K) + \mu(G/K) = \mu(G/K) = \mu(AK/K) = \mu(A/A(\cap K)) = \mu(A),$$

and so obviously

$$\mu(G) = \mu(A) \leqslant \mu(A) + \mu(B) - \mu(A \cap B).$$

Suppose now that K is a torsion-free abelian group such that $\mu(K) \neq 0$. The subgroup $A \cap K$ is normal in $G = AK$; and it follows from Lemma 1 that the μ-inequality does not hold for the factor-group

$$G/(A \cap K) = (A/(A \cap K))(B(A \cap K)/(A \cap K)),$$

so that $A \cap K = 1$. The centralizer $C_A(K)$ is normal in G, and also the group

$$G/C_A(K) = (A/C_A(K))(BC_A(K)/C_A(K))$$

is a counterexample by Lemma 1. Replacing G by $G/C_A(K)$, we may suppose that $C_A(K) = 1$. Therefore A is isomorphic with a group of automorphisms of K, and so is linear over the field of rational numbers. Since G has no free abelian sections of infinite rank, we obtain that A is soluble-by-finite by a theorem of Tits (see [8, p. 145]). Then also G is soluble-by-finite. As the μ-inequality does not hold for the group

$$G/(B \cap K) = (A(B \cap K)/(B \cap K))(B/(B \cap K)),$$

the intersection $B \cap K$ must be trivial, and Lemma 3 shows that G does not act rationally irreducibly on K. Let L be a proper non-trivial G-invariant subgroup of K such that K/L is torsion-free. Clearly $\mu_0(G/L) < \mu_0(G)$, and thus the μ-ineaquality holds for $G/L = (AL/L)(BL/L)$. Consider the factorizer $X(L)$ of L in $G = AB$. Since $X(L) \cap K = L(A \cap BL) \cap K = L$, we have that $X(L)/L$ is isomorphic with a subgroup of G/K, and hence $\mu_0(X(L)) < \mu_0(G)$. Therefore the μ-inequality holds for the factorized group $X(L) = (A \cap BL) \cdot (B \cap AL)$, and Lemma 2 proves that also $G = AB$ satisfies the μ-inequality. This contradiction shows that K cannot be torsion-free.

Suppose finally that K is an abelian group satisfying the minimal condition. Then there exists a finite G-invariant subgroup E of K such tha K/E is radicable. Since $\mu(E)=0$ it is clear that the μ-inequality does not hold fort the factor-group $G/E=(AE/E)(BE/E)$, and without loss of generality it can be assumed that K is radicable. Let M be an infinite G-invariant subgroup of K with minimal total rank. Then M is radicable, and by induction on the total rank of K the μ-inequality holds for the group $G/M = (AM/M)(BM/M)$. It follows from Lemma 2 that the factorizer $X(M)$ of M in $G = AB$ is also a counterexample, so that $M=K$. Therefore each proper G-invariant subgroup of K is finite, and in particular K is a p-group for some prime p. The factor-group

$$G/A_G = (A/A_G)(BA_G/A_G)$$

is also a counterexample by Lemma 1, and hence we may suppose that there are no nontrivial normal subgroups of G contianed in A. It follows that $A \cap K = C_A(K) = 1$, and so A is isomorphic with a group of automorphisms of K. Since K has no infinite proper A-invariant subgroups, A is an irreducible linear group (see [9], Lemma 5). Moreover A has no free abelian sections of infinite rank, so that it is soluble-by-finite (see [8, p. 145]), and hence even abelian-by-finite (see [6, p. 75], Pt. 1). This is a contradiction by Proposition 1 of [5]. The theorem is proved.

Theorem B also is an easy consequence of Theorem 1.

P r o o f o f T h e o r e m B. Let W be an abelian section of G, and let P be the Sylow p-subgroup of W. Then $P = D \times R$, where D is a radicable

p-group satisfying the minimal condition and R is reduced. A basic subgroup S of R has no radicable quotients of infinite rank, since G has finite p^∞-rank. Hence S has finite exponent (see [10, p. 91], Vol. I), so that $R=S$, and in particular R has no sections of type p^∞. This agrument shows that the derived series of the soluble radical of G can be refined to a normal series of finite length of G whose factors are either torsion-free abelian groups or abelian groups with the minimal condition or groups without section of type p^∞. Application of Theorem 1 completes the proof of Theorem B.

3. Rank equalities. In order to prove Theorem C we need the following two technical lemmas.

Lemma 4. *Let G be a group with finite p^∞-rank for a certain prime p, and let H be a subgroup of G such that for every element x of G there exists a positive integer $m = m(x)$ prime to p for which $x^m \in H$. Then $r_p(H) = r_p(G)$.*

Proof. Let K/L be a normal section of type p^∞ of G, and assume that $(H \cap K)/(H \cap L)$ has finite order p^n. If x is an element of K, there exists a positive integer m prime to p such that $x^m \in H$. Then x^{mp^n} belongs to $H \cap L$, and hence x^{p^n} belongs to L. This contradiction shows that $(H \cap K)/(H \cap L)$ must be infinite, and so of type p^∞. It follows that $r_p(H) = r_p(G)$.

Lemma 5. *Let G be a locally nilpotent group whose commutator subgroup G' is periodic and has no elements of order p, for a certain prime p. If a and b are elements of G such that $a = bx$, where x is an element whose order is finite and prime to p, then there exists a positive integer m prime to p such that $a^m = b^m$.*

Proof. It is clearly enough to prove the lemma for the subgroup $\langle a, b\rangle$, so that we may suppose that G is a finitely generated nilpotent group. In particular G' is a finite group whose order n is prime to p. Obviously we may also assume that G is not abelian. Then $G/(G' \cap Z(G))$ has nilpotency class less than G, and by induction there exists a positive integer k prime to p such that $a^k = b^k u$, where u belongs to $G' \cap Z(G)$. Therefore

$$a^{kn} = (b^k u)^n = b^{kn} u^n = b^{kn},$$

and the lemma is proved, since kn is prime to p.

It is well-known that a soluble-by-finite S_1-group is hypercentral-by-polycyclic-by-finite (see for instance [11], Corollary 2.4). Thus Theorem C will follow from our next result.

Theorem 2. *Let the soluble-by-finite group $G = AB$ with finite abelian section rank be the product of two subgroups A and B. If at least one of the subgroups A and B is hypercentral-by-polycyclic-by-finite, then*

$$r_p(G) = r_p(A) + r_p(B) - r_p(A \cap B)$$

for every prime p.

Proof. Assume that the theorem is false, and among the counterexamples with minimal torsion-free rank consider those for which the p^∞-rank is minimal. Choose finally one of these $G = AB$ having a normal series Σ of minimal length whose factors are either torsion-free abelian groups (of finite rank) or radicable abelian p-groups (with the minimal condition) or periodic abelian groups without elements of order p or finite groups. If K is the smallest non-trivial term of Σ, the p^∞-rank equality holds for the factor-group $G/K = (AK/K)(BK/K)$, and Lemma 2 shows that the factorizer $X(K) = (A \cap BK)(B \cap AK)$ of K is also a counterexample. Therefore we may suppose that G has a triple factorization

$$G = AB = AK = BK,$$

where K is normal in G. If K is either a torsion-free abelian group or a radicable abelian p-group, then a contradiction can be obtained as in the proof of Theorem 1. Assume that K is finite. Then A and B have finite indices is G, so that also $A \cap B$ has finite index in G. It follows that $r_p(G) = r_p(A) = r_p(B) = r_p(A \cap B)$, and hence clearly

$$r_p(G) = r_p(A) + r_p(B) - r_p(A \cap B).$$

This contradiction proves that K must be a periodic abelian group without elements of order p.

Suppose that A is hypercentral-by-polycyclic-by-finite. As the factor-group

$$G/B_G = (AB_G/B_G)(B/B_G)$$

is also a counterexample by Lemma 1, we may assume that B contains no non-trivi al normal subgroups of G. In particular $B \cap K = 1$, and hence B is hyperc entral-by-polycyclic-by-finite. Application of Theorem B of [11] yields that also the group G is hypercentral-by-polycyclic-by-finite. Let N be a hypercentral normal subgroup of G such that the factor-group G/N is polycyclic-by-finite. As KN is also hypercentral, N can be chosen containing K. Then $N = K(A \cap N) = K(B \cap N)$, so that $r_p(N) = r_p(A \cap N) = r_p(B \cap N)$, since K has no elements of order p. Let a be an element of $A \cap N$, and write $a = bx$, where $b \in B \cap N$ and $x \in K$. Clearly $\langle a, b\rangle$ is a nilpotent group whose commutator subgroup $\langle a, b\rangle'$ is contained in K, and so has finite order prime to p. By Lemma 5 there exists a positive integer m prime to p such that $a^m = b^m$, so that a^m belongs to $A \cap B \cap N$. Thus it follows from Lemma 4 that $r_p(A \cap N) = r_p(A \cap B \cap N)$, and hence $r_p(N) \leqslant r_p(A \cap B)$. Since the factor-group G/N is polycyclic-by-finite, we have also that $r_p(G) = r_p(N)$. Therefore

$$r_p(G) = r_p(A) = r_p(B) = r_p(A \cap B),$$

and this contradiction proves the theorem.

Is should be noted that the hypotheses of Theorem C can be weakened, assuming that the soluble-by-finite group G has finite p^∞-rank and at least one of the factors A and B is a soluble S_1-group. In fact, in this situation, one can quickly reduce to the case of a triply factorized group

$$G = AB = AK = BK,$$

where K is an abelian normal subgroup of G such that $A \cap K = B \cap K = 1$, and both the subgroups A and B are hypercentral-by-polycyclic and have finite abelian section rank. Thus it follows from a recent result of Sysak [12] and Wilson [13] that also the soluble group G has finite abelian section rank, and hence Theorem 2 can be applied.

Our last result gives another condition under which the p^∞-rank equality holds.

Theorem 3. *Let the soluble-by-finite group $G = AB$ with finite abelian section rank be the product of two subgroups A and B. If at least one of the subgroups A and B is periodic by-polycyclic-by-finite, then*

$$r_p(G) = r_p(A) + r_p(B) - r_p(A \cap B)$$

for every prime p.

Proof. Assume that the theorem is false. As in the proof of Theorem 2 it can be assumed that G has a triple factorization

$$G = AB = AK = BK,$$

where K is a periodic abelian normal subgroup of G having no elements of order p. As one of the factors A and B is periodic-by-polycyclic-by-finite, it follows that also G is periodic-by-polycyclic-by-finite. The factorized group

$$G/A_G = (A/A_G)(BA_G/A_G)$$

is a counterexample, so that we may suppose that A contains no non-trivial normal subgroups of G, and in particular $C_A(K) = 1$. Let T be a periodic normal subgroup of G such that G/T is polycyclic-by-finite, and put $A_0 = A \cap T$. For each prime number q, the q-component K_q of K is an abelian group satisfying the minimal condition, so that its periodic group of automorphisms $A_0/C_{A_0}(K_q)$ is finite (see [6, p. 85], Pt. 1). On the other hand

$$\bigcap_q C_{A_0}(K_q) = C_{A_0}(K) = 1,$$

and hence A_0 is residually finite. Thus the Sylow subgroups of A_0 are finite, since G has finite abelian section rank, and so A_0 has no sections which are infinite p-groups. Moreover A/A_0 is polycyclic-by-finite and $G = AK$, where K is a periodic normal subgroup without elements of order p, so that also G has no infinite sections which are p-groups. In particular $r_p(G) = 0$, and this contradiction proves the theorem.

1. *Amberg B.* Factorizations of infinite soluble groups // Rocky Mountain J. Math.— 1977.— 7. N 1.— P. 1—17.
2. *Amberg B.* Produkte von Gruppen mit endlichem torsionfreiem Rang // Arch. Math.— 1985.— **45**, N 4.— P. 398—406.
3. *Černikov N. S.* Products of groups of finite torsion-free rank// Groups and Systems of Subgroups. —Kiev, 1983.— P. 42—56.
4. *Robinson D. J. S.* Soluble products of nilpotent groups / J. Algebra.—1986.— **98**, N 1.— P. 183—196.
5. *Wilson J. S.* On products of soluble groups of finite rank / Comment. Math. Helv.— 1985.— **60**, N 3.— P. 337—353.
6. *Robinson D. J. S.* FinitenessConditions and Generalized Soluble Groups.— Berlin: Springer, 1972.—464 p.
7. *Zaičev D. I.* Factorizations of polycyclic groups // Mat. Zametki.—1981.—**29**, N 4.—P.481—490; Math. Notes.— 1981.— **29**.— P. 247—252.
8. *Wehrfritz B. A. F.* Infinite Linear Groups.— Berlin: Springer, 1973.—229 p.
9. *Newell M. L.* Supplements in abelian-by-nilpotent groups // J. London Math. Soc.— 1975.— **11**, N 1.— P. 74—80.
10. *Fuchs L.* Infinite Abelian Groups.— New York: Acad. Press, 1970.
11. *Amberg B., Franciosi S., Giovanni F.* Triply factorized groups//Comm. Algerba.— 1990.— 18.— P. 789—809.
12. *Sysak Y. P.* Radical modules over groups of finite rank.— Kiev, 1989.— (Preprint / In-te Math. Acad. Sci. UkrSSR; 89.18).
13. *Wilson J. S.* Soluble groups which are products of groups of finite rank // To appear.

Received 31.01.91