Algebra and Discrete Mathematics
Volume 12 **(2011)**. Number 2. pp. 1 – 24

RESEARCH ARTICLE

Generalized multiplicative bases for one-sided bimodule problems

Vyacheslav Babych, Nataliya Golovashchuk and Sergiy Ovsienko

Communicated by A. V. Zhuchok

Abstract. We consider a class of normal bimodule problems satisfying some structure, triangularity and finiteness conditions. For a bimodule problem from this class we construct explicitly an analogue of multiplicative basis which we call quasi multiplicative.

Introduction

A classification of bimodule problems of finite and tame representation type and their indecomposable representations and description of their representation categories belongs to important problems of representation theory [6, 7, 8]. A useful tool for a solution of the finiteness problem is so called "covering method" ([5, 3]), which is especially effective when the basis of associative algebra ([4]) or bimodule problem ([9]) is multiplicative. We give a generalization of the notion of a scalarly multiplicative basis from [9] and apply it for a wider class of bimodule problems. For a faithful bimodule problem from our class we construct explicitly the quasi multiplicative basis using mainly geometrical techniques.

1. Preliminaries

Let $\Bbbk$ be algebraically closed field. Unless otherwise stated, all the categories we consider are the categories over $\Bbbk$, all morphism spaces are finite dimensional, and all functors are $\Bbbk$-linear. A category K is called *local*,

2000 Mathematics Subject Classification: *16D90, 16P60, 15A04, 16G20.*
Key words and phrases: *bimodule problem, quasi multiplicative basis.*

provided for every $X \in \operatorname{Ob} \mathsf{K}$ the endomorphism algebra $\mathsf{K}(X,X)$ is local, and *regular*, if, in addition, every invertible morphism is automorphism. A category K is called *fully additive* or *Krull-Schmidt category* if K is a category with finite direct sums and every idempotent from K splits, i. e. it has kernel and cokernel. A full subcategory $\mathsf{K}_0 \subset \mathsf{K}$ will be called *an additive skeleton* of K, provided K_0 is regular and every $X \in \operatorname{Ob} \mathsf{K}$ is isomorphic to a finite direct sum of objects from K_0.

For a local category K and for every $X \in \operatorname{Ob} \mathsf{K}$ there exists the decomposition $\mathsf{K}(X,X) = \Bbbk \mathbb{1}_X \oplus \operatorname{Rad} X$, where $\operatorname{Rad} X$ is the Jacobson radical of the algebra $\mathsf{K}(X,X)$. If K is regular, then we denote by $\operatorname{Rad} \mathsf{K}$ the *radical* of K, i. e. an ideal in K such that $\operatorname{Rad} \mathsf{K}(X,Y) = \mathsf{K}(X,Y)$ for $X \neq Y$, and $\operatorname{Rad} \mathsf{K}(X,X) = \operatorname{Rad} X$, $X, Y \in \operatorname{Ob} \mathsf{K}$.

Let V be a K-bimodule ([1]). A category K (a bimodule V) is called *locally finite dimensional*, if for any $X \in \operatorname{Ob} \mathsf{K}$ the spaces $\bigoplus\limits_{Y \in \operatorname{Ob} \mathsf{K}} \mathsf{K}(X,Y)$ and $\bigoplus\limits_{Y \in \operatorname{Ob} \mathsf{K}} \mathsf{K}(Y,X)$ ($\bigoplus\limits_{Y \in \operatorname{Ob} \mathsf{K}} \mathsf{V}(X,Y)$ and $\bigoplus\limits_{Y \in \operatorname{Ob} \mathsf{K}} \mathsf{V}(Y,X)$) are finite dimensional, and *finite dimensional*, provided the spaces $\bigoplus\limits_{X,Y \in \operatorname{Ob} \mathsf{K}} \mathsf{K}(X,Y)$ ($\bigoplus\limits_{X,Y \in \operatorname{Ob} \mathsf{K}} \mathsf{V}(X,Y)$) are finite dimensional.

Given a category K, we denote by $\operatorname{add} \mathsf{K}$ an *additive hull* of K, i. e. a minimal fully additive category which contains K. For a K-bimodule V, we denote by $\operatorname{add} \mathsf{V}$ the corresponding $\operatorname{add} \mathsf{K}$-bimodule.

A pair $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ consisting of a category K and a K-bimodule V is called a *bimodule problem* over K or shortly bimodule problem. A bimodule problem $\mathcal{A}$ will be called *normal*, provided the category K is regular, and both K and V are locally finite dimensional. All the bimodule problems we will consider are assumed to be normal. Given some $S \subset \operatorname{Ob} \mathsf{K}$ denote by K_S the full subcategory of K with $\operatorname{Ob} \mathsf{K}_S = S$, and by V_S the subbimodule $\mathsf{V}_S = \mathsf{K}_S \mathsf{V} \mathsf{K}_S$. A bimodule problem $\mathcal{A}_S = (\mathsf{K}_S, \mathsf{V}_S)$ is called the *restriction* of $\mathcal{A}$ to S.

For a bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$, *a representation* M of $\mathcal{A}$ is a pair $M = (M_{\mathsf{K}}, M_{\mathsf{V}})$, where $M_{\mathsf{K}} \in \operatorname{Ob} \operatorname{add} \mathsf{V} = \operatorname{Ob} \operatorname{add} \mathsf{K}$ and $M_{\mathsf{V}} \in \operatorname{add} \mathsf{V}(M_{\mathsf{K}}, M_{\mathsf{K}})$. If M, N are two representations of $\mathcal{A}$, then a *morphism* f from M to N is a morphism $f \in \operatorname{add} \mathsf{K}(M_{\mathsf{K}}, N_{\mathsf{K}})$ such that $N_{\mathsf{V}} \cdot f - f \cdot M_{\mathsf{V}} = 0$. The composition of morphisms and the unit morphisms in the representation category $\operatorname{rep} \mathcal{A}$ and in the category $\operatorname{add} \mathsf{K}$ coincide.

Given two bimodule problems $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ and $\mathcal{A}' = (\mathsf{K}', \mathsf{V}')$, a *morphism* of bimodule problems $\theta : \mathcal{A} \to \mathcal{A}'$ is a pair $\theta = (\theta_0, \theta_1)$, where $\theta_0 : \mathsf{K} \to \mathsf{K}'$ is a $\Bbbk$-functor, $\theta_1 : \mathsf{V} \to \mathsf{V}'$ is a K-bimodule morphism with the K-bimodule structure on V' induced by θ_0 ([1]).

Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a normal bimodule problem. Bigraph $\Sigma (= \Sigma_{\mathcal{A}}) = (\Sigma_0, \Sigma_1)$ is called a *basis* of the bimodule problem $\mathcal{A}$, if $\Sigma_0 = \operatorname{Ob} \mathsf{K}$,

$\Sigma_1^0(X,Y)$ is a basis of $\mathsf{V}(X,Y)$, and $\Sigma_1^1(X,Y)$ is a basis of $\operatorname{Rad}\mathsf{K}(X,Y)$, $X,Y \in \operatorname{Ob}\mathsf{K}$. For all $x,y \in \Sigma_1$ such that the product xy is not specified, we assume $xy = 0$. A bimodule problem is called connected, if its bigraph is connected.

Let V be a K-bimodule. We say that $x \in \operatorname{Rad}\mathsf{K}(X,Y)$ *annihilates the bimodule* V, if $xa = 0$, $bx = 0$ for any $Z \in \operatorname{Ob}\mathsf{K}$, $a \in \mathsf{V}(Z,X)$, $b \in \mathsf{V}(Y,Z)$. The ideal of the category K consisting of all elements annihilating the bimodule V is called the *annihilator* of V and is denoted by $\operatorname{Ann}_{\mathsf{K}}(\mathsf{V})$. A bimodule V is called *faithful* provided $\operatorname{Ann}_{\mathsf{K}}(\mathsf{V}) = 0$. We call a bimodule problem $\mathcal{A} = (\mathsf{K},\mathsf{V})$ *faithful*, if the bimodule V is faithful. For a bimodule problem $\mathcal{A}$, a *faithful part* of $\mathcal{A}$ is defined as the faithful bimodule problem $\mathcal{A}_{\mathrm{red}}$, $\mathcal{A}_{\mathrm{red}} = (\mathsf{K}_{\mathrm{red}},\mathsf{V})$, where $\mathsf{K}_{\mathrm{red}} = \mathsf{K}/\operatorname{Ann}_{\mathsf{K}}\mathsf{V}$. Remark that a restriction of faithful bimodule problem may not be faithful. Faithful part of restriction of a bimodule problem is called *faithful restriction*.

Let $\mathcal{A} = (\mathsf{K},\mathsf{V})$ be a bimodule problem, and $\mathsf{V}' \subset \mathsf{V}$ be a subbimodule of V such that $\mathsf{V}' \neq 0$, $\mathsf{V}' \neq \mathsf{V}$. Denote by $\succ$ the minimal relation of (strict) partial order on the set of bimodule problems such that $\mathcal{A} \succ \mathcal{A}'$ and $\mathcal{A} \succ \mathcal{A}''$, where $\mathcal{A}' \simeq (\mathsf{K},\mathsf{V}')$, $\mathcal{A}'' \simeq (\mathsf{K},\mathsf{V}/\mathsf{V}')$, and $\mathcal{A} \succ \mathcal{A}_S$ for any proper subset $S \subset \operatorname{Ob}\mathsf{K}$. Similarly we denote by $\sim$ the minimal equivalence such that for every ideal $\mathcal{I} \subset \mathsf{K}$, $\mathcal{I}\mathsf{V} = \mathsf{V}\mathcal{I} = 0$, holds $\mathcal{A} \sim \mathcal{A}_{\mathcal{I}} = (\mathsf{K}/\mathcal{I},\mathsf{V})$, and if $\mathcal{A} \simeq \mathcal{B}$ then $\mathcal{A} \sim \mathcal{B}$. The transitive closure of $\succ$ and $\sim$ defines a preorder on the set of bimodule problems, which defines the strict order, denoted again by $\succ$. The relations $\succ$ and $\sim$ are obviously defined on the set of isoclasses of bimodule problems. If for bimodule problems $\mathcal{A}$, $\mathcal{B}$ holds $\mathcal{A} \succ \mathcal{B}$ then we say $\mathcal{B}$ is a *subproblem* of $\mathcal{A}$.

Let $\mathcal{A} = (\mathsf{K},\mathsf{V})$ be a normal bimodule problem, $\mathsf{R} = \operatorname{Rad}\mathsf{K}$, Σ be a basis of $\mathcal{A}$. Radical R is called *nilpotent* if $\mathsf{R}^n = 0$ for some $n \in \mathbb{N}$. The integer N is called the *nilpotence degree* of $\mathcal{A}$ if $\mathsf{R}^N = 0$, but $\mathsf{R}^{N-1} \neq 0$. Denote by $\mathsf{V}_i = \mathsf{R}^{i-1}\mathsf{V}$, $i = 1,\dots,N$. We have two filtrations

$$\mathsf{R} \supset \mathsf{R}^2 \supset \ldots \supset \mathsf{R}^{N-1} \supset 0, \qquad \mathsf{V}_1 \supset \mathsf{V}_2 \supset \ldots \supset \mathsf{V}_N \supset 0. \tag{1}$$

Remark that all inclusions in (1) are strict, and for $\mathcal{A}$ faithful $V_i \neq 0$ for all $i = 1,\dots,N$.

The map $h : \mathsf{R} \cup \mathsf{V} \to \mathbb{N}$ such that $h(x) = \max\{i \in \mathbb{N} \mid x \in \mathsf{R}^i \cup \mathsf{V}_i\}$ is called the *height* of an element. Let $h(0) = \infty$. Then $h(xy) \geqslant h(x) + h(y)$ and $h(x+y) \geqslant \min\{h(x),h(y)\}$ for $x,y \in \mathsf{R} \cup \mathsf{V}$. Let $\Sigma_1^{k(i)} = \Sigma_1^k \cap h^{-1}(i)$, $i = 1,\dots,N$, $k = 0,1$. Clearly, the set $\{\Sigma_1^{k(i)}, i = 1,\dots,N\}$ is a partition of Σ_1^k, $k = 0,1$.

Definition 1. *The basis Σ of bimodule problem $\mathcal{A}$ we call triangled (with respect to the filtration (1)), if $\bigcup\limits_{l=i}^{N-1} \Sigma_1^{1(l)}$ is a basis of R^i and $\bigcup\limits_{l=i}^{N} \Sigma_1^{0(l)}$ is a basis of V_i, $i = 1,\dots,N$.*

Lemma 1 ([6]). *Every normal finite dimensional bimodule problem $\mathcal{A}$ with the nilpotent radical has a triangled basis.*

Remark 1. For a triangled basis Σ of a normal bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ with nilpotent radical $\mathsf{R} = \operatorname{Rad}\mathsf{K}$, the following properties hold:

1) $\Sigma_1^{1(i)}$ is a basis of $\mathsf{R}^i/\mathsf{R}^{i+1}$ modulo R^{i+1}, $i = 1, \ldots, N-1$, $\Sigma_1^{0(i)}$ is a basis of $\mathsf{V}_i/\mathsf{V}_{i+1}$ modulo V_{i+1}, $i = 1, \ldots, N$;

2) for $x \in \mathsf{R} \cup \mathsf{V}$ the equality $x = \sum\limits_{y \in \Sigma_1} \lambda_y y$, $\lambda_y \in \Bbbk$, implies $h(y) \geqslant h(x)$ for any $y \in \Sigma_1$ with $\lambda_y \neq 0$.

Definition 2. *A normal bimodule problem $\mathcal{A}$ with nilpotent radical $\mathsf{R} = \operatorname{Rad}\mathsf{K}$ we call* admitted *if the set $\operatorname{Ob}\mathsf{K}$ can be decomposed to a disjoint union $\operatorname{Ob}\mathsf{K} = \operatorname{Ob}\mathsf{K}^+ \cup \operatorname{Ob}\mathsf{K}^-$ such that inequality $\mathsf{V}(X,Y) \neq 0$ implies $X \in \operatorname{Ob}\mathsf{K}^-$, $Y \in \operatorname{Ob}\mathsf{K}^+$, and $\mathsf{R}(X,Y) \neq 0$ implies $X, Y \in \operatorname{Ob}\mathsf{K}^+$. The property of a bimodule problem $\mathcal{A}$ to be admitted depends only on the bigraph $\Sigma_{\mathcal{A}}$, therefore we will use the notation $\Sigma_0^+ = \operatorname{Ob}\mathsf{K}^+$ and $\Sigma_0^- = \operatorname{Ob}\mathsf{K}^-$.*

Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be an admitted bimodule problem with nilpotent radical $\mathsf{R} = \operatorname{Rad}\mathsf{K}$ and a triangled basis Σ.

Remark 2. There are the decompositions

$$\mathsf{V}_i = \bigoplus_{E \in \Sigma_0^-, A \in \Sigma_0^+} \mathsf{V}_i(E, A), \quad \mathsf{R}^i = \bigoplus_{A, B \in \Sigma_0^+} \mathsf{R}^i(A, B), \quad i = 1, \ldots, N$$

of $\Bbbk$-vector spaces with the multiplications

$$\begin{aligned} &\mathsf{R}^i(A,B) \times \mathsf{V}_j(E,A) \to \mathsf{V}_{i+j}(E,B), && A, B \in \Sigma_0^+, E \in \Sigma_0^-, \\ &\mathsf{R}^i(B,C) \times \mathsf{R}^j(A,B) \to \mathsf{R}^{i+j}(A,C), && A, B, C \in \Sigma_0^+. \end{aligned}$$

Definition 3. *For $a, b \in \mathsf{V}$, we say that $a \underset{\mathsf{R}}{<} b$ (or simply $a < b$), if $b \in \mathsf{R}a$. In other words, $a < b$ if and only if $ra = b$ for some $r \in \mathsf{R}(A,B)$, in this case we write $a <_r b$. Obviously, the order $\underset{\mathsf{R}}{<}$ on V is non-reflexive and transitive. Two elements $a, b \in \mathsf{V}$ are called comparable is either $a \underset{\mathsf{R}}{<} b$ or $b \underset{\mathsf{R}}{<} a$. For $A \in \Sigma_0^+$ let $\operatorname{ord} A = \sum\limits_{E \in \Sigma_0^-} \dim_{\Bbbk} \mathsf{V}(E,A) = \sum\limits_{E \in \Sigma_0^-} |\Sigma_1(E,A)|$. It is clear, that $h(a) < h(b)$ for any $a, b \in \Sigma_1^0$, $a < b$, i. e. h is monotonous with respect to $<$.*

Lemma 2. *The admitted bimodule problems $\mathcal{A}^k = (\mathsf{K}^k, \mathsf{V}^k)$, $k = 1, ., 4$, given respectively by the following bases (see bigraphs below)*

1) $\Sigma_0^+ = \{A\}$, $\Sigma_0^- = \{E_1, E_2, E_3, E_4\}$, $\Sigma_1^0 = \{a_i : E_i \to A, i = 1, \ldots, 4\}$, $\Sigma_1^1 = \varnothing$;

2) $\Sigma_0^+ = \{A\}$, $\Sigma_0^- = \{E\}$, $\Sigma_1^0 = \{a_1, a_2 : E \to A\}$, $\Sigma_1^1 = \varnothing$*;*

3) $\Sigma_0^+ = \{A, B\}$, $\Sigma_0^- = \{E_1, E_1', E_2, E_2', E_3 = E_3'\}$, $\Sigma_1^0 = \{a_i : E_i \to A, b_i : E_i' \to B, i = 1, 2, 3\}$, $\Sigma_1^1 = \varnothing$*;*

4) $\Sigma_0^+ = \{A, B\}$, $\Sigma_0^- = \{E_1, E_2, E_3\}$, $\Sigma_1^0 = \{a_i : E_i \to A, b_i : E_i \to B, i = 1, 2, 3\}$, $\Sigma_1^1 = \{\varphi : A \to B\}$, *and* $\varphi a_i = b_i$, $i = 1, 2, 3$*;*

$$\begin{tikzcd}
& E_1 \arrow[d, "a_1"'] & \\
E_4 \arrow[r, "a_4"'] & A & E_2 \arrow[l, "a_2"'] \\
& E_3 \arrow[u, "a_3"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \\
\\
E \arrow[uu, bend left, "a_1"] \arrow[uu, bend right, "a_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E_1 \arrow[d, "a_1"'] & & E_1' \arrow[d, "b_1"] \\
A & E_3 \arrow[l, "a_3"] \arrow[r, "b_3"'] & B \\
E_2 \arrow[u, "a_2"] & & E_2' \arrow[u, "b_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& E_1 \arrow[dl, "a_1"'] \arrow[dr, "b_1"] & \\
A \arrow[rr, dashed, "\varphi"] & & B \\
E_2 \arrow[u, "a_2"] \arrow[urr, "b_2"'] & & E_3 \arrow[u, "b_3"'] \arrow[ull, "a_3"]
\end{tikzcd}$$

$\mathcal{A}^1$ $\qquad$ $\mathcal{A}^2$ $\qquad$ $\mathcal{A}^3$ $\qquad$ $\mathcal{A}^4$

are of strictly unbounded representation type (see [6]).

Definition 4. *Define the class* $\mathcal{C}$ *of admitted bimodule problems* $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ *with nilpotent radical* $\mathsf{R} = \operatorname{Rad} \mathsf{K}$ *and a triangled basis* Σ *such that for any* $E \in \Sigma_0^-$, $A, B \in \Sigma_0^+$, $A \neq B$*:*

1) $\operatorname{ord} A \leqslant 3$*;*

2) any $a_1, a_2 \in \Sigma_1^0(E, A)$ *are comparable;*

3) if $\operatorname{ord} A = \operatorname{ord} B = 3$*, then any* $a \in \Sigma_1^0(E, A)$, $b \in \Sigma_1^0(E, B)$ *are comparable;*

4) if $\varphi \in \mathsf{R}(A, B)$*, then* $\sum\limits_{E \in \Sigma_0^-} \dim_{\Bbbk} \varphi \mathsf{V}(E, A) < 3$.

If one of the conditions 1)–4) does not hold, then, using Lemma 2, it is easy to check that the bimodule problem is of strictly unbounded representation type.

Let $\mathcal{A} \in \mathcal{C}$. For any $x \in \mathsf{R} \cup \mathsf{V}$ there is a basis decomposition $x = \sum\limits_{y \in \Sigma_1} \lambda_y \, y$, where almost all $\lambda_y \in \Bbbk$ equals 0. Denote by $\operatorname{con}_y x = \lambda_y$ the content of y in x. Two nonzero elements $x, y \in \mathsf{R} \cup \mathsf{V}$ are called *collinear* if $\Bbbk^* x = \Bbbk^* y$, in this case we write $x \| y$. Given $a, b \in \Sigma_1^0$ we shall write $a \underset{\Sigma}{<} b$ and $a \underset{\xi}{<} b$ if there exists $\xi \in \Sigma_1^1$ such that $b \| \xi a$.

Definition 5. *Given* $A, B \in \operatorname{Ob} \Sigma_0^+$*, denote by* $\mathcal{A}^{(A,B)} = (\mathsf{K}^{(A,B)}, \mathsf{V}^{(A,B)})$ *the restriction of the bimodule problem* $\mathcal{A}$ *to the set*

$$S_{A,B} = \{A, B\} \cup \{E \in \Sigma_0^- \mid \mathsf{V}(E, A) \neq 0 \text{ or } \mathsf{V}(E, B) \neq 0\}.$$

Denote by $\Sigma^{(A,B)} \subset \Sigma$ *the basis of* $\mathcal{A}^{(A,B)}$ *which is the restriction of* Σ *to* $\mathcal{A}^{(A,B)}$*. We will write* $\mathcal{A}^{(A)}$ *instead of* $\mathcal{A}^{(A,A)}$ *in the case* $A = B$*. The bimodule problem* $\mathcal{A}^{(A,B)}$ *inherits the triangled structure from* $\mathcal{A}$ *(and may have the proper one).*

Remark 3. Let $A,B\in\Sigma_0^+$. If $\mathcal{A}$ is faithful, then bimodule problem $\mathcal{A}^{(A,B)}$ is faithful as well. Moreover, $\mathsf{R}(A,B)=\mathrm{Rad}(\mathsf{K}^{(A,B)})(A,B)$. This fact follows from the equality

$$\mathrm{Ann}_{\mathsf{K}}(\mathsf{V})=\bigcup_{A,B\in\mathrm{Ob}\,\Sigma_0^+}\mathrm{Ann}_{\mathsf{K}^{(A,B)}}(\mathsf{V}^{(A,B)}).$$

2. Quasi multiplicative basis. Main result

Let $\mathcal{A}=(\mathsf{K},\mathsf{V})$ be a bimodule problem from $\mathcal{C}$ with nilpotent radical $\mathsf{R}=\mathrm{Rad}\,\mathsf{K}$ and a triangled basis Σ.

A change of basis $\Sigma_1=\Sigma_1^0\cup\Sigma_1^1$ consists of a family of changes of bases in all $\mathsf{V}(E,A)$ (the change of Σ_1^0) and in all $\mathsf{R}(A,B)$ (the change of Σ_1^1), $A,B\in\Sigma_0^+$, $E\in\Sigma_0^-$. These new bases gives the new basis Σ' of $\mathcal{A}$. The change of basis from Σ to Σ' we call *triangled*, provided both Σ and Σ' are triangled.

Definition 6. *Let $x,y\in\Sigma_1^i(X,Y)$, $i=0,1$. For $\lambda_x\in\Bbbk$, $\lambda_y\in\Bbbk^*$, the change of basis from Σ to Σ' such that $y'=\lambda_y y+\lambda_x x$, and $z'=z$ for all $z\in\Sigma_1\backslash\{y\}$ we call* elementary. *An elementary change is called* correct, *if $h(x)\geqslant h(y)$. Denote by $\mathfrak{C}_\lambda(x,y)$ and $\mathfrak{C}_\lambda(y)$ elementary changes $y'=y+\lambda x$, $\lambda\in\Bbbk$, and $y'=\lambda y$, $\lambda\in\Bbbk^*$, respectively.*

The change of basis from Σ to Σ' is called standard *if it is the superposition of correct elementary changes. We use only standard changes of basis. Usually we do not modify the notations of basis and its elements after change, and write Σ and y instead of Σ' and y' respectively.*

Definition 7. *For $A,B\in\Sigma_0^+$, $E\in\Sigma_0^-$, $a\in\Sigma_1^0(E,A)$, $b\in\Sigma_1^0(E,B)$ let*

$$\begin{aligned}\mathsf{S}(a,b)&=\{\xi\in\Sigma_1^1(A,B)\mid \mathrm{con}_b(\xi a)\neq 0\},\\ \mathsf{C}(a,b)&=\{\xi\in\Sigma_1^1(A,B)\mid \xi a\|b\}\subset\mathsf{S}(a,b).\end{aligned}\tag{2}$$

A pair (a,b) is called adjusted *if $\mathsf{S}(a,b)=\mathsf{C}(a,b)$. For any $\varphi\in\Sigma_1^1$, denote $\mathrm{P}_\varphi=\{(a,b)\in\Sigma_1^0\times\Sigma_1^0\mid\varphi\in\mathsf{S}(a,b)\}$. A $\varphi\in\Sigma_1^1$ is called* single *provided $\mathrm{P}_\varphi=\{(a,b)\}$ and the pair (a,b) is adjusted, and* joint *if $\mathrm{P}_\varphi=\{(a_1,b_1),(a_2,b_2)\}$ with $a_1\neq a_2$, $b_1\neq b_2$, and the pairs (a_1,b_1), (a_2,b_2) are adjusted. Obviously, if $\mathcal{A}$ is faithful, $\mathrm{P}_\varphi\neq\varnothing$ for any $\varphi\in\Sigma_1^1$.*

Remark 4. Let $A,B\in\Sigma_0^+$, $A\neq B$, $\varphi\in\Sigma_1^1(A,B)$. If three adjusted pairs (a_i,b_i), $i=1,2,3$, lies in P_φ, then $\mathcal{A}\notin\mathcal{C}$ by Definition 4, item 4).

Lemma 3. *Let $a\in\Sigma_1^0(E,A)$, $b\in\Sigma_1^0(E,B)$, and $a<b$.*

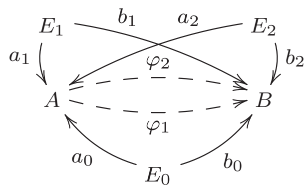
1) If $\varphi,\psi\in\mathsf{S}(a,b)$ and $h(\varphi)\geqslant h(\psi)$, then there is a correct elementary change of basis $\mathfrak{C}_\lambda(\varphi,\psi)$, $\lambda\in\Bbbk^*$, such that $\mathsf{S}'(a,b)=\mathsf{S}(a,b)\backslash\{\psi\}$.

2) There is a standard change of basis from Σ_1 to Σ'_1 on $\Sigma_1^1(A,B)$, such that $|\mathsf{S}'(a,b)|=1$.

Proof. Since $h(\varphi)\geqslant h(\psi)$, then the elementary change of basis $\psi'=\psi-\frac{\mathrm{con}_b(\psi a)}{\mathrm{con}_b(\varphi a)}\varphi$ is correct and leads to the condition $\mathsf{S}'(a,b)=\mathsf{S}(a,b)\backslash\{\psi\}$. The second item follows from the first by induction algorithm. $\square$

We call elements $\varphi_1,\varphi_2\in\Sigma_1^1(A,B)$, $A,B\in\Sigma_0^+$, $A\neq B$, *joint parallel* if $\mathrm{ord}(A)=\mathrm{ord}(B)=3$, and there are $E_0,E_1,E_2\in\Sigma_0^-$, $a_i\in\Sigma_1^0(E_i,A)$, $b_i\in\Sigma_1^0(E_i,B)$, $i=0,1,2$, such that the following hold:

1) $\mathsf{C}(a_0,b_0)=\{\varphi_1,\varphi_2\}$, $\mathsf{C}(a_i,b_i)=\{\varphi_i\}$, $i=1,2$;
2) $\mathrm{P}_{\varphi_i}=\{(a_0,b_0),(a_i,b_i)\}$, $i=1,2$ (see diagram below).



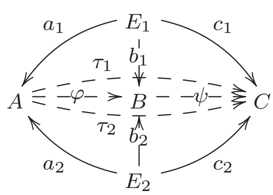
Here some of vertices E_0,E_1,E_2 may be equal, but the arrows a_0,a_1,a_2 (b_0,b_1,b_2) are pairwise different.

Given $a_1,\ldots,a_t\in\Sigma_1$, define $\Bbbk^*\langle a_1,\ldots,a_t\rangle=\big\{\sum\limits_{i=1}^{t}\lambda_i a_i\mid\lambda_i\in\Bbbk^*\big\}$.

Definition 8. *We say that the* multiplication rule *holds on $\mathcal{A}$ if given any $\varphi,\psi\in\Sigma_1^1$ with $\psi\varphi\neq 0$, one of the following conditions holds:*

1) there is $\tau\in\Sigma_1^1$ s.t. $\psi\varphi\|\tau$;

2) φ,ψ are joint, and there are single $\tau_1,\tau_2\in\Sigma_1^1$ such that $\psi\varphi\in\Bbbk^*\langle\tau_1,\tau_2\rangle$, and there are $E_1,E_2\in\Sigma_0^-$, with, possibly, $E_1=E_2$, $A,B,C\in\Sigma_0^+$, where two of the vertices A, B, C may be equal, $a_i\in\Sigma_1^0(E_i,A)$, $b_i\in\Sigma_1^0(E_i,B)$, $c_i\in\Sigma_1^0(E_i,C)$, such that $\varphi a_i\,\|\,b_i$, $\psi b_i\,\|\,c_i$, $i=1,2$, and $\tau_j a_i\|\delta_{ij}c_i$, $i,j=1,2$ where δ_{ij} is the Kronecker delta:



Definition 9. *The triangled basis Σ of a bimodule problem $\mathcal{A}\in\mathcal{C}$ is called* quasi multiplicative *if the following properties hold:*

1) Any pair $(a,b)\in\Sigma_1^0\times\Sigma_1^0$ with $\mathsf{S}(a,b)\neq\varnothing$ is adjusted.

2) Any $\varphi\in\Sigma_1^1$ with $\mathrm{P}_\varphi\neq\varnothing$ is either single or joint.

3) For any $a \in \Sigma_1^0(E,A)$, $b \in \Sigma_1^0(E,B)$, the inequality $|\mathsf{C}(a,b)| \leqslant 2$ holds. If $\mathsf{C}(a,b) = \{\varphi_1, \varphi_2\}$, then φ_1, φ_2 are joint parallel.

4) The multiplication rule holds on $\mathcal{A}$.

It is an approximation of the notion of multiplicative basis ([4], [9]).

Theorem 1 (Main result)**.** *Let $\mathcal{A}$ be a faithful connected finite dimensional bimodule problem from class $\mathcal{C}$ with nilpotent radical. Then there exists a standard change of triangled basis to a quasi multiplicative one.*

The rest of this paper is devoted to the proof of Theorem 1.

3. Bimodule problems with $|\Sigma_0^+| = 1$

Proposition 1. *Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a faithful connected bimodule problem from $\mathcal{C}$ with nilpotent radical $\mathsf{R} = \operatorname{Rad}\mathsf{K}$ and a triangled basis Σ such that $\Sigma_0^+ = \{A\}$. Then there exists a standard change of basis such that Σ becomes multiplicative and one the following conditions hold:*

1) $\Sigma_1^1(A,A) = \varnothing$;

2) $|\Sigma_1^1(A,A)| = 1$, $|\Sigma_0^-| \leqslant 2$, and there exists $E \in \Sigma_0^-$ such that $\Sigma_1^0(E,A) = \{a_1, a_2\}$, $\varphi_{12} a_1 = a_2$ for $\varphi_{12} \in \Sigma_1^1(A,A)$;

3) $2 \leqslant |\Sigma_1^1(A,A)| \leqslant 3$, $\Sigma_0^- = \{E\}$, $\Sigma_1^0(E,A) = \{a_1, a_2, a_3\}$, and $\Sigma_1^1(A,A) = \{\varphi_{12}, \varphi_{23}, \varphi_{13}\}$ with, possibly, $\varphi_{12} = \varphi_{23}$, where $\varphi_{13} = \varphi_{23}\varphi_{12}$, $\varphi_{12} a_1 = a_2$, $\varphi_{23} a_2 = a_3$, $\varphi_{13} a_1 = a_3$ (all other products are zero).

Proof. By Definition 4, $\operatorname{ord} A \leqslant 3$. Thus $\dim_{\Bbbk}\mathsf{V} \leqslant 3$ and $|\Sigma_0^-| \leqslant 3$. If $\mathsf{R} = 0$, then, obviously, $\Sigma_1^1(A,A) = \varnothing$. Assume $\mathsf{R} \neq 0$.

Consider the case $|\Sigma_0^-| = 1$. Here $\dim_{\Bbbk}\mathsf{V}_i/\mathsf{V}_{i+1} \leqslant 1$, $i = 1, 2$, $\mathsf{R}^3\mathsf{V} = \mathsf{V}_4 = 0$ and so $\mathsf{R}^3 = 0$. Let N be nilpotence degree of $\mathcal{A}$. Then $2 \leqslant N \leqslant 3$. We have $\mathsf{V}_1 \supsetneqq \mathsf{V}_2 \supsetneqq \mathsf{V}_3$ and $\dim_{\Bbbk}\mathsf{V}_i/\mathsf{V}_{i+1} = 1$, $i = 1, 2$. Let $a_i \in \Sigma_1^{0(i)}$, $i = 1, \ldots, N$.

Let $N = 2$. If $\varphi \in \Sigma_1^1$, then $\varphi a_1 = \lambda_\varphi a_2$ for some $\lambda_\varphi \in \Bbbk^*$ and $\varphi a_2 = 0$. If there is another $\psi \in \Sigma_1^1$, then $\lambda_\psi \varphi - \lambda_\varphi \psi = 0$ due to faithfulness of $\mathcal{A}$. Hence, $|\Sigma_1^1| = 1$. Applying correct elementary change $\mathfrak{C}_{\lambda_\varphi^{-1}}(\varphi)$, we obtain $\Sigma_1^1 = \{\varphi_{12}\}$ such that $\varphi_{12} a_1 = a_2$.

Let $N = 3$. Then for any $\varphi \in \Sigma_1^{1(2)}$, we have $\varphi a_1 \parallel a_3$, $\varphi a_k = 0$, $k = 2, 3$. Hence, as above, $\Sigma_1^{1(2)} = \{\varphi_{13}\}$ and $\varphi_{13} a_1 = a_3$, $\varphi_{13} a_2 = 0$. Now if $\varphi \in \mathsf{S}(a_1, a_3) \backslash \{\varphi_{13}\}$, then there exists correct elementary change $\mathfrak{C}_\lambda(\varphi_{13}, \varphi)$ such that $\operatorname{con}_{a_3}(\varphi a_1) = 0$, i. e.

$$\mathsf{S}(a_1, a_3) = \mathsf{C}(a_1, a_3) = \{\varphi_{13}\}. \tag{3}$$

Further, there is $\varphi \in \Sigma_1^{1(1)}$ such that $\varphi a_2 = \lambda_\varphi a_3$, $\lambda_\varphi \in \Bbbk^*$. For another $\psi \in \Sigma_1^{1(1)}$ with $\psi a_2 = \lambda_\psi a_3$, $\lambda_\psi \in \Bbbk^*$, we have $(\lambda_\psi \varphi - \lambda_\varphi \psi) a_2 = 0$.

Applying correct elementary changes $\mathfrak{C}_{-\lambda_\psi/\lambda_\varphi}(\varphi,\psi)$ and $\mathfrak{C}_{\lambda_\varphi^{-1}}(\varphi)$ (that does not change $\mathsf{C}(a_1,a_3)$), we obtain that $\Sigma_1^{1(1)}$ contains φ_{23} such that $\varphi_{23}a_2=a_3$ and $\varphi a_2=0$ for any $\varphi\in\Sigma_1^{1(1)}\backslash\{\varphi_{23}\}$, i. e.

$$\mathsf{S}(a_2,a_3)=\mathsf{C}(a_2,a_3)=\{\varphi_{23}\}. \tag{4}$$

Now it is obvious that $\Sigma_1^1\backslash\{\varphi_{13},\varphi_{23}\}\subset\mathsf{C}(a_1,a_2)\subset\Sigma_1^{1(1)}$. For $\varphi,\psi\in\mathsf{C}(a_1,a_2)$, $\varphi\neq\varphi_{23}$, $\mathfrak{C}_\lambda(\varphi,\psi)$ does not change (3) and (4). Then, similarly, $\mathsf{S}(a_1,a_2)=\mathsf{C}(a_1,a_2)=\{\varphi_{12}\}$. If $\varphi_{12}\neq\varphi_{23}$, then there is some $\mathfrak{C}_\lambda(\varphi_{12})$ such that $\varphi_{12}a_1=a_2$, $\varphi_{12}a_2=0$. In this case $\varphi_{23}a_1=0$. Otherwise, $\varphi_{23}a_1=\lambda a_2$. Applying correct elementary changes $\mathfrak{C}_\lambda(\varphi_{13})$ and $\mathfrak{C}_{\lambda^{-1}}(a_1)$, we obtain $\varphi_{23}a_1=a_2$, $\varphi_{23}a_2=a_3$ and $\varphi_{13}a_1=a_3$. In both cases $\varphi_{23}\varphi_{12}a_1=a_3=\varphi_{13}a_1$, and hence $\varphi_{23}\varphi_{12}=\varphi_{13}$.

It remains to consider the case $|\Sigma_0^-|>1$. If $|\Sigma_0^-|=|\Sigma_1^0|$, then $\mathsf{R}=0$. So $\Sigma_0^-=\{E_1,E_2\}$, and $|\Sigma_1^0|=3$. We have (up to renumbering) $\Sigma_1^0(E_1,A)=\{a_1,a_2\}$, $\Sigma_1^0(E_2,A)=\{a\}$. Since a_1 and a_2 are comparable by Definition 4 and $\mathcal{A}$ is faithful, then $\Sigma_1^1(A,A)=\{\varphi\}$, and $a_2\|\varphi a_1$. Therefore, we can obtain $\varphi a_1=a_2$. $\square$

4. Bimodule problems with $|\Sigma_0^+|=2$

Proposition 2. *Let $\mathcal{A}=(\mathsf{K},\mathsf{V})$ be a faithful connected bimodule problem from $\mathcal{C}$ with nilpotent radical $\mathsf{R}=\operatorname{Rad}\mathsf{K}$ and a triangled basis Σ such that $\Sigma_0^+=\{A,B\}$, $A\neq B$, and $\Sigma_0^-=\{E\}$. Then there exists a standard change $\mathfrak{C}$ of basis such that Σ becomes quasi multiplicative. If $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ are endowed with quasi multiplicative basis, then $\mathfrak{C}$ leaves these bases unchangeable.*

We give the proof by series of lemmas under conditions of Proposition 2. First of all we note that since $\mathcal{A}\in\mathcal{C}$ is a faithful connected bimodule problem, then the bimodule problems $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ are faithful connected as well by Remark 3. We assume that bases of $\mathcal{A}^{(A)}$, $\mathcal{A}^{(B)}$ are quasi multiplicative (they exist by Proposition 1). So, it is sufficient to change the basic elements from $\Sigma_1^1(A,B)\cup\Sigma_1^1(B,A)$ without changing the rest of them from Σ_1 in order to make Σ quasi multiplicative.

By Proposition 1, we can assert that $\Sigma_1^0(E,A)=\{a_1,\dots,a_{q_A}\}$, where $1\leqslant q_A=\dim_{\Bbbk}\mathsf{V}(E,A)\leqslant 3$ due to Definition 4, $\Sigma_1^1(A,A)=\{\alpha_{ij}\mid 1\leqslant i<j\leqslant q_A\}$ where, possibly, $\alpha_{12}=\alpha_{23}$ for the case $q_A=3$, and $\mathsf{C}(a_i,a_j)=\{\alpha_{ij}\}$, $\alpha_{13}=\alpha_{23}\alpha_{12}$ (if $q_A=3$), $\alpha_{ij}a_i=a_j$, $i<j$, and all other products equals zero.

Similarly, $\Sigma_1^0(E,B)=\{b_1,\dots,b_{q_B}\}$, where $1\leqslant q_B=\dim_{\Bbbk}\mathsf{V}(E,B)\leqslant 3$, $\Sigma_1^1(B,B)=\{\beta_{ij}\mid 1\leqslant i<j\leqslant q_B\}$ (possibly, $\beta_{12}=\beta_{23}$) and $\mathsf{C}(b_i,b_j)=$

$\{\beta_{ij}\}$, $\beta_{13} = \beta_{23}\beta_{12}$ (if $q_B = 3$), $\beta_{ij} b_i = b_j$, $i < j$, and all other products are zero.

Let h, h_A and h_B be heights of elements of bimodule problems $\mathcal{A}$, $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ respectively.

Remark 5. Any nonzero $a \in \mathsf{V}(E, A)$ is uniquely decomposed to a sum $a = \sum_{k=i}^{q_A} \lambda_k a_k$ with $i = h_A(a)$, $\lambda_i \in \Bbbk^*$, $1 \leqslant i \leqslant q_A$. For $a, a' \in \mathsf{V}(E, A)$, $a \underset{\mathsf{R}}{<} a'$ if and only if $h_A(a) < h_A(a')$ (see Proposition 1).

Lemma 4. *Let $a \in \mathsf{V}(E, A)$, $b \in \mathsf{V}(E, B)$, and $a \underset{\mathsf{R}}{<} b$. Then $a_u \underset{\mathsf{R}}{<} b_v$ for all u, v such that $1 \leqslant u \leqslant h_A(a)$ and $h_B(b) \leqslant v \leqslant q_B$. Besides, $\mathsf{S}(a_i, b_j) \neq 0$ if and only if $a_i \underset{\mathsf{R}}{<} b_j$.*

Proof. Denote $i = h_A(a)$, $j = h_B(b)$. Due to transitivity of $\underset{\mathsf{R}}{<}$ it is sufficient to show that $a_i \underset{\mathsf{R}}{<} b_j$. By Remark 5, we can assume $a = a_i + a'$, where $a' = 0$ or $h_A(a') > i$, and $r' a_i = a'$ for some $r' \in \mathsf{R}$. Similarly, $b = b_j + b'$, where $b' = 0$ or $h_B(b') > j$. Since $ra = b$ for some $r \in \mathsf{R}$, then $a_i <_s b$ for $s = r + rr'$, and so $a_i \underset{\mathsf{R}}{<} b$. While $b \underset{\mathsf{R}}{<} b'$, then $s' a_i = b'$ for some $s' \in \mathsf{R}$. So $a_i <_{s-s'} b_j$, and hence $a_i \underset{\mathsf{R}}{<} b_j$.

To prove second statement consider $\xi \in \mathsf{S}(a_i, b_j)$. Then $h_B(\xi a_i) \leqslant h_B(b_j) = j$. Since $a_i \underset{\mathsf{R}}{<} \xi a_i$, then $a_i \underset{\mathsf{R}}{<} b_j$ as proved above. Conversely, while $r a_i = b_j$ with $r = \sum_{\xi \in \Sigma_1^1} \lambda_\xi \xi$, then $\mathsf{S}(a_i, b_j) \neq \varnothing$. $\square$

4.1. The case $q_A = q_B = 3$

We have $\Sigma_1^0 = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ with $a_1 \underset{\alpha_{12}}{<} a_2 \underset{\alpha_{23}}{<} a_3$, $\alpha_{13} = \alpha_{23}\alpha_{12}$, and $b_1 \underset{\beta_{12}}{<} b_2 \underset{\beta_{23}}{<} b_3$, $\beta_{13} = \beta_{23}\beta_{12}$. Any $a_k \in \Sigma_1^0(E, A)$ and $b_l \in \Sigma_1^0(E, B)$, $1 \leqslant k, l \leqslant 3$, are comparable by Definition 4. Then, due to the triangularity, there is the linear order on the set Σ_1^0. Without loss of generality we can assume a_1 the smallest element. For $x, y \in \Sigma_1^0$, let $n(x, y) = |\{z \in \Sigma_1^0 \mid x < z < y\}|$. If $\alpha_{12} = \alpha_{23} = \alpha$ then $\Sigma_1^1(A, A) = \{\alpha, \alpha^2\}$, and $\alpha^3 = 0$.

Lemma 5. *If $q_A = q_B = 3$, $\alpha_{12} = \alpha_{23} = \alpha$, and $\beta_{12} = \beta_{23} = \beta$, then $n(a_1, a_2) = n(a_2, a_3) = 0$ and $n(b_1, b_2) = n(b_2, b_3) = 0$. Therefore the following case occurs: $a_1 <_\alpha a_2 <_\alpha a_3 < b_1 <_\beta b_2 <_\beta b_3$.*

Proof. If there exists $b \in \mathsf{V}(E, B)$ such that $a_i < b < a_{i+1}$, $i = 1, 2$, then due to Lemma 4, there is $b_j \in \Sigma_1^0(E, B)$ such that $a_i < b_j < a_{i+1}$, in particular, $n(a_i, a_{i+1}) > 0$.

Assume $n(a_2,a_3)>1$ (the cases $n(a_1,a_2)$, $n(b_1,b_2)$, $n(b_2,b_3)>1$ can be considered similarly). Then $a_2 <_r b_j <_\beta b_{j+1} <_s a_3$ for some $r\in \mathsf{R}(A,B)$, $s\in\mathsf{R}(B,A)$, $j\in\{1,2\}$, and we have $s\beta r=\alpha+\lambda\alpha^2\in\mathsf{R}(A,A)$. We can assume $\lambda=0$. Indeed, otherwise $s\beta r\alpha=\alpha^2$ and $s\beta r'=\alpha$ for $r'=r-\lambda r\alpha\in\mathsf{R}(A,B)$. Since $\mathsf{C}(a_2,a_3)=\mathsf{S}(a_2,a_3)=\{\alpha\}$, then there are $b',b''\in\mathsf{V}(E,B)$ such that $a_1 <_r b' <_\beta b'' <_s a_2$. But this is impossible while $q_B=3$.

Hence $n(a_1,a_2)=n(a_2,a_3)=1$, and $a_1<_r b_1<_s a_2$ with $r\in\mathsf{R}(A,B)$, $s\in\mathsf{R}(B,A)$. As above, we can assume $sr=\alpha$, $a_2<b_2<a_3$. Then $a_1<b_1<a_2<b_2<a_3<b_3$.

Note that $h(\varphi)=1$ for any $\varphi\in\bigcup\limits_{i=1,2,3}\mathsf{S}(a_i,b_i)$. By Lemma 3, there is standard change of basis such that $\mathsf{S}(a_3,b_3)=\mathsf{C}(a_3,b_3)=\{\varphi_3\}$ and $\mathsf{S}(a_2,b_2)=\mathsf{C}(a_2,b_2)=\{\varphi_2\}$ where, possibly, $\varphi_2=\varphi_3$. Let $\varphi_i a_i=\lambda_i b_i$, $\lambda_i\in\Bbbk^*$, $i=2,3$. If there is $\varphi\in\mathsf{S}(a_1,b_1)$, $\varphi\neq\varphi_2,\varphi_3$, then we can do standard change of basis such that $\mathsf{S}(a_1,b_1)=\mathsf{C}(a_1,b_1)=\{\varphi_1\}$, $\varphi_1\neq\varphi_2,\varphi_3$. Otherwise, we can assume $\mathsf{S}(a_1,b_1)\subset\{\varphi_2,\varphi_3\}$. Let $\varphi_3\in\mathsf{S}(a_1,b_1)$ and $\varphi_3a_1=\lambda_1b_1+\lambda'_2b_2+\lambda'_3b_3$, $\lambda_1\neq0$. If $\varphi_3\neq\varphi_2$, then standard change $\varphi'_3=\varphi_3-\frac{\lambda'_2}{\lambda_2}\varphi_2\alpha-\frac{\lambda'_3}{\lambda_3}\varphi_3\alpha^2$ leads to equalities $\varphi'_3a_1=\lambda_1b_1$, $\varphi'_3a_2=\varphi_3a_2=0$, $\varphi'_3a_3=\lambda_3b_3$. If $\varphi_2\in\mathsf{S}(a_1,b_1)$, then similarly we obtain $\varphi'_2a_1=\mu_1b_1$, $\varphi'_2a_2=\varphi_2a_2=\lambda_2b_2$, $\varphi'_2a_3=0$, $\mu_1\in\Bbbk^*$. Hence $\mathsf{S}(a_i,b_i)=\mathsf{C}(a_i,b_i)$, $i=1,2,3$, and we have one of the cases:

1) $\mathsf{C}(a_i,b_i)=\{\varphi_i\}$, $i=1,2,3$, where two of $\varphi_1,\varphi_2,\varphi_3$ may be equal;
2) $\mathsf{C}(a_i,b_i)=\{\varphi_i\}$, $i=2,3$, $\mathsf{C}(a_1,b_1)=\{\varphi_2,\varphi_3\}$.

In the case $\varphi_2=\varphi_3$ we have $\varphi_1=\varphi_3$ as well, and similarly $\mathsf{S}(a_1,b_1)=\mathsf{C}(a_1,b_1)=\{\varphi_1\}$. But this case contradicts $\mathcal{A}\in\mathcal{C}$ (see Remark 4).

Similarly, we can find $\psi_1,\psi_2\in\Sigma_1^1$ such that $\mathsf{S}(b_1,a_2)=\mathsf{C}(b_1,a_2)=\{\psi_1\}$, $\mathsf{S}(b_2,a_3)=\mathsf{C}(b_2,a_3)=\{\psi_2\}$, where possibly $\psi_1=\psi_2$, and finally

$$a_1\underset{\varphi_1}{<}b_1\underset{\psi_1}{<}a_2\underset{\varphi_2}{<}b_2\underset{\psi_2}{<}a_3\underset{\varphi_3}{<}b_3\quad\text{or}\quad a_1\underset{\varphi_2,\varphi_3}{<}b_1\underset{\psi_1}{<}a_2\underset{\varphi_2}{<}b_2\underset{\psi_2}{<}a_3\underset{\varphi_3}{<}b_3.$$

Let $\varphi\in\mathsf{C}(a_1,b_1)$, then $\psi_1\varphi\parallel\alpha$. Since $\alpha a_2=a_3$ then $\varphi\in\mathsf{C}(a_2,b_2)$, and therefore $\varphi=\varphi_2$. Furthermore, $\varphi_2\psi_1\|\beta$, then $\psi_1=\psi_2$ and $\varphi_2a_3\neq0$, hence $\varphi_2=\varphi_3$. But this contradicts Remark 4. □

4.2. Comparable pairs

Definition 10. *Denote by $\Pi=\Pi_{A,B}$ the set*

$$\Pi=\big\{(a_i,b_j)\in\Sigma_1^0(E,A)\times\Sigma_1^0(E,B)\mid a_i\underset{\mathsf{R}}{<}b_j\big\}.$$

Let us define partial order on Π: given different pairs (a_i,b_j), $(a_u,b_v)\in\Pi$, we say that $(a_i,b_j)\prec(a_u,b_v)$ if $a_u\leqslant a_i<b_j\leqslant b_v$. We say that two

pairs (a_i,b_j), $(a_u,b_v)\in\Pi$ *are comparable, provided* $(a_i,b_j)\prec(a_u,b_v)$ *or* $(a_u,b_v)\prec(a_i,b_j)$. *We call the map* $\Pi\ni(a_i,b_j)\overset{h_m}{\longmapsto}\max\limits_{\xi\in\mathsf{S}(a_i,b_j)}h(\xi)\in\mathbb{N}$ *the maximal height, and denote by* $\mathsf{S}_m(a_i,b_j)=\{\xi\in\mathsf{S}(a_i,b_j)\mid h(\xi)=h_m(a_i,b_j)\}$. *For* $(a_i,b_j)\in\Pi$, $\mathsf{S}_m(a_i,b_j)\neq\varnothing$.

Lemma 6. *If* $(a_i,b_j)\in\Pi$, $\xi\in\mathsf{S}_m(a_i,b_j)$, *then* $h_B(\xi a_i)=h_B(b_j)=j$. *In particular,* $h(\xi a_i)=h(b_j)$, *and* $h(b_j)-h(a_i)\geqslant h_m(a_i,b_j)$.

Proof. If $\xi a_i\parallel b_l+b'$ for $l<j$ and $h_B(b')>l$, then $\beta_{lj}\xi a_i\parallel b_j+b''$ and $h_B(b'')>j$. Hence there exists $\eta\in\mathsf{S}(a_i,b_j)$ such that $h(\eta)\geqslant h(\beta_{lj}\xi)>h(\xi)=h_m(a_i,b_j)$, and we get contradiction. Then $h(b_j)=h(\xi a_i)\geqslant h(\xi)+h(a_i)$, hence $h(b_j)-h(a_i)\geqslant h_m(a_i,b_j)$. □

Lemma 7. *If* $(a_i,b_j),(a_u,b_v)\in\Pi$ *and* $h_m(a_i,b_j)=h_m(a_u,b_v)$ *then the pairs* (a_i,b_j), (a_u,b_v) *are incomparable.*

Proof. If $\mathrm{con}_{b_j}(ra_i)\neq0$, $r\in\mathsf{R}(A,B)$, then $h_m(a_i,b_j)\geqslant h(r)$. Indeed, if $r=\sum\limits_{\xi\in\Sigma_1^1(A,B)}\lambda_\xi\xi$, then $h(\xi)\geqslant h(r)$ whenever $\lambda_\xi\neq0$. Since $\mathrm{con}_{b_j}(ra_i)\neq0$, then there exists $\xi\in\mathsf{S}(a_i,b_j)$ such that $\lambda_\xi\neq0$, and therefore $h_m(a_i,b_j)=\max\limits_{\eta\in\mathsf{S}(a_i,b_j)}h(\eta)\geqslant h(\xi)\geqslant h(r)$.

For $(a_i,b_j),(a_u,b_v)\in\Pi$, the inequality $h_m(a_i,b_j)<h_m(a_u,b_v)$ holds whenever $(a_i,b_j)\prec(a_u,b_v)$. Let $(a_i,b_j)\prec(a_u,b_v)$, $\varphi\in\mathsf{S}_m(a_i,b_j)$. If $i=u$ and $j<v$, then $\mathrm{con}_{b_v}(ra_i)\neq0$ for $r=\beta_{jv}\varphi$. If $u<i$, $j=v$, then $\mathrm{con}_{b_j}(ra_u)\neq0$ for $r=\varphi\alpha_{ui}$. If $u<i$, $j<v$, then $\mathrm{con}_{b_v}(ra_u)\neq0$ for $r=\beta_{jv}\varphi\alpha_{ui}$. For all the cases, $h_m(a_u,b_v)\geqslant h(r)>h(\varphi)=h_m(a_i,b_j)$. □

Further, writing $(a_i,b_j),(a_u,b_v)\in\Pi$, *we assume* $(a_i,b_j)\neq(a_u,b_v)$.

Remark 6. If $(a_i,b_j),(a_u,b_v)\in\Pi$ are incomparable, then either $i<u$, $j<v$ or $i>u$, $j>v$.

Proof. The pairs (a_i,b_j), (a_i,b_v), $j\neq v$, are comparable as well as the pairs (a_i,b_j), (a_u,b_j), $i\neq u$. Hence $i\neq u$, $j\neq v$. Given $i<u$, assume that $j>v$, then $a_i<a_u<b_v<b_j$, and hence $(a_u,b_v)\prec(a_i,b_j)$. □

Definition 11. *For any subset* $\mathfrak{X}\subset\Pi$, *put* $\mathsf{C}(\mathfrak{X})=\bigcup\limits_{(a_i,b_j)\in\mathfrak{X}}\mathsf{C}(a_i,b_j)$. *Given* $\varphi\in\Sigma_1^1(A,B)$, *let* $\mathfrak{X}_\varphi=\{(a_i,b_j)\in\mathfrak{X}\mid\varphi\in\mathsf{C}(a_i,b_j)\}$. *Obviously,* $\mathfrak{X}_\varphi\neq\varnothing$ *if and only if* $\varphi\in\mathsf{C}(\mathfrak{X})$. *A subset* $\mathfrak{X}\subset\Pi$ *is named* upper closed *if for any* $(a_i,b_j)\in\mathfrak{X}$ *and* $(a_u,b_v)\in\Pi$ *there hold:*

- *if* $h_m(a_u,b_v)>h_m(a_i,b_j)$, *then* $(a_u,b_v)\in\mathfrak{X}$;
- *if* $h_m(a_u,b_v)=h_m(a_i,b_j)$ *and* $u>i$, *then* $(a_u,b_v)\in\mathfrak{X}$.

The pair $(a_i, b_j) \in \Pi \backslash \mathfrak{X}$ *is called* boundary *for the upper closed set* $\mathfrak{X} \subset \Pi$ *if* $\mathfrak{X} \cup \{(a_i, b_j)\}$ *is upper closed too. A upper closed subset* $\mathfrak{X} \subset \Pi$ *is said to be* canonical *if the following conditions hold:*

1) if $(a_i, b_j) \in \mathfrak{X}$ *then we have* $|\mathsf{S}(a_i, b_j)| = |\mathsf{C}(a_i, b_j)| = 1$*;*

2) $1 \leqslant |\mathfrak{X}_\varphi| \leqslant 2$ *for each* $\varphi \in \mathsf{C}(\mathfrak{X})$*;*

3) if $\mathfrak{X}_\varphi = \{(a_i, b_j), (a_u, b_v)\}$*, then* $a_i \neq a_u$*,* $b_j \neq b_v$*, and* $h_m(a_i, b_j) = h_m(a_u, b_v) = 1$*.*

Remark that $\mathfrak{X} = \varnothing$ is canonical.

A standard basis change is called *careful* for the canonical subset $\mathfrak{X} \subset \Pi$ if $\mathfrak{X}$ is a canonical subset with respect to the new basis as well.

Lemma 8. *Let* $\mathfrak{X} \subset \Pi$ *be a canonical subset and let* $(a_i, b_j) \in \Pi \backslash \mathfrak{X}$ *be a boundary pair. Then* $\mathsf{S}_m(a_i, b_j) \subset \mathsf{C}(a_i, b_j)$*.*

Proof. If $\mathrm{con}_{b_k}(\xi a_i) \neq 0$, $\xi \in \mathsf{S}(a_i, b_j)$, $k > j$, then $\xi \in \mathsf{S}(a_i, b_k)$, and so $(a_i, b_k) \in \mathfrak{X}$ since $(a_i, b_k) \succ (a_i, b_j)$. We have $\mathsf{S}(a_i, b_k) = \mathsf{C}(a_i, b_k) = \{\xi\}$ by the definition of $\mathfrak{X}$, hence $\xi a_i \| b_k$ that contradicts to inclusion $\xi \in \mathsf{S}(a_i, b_j)$. Therefore $\mathrm{con}_{b_k}(\xi a_i) = 0$ for all $\xi \in \mathsf{S}(a_i, b_j)$ and $k > j$.

Let $\varphi \in \mathsf{S}_m(a_i, b_j)$. If $\varphi a_i \nparallel b_j$, then we conclude $\varphi a_i = \mu_k b_k + \ldots + \mu_j b_j$ with $k < j$ and $\mu_k \neq 0$, $\mu_j \neq 0$. Therefore $(\beta_{kj}\varphi) a_i = \mu_k b_j + \ldots + \mu_j (\beta_{kj} b_j)$. There exists ξ in decomposition $\beta_{kj}\varphi = \sum\limits_{\xi \in \Sigma_1^1(A,B)} \lambda_\xi \xi$ such that $\lambda_\xi \neq 0$ and $\mathrm{con}_{b_j}(\xi a_i) \neq 0$, so $\xi \in \mathsf{S}(a_i, b_j)$. But $h(\xi) \geqslant h(\beta_{kj}\varphi) > h(\varphi) = h_m(a_i, b_j)$, and we obtain contradiction. □

Lemma 9. *If* $q_A = q_B = 3$*,* $\alpha_{12} = \alpha_{23} = \alpha$*, and* $\beta_{12} = \beta_{23} = \beta$*, then* $\Pi_{A,B} = \{(a_i, b_j) \mid i, j = 1, 2, 3\}$ *is canonical, each* $\varphi \in \Sigma_1^1(A, B)$ *is single, and multiplication rule holds.*

Proof. By Lemma 5, $a_1 <_\alpha a_2 <_\alpha a_3 <_\tau b_1 <_\beta b_2 <_\beta b_3$. Hence, $\Pi_{A,B} = \{(a_i, b_j) \mid i, j = 1, 2, 3\}$ is upper closed. By Lemma 3 there is standard basis change that leads to condition $\mathsf{S}(a_3, b_1) = \mathsf{C}(a_3, b_1) = \{\tau\}$. Obviously, $h(\tau) = 1$.

Let $\tau a_2 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$, then standard basis change $\tau' = \tau - \lambda_1 \tau\alpha - \lambda_2 \beta\tau\alpha - \lambda_3 \beta^2 \tau\alpha$ implies $\tau' a_2 = 0$, $\tau' a_3 = b_1$. Similarly, we can obtain $\tau a_1 = 0$, $\tau a_2 = 0$, $\tau a_3 = b_1$.

Denote $\varphi_{ij} = \beta^{j-1} \tau \alpha^{3-i}$, $i, j = 1, 2, 3$. Then $\varphi_{ij} = \varphi_{kl}$ implies $i = k$, $j = l$. Indeed, if $k < i$, then $0 = \beta^{l-1} \tau \alpha^{3-k} a_i = \beta^{j-1} \tau \alpha^{3-i} a_i = b_j$. So $i = k$. Similarly, $j = l$. It is easy to check that $\varphi_{ij} a_k = \delta_{ik} b_j$.

If there is $\varphi \in \mathsf{S}(a_i, b_j) \backslash \{\varphi_{ij}\}$, then $h(\varphi) \leqslant h(\varphi_{ij})$ and $\varphi \neq \varphi_{kl}$ for all k, l. So basis change $\mathfrak{C}_\lambda(\varphi_{ij}, \varphi)$ with $\lambda = \mathrm{con}_{b_j}(\varphi a_i)$ is correct and leads to condition $\varphi \notin \mathsf{S}(a_i, b_j)$. Thus, $\mathsf{C}(a_i, b_j) = \mathsf{S}(a_i, b_v) = \{\varphi_{ij}\}$ and $\Sigma_1^1(A, B) = \{\varphi_{ij}, i, j = 1, 2, 3\}$. Multiplication rule holds obviously. □

4.3. Induction step

We exclude the case of Lemma 9 from further consideration in this subsection. Let us fix a canonical subset $\mathfrak{X} \subset \Pi$ and a boundary pair $(a_i, b_j) \in \Pi \backslash \mathfrak{X}$.

Lemma 10. *If there exists $\varphi \in \mathsf{S}_m(a_i, b_j) \backslash \mathsf{C}(\mathfrak{X})$, then there is a careful basis change such that $\mathsf{S}(a_i, b_j) = \mathsf{C}(a_i, b_j) = \{\varphi\}$, $\varphi \notin \mathsf{C}(\mathfrak{X})$.*

Proof. By Lemma 8, $\varphi \in \mathsf{C}(a_i, b_j)$. For a $\psi \in \mathsf{S}(a_i, b_j) \backslash \{\varphi\}$, we have $h(\psi) \leqslant h(\varphi)$. Hence, there exists $\lambda \in \Bbbk^*$ such that the elementary basis change $\psi' = \psi + \lambda\varphi$ is correct and leads to the condition $\mathsf{S}(a_i, b_j) = \mathsf{S}(a_i, b_j) \backslash \{\psi\}$ by Lemma 3.

Let us show that this change is careful. If $\psi \notin \mathsf{C}(\mathfrak{X})$ then it is clear. Otherwise, there is the pair $(a_u, b_v) \in \mathfrak{X}$ such that $\mathsf{S}(a_u, b_v) = \mathsf{C}(a_u, b_v) = \{\psi\}$. If $\varphi a_u \neq 0$, then $\varphi \in \mathsf{S}(a_u, b_w)$ for some $v \leqslant w \leqslant q_B$ since $h(\varphi a_u) \geqslant h(\psi a_u) = h(b_v)$. Then $(a_u, b_w) \in \mathfrak{X}$ while $(a_u, b_w) \succ (a_u, b_v)$, and therefore $\varphi \in \mathsf{C}(\mathfrak{X})$. We have a contradiction. So $\varphi a_u = 0$, and $\psi' a_u = \psi a_u$, hence this change of basis is careful. □

Corollary 1. *If $\mathsf{S}_m(a_i, b_j) \not\subset \mathsf{C}(\mathfrak{X})$ then $\mathfrak{X} \cup \{(a_i, b_j)\}$ is a canonical subset as well.*

Lemma 11. *Let $\mathcal{Y} = \{(a_{i_k}, b_{j_k}), k = 1, \ldots, n\} \subset \Pi$ be any set of pairwise incomparable pairs in Π. Then $n \leqslant 3$, and if $n = 3$, then $q_A = q_B = 3$, $\mathcal{Y} = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$.*

Proof. By Remark 6 $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ up to renumbering. Then $n \leqslant q_A \leqslant 3$. If $n = 3$, then $i_1 = j_1 = 1$, $i_2 = j_2 = 2$, $i_3 = j_3 = 3$. □

Lemma 12. *If $\mathsf{S}_m(a_i, b_j) \subset \mathsf{C}(\mathfrak{X})$, then $h_m(a_i, b_j) = 1$.*

Proof. Assume $h = h_m(a_i, b_j) > 1$. Denote $\mathcal{Y} = \bigcup_{\varphi \in \mathsf{S}_m(a_i, b_j)} \mathfrak{X}_\varphi$. By definition 11, $|\mathcal{Y}| \geqslant |\mathsf{S}_m(a_i, b_j)|$, and $h_m(a_u, b_v) = h$ for any $(a_u, b_v) \in \mathcal{Y}$, and hence the pairs from $\mathcal{Y} \cup \{(a_i, b_j)\}$ are pairwise incomparable by Lemma 7. By Lemma 11, $|\mathcal{Y}| \leqslant 2$.

Assume $\mathsf{S}_m(a_i, b_j) = \{\varphi, \psi\}$, and $|\mathfrak{X}_\varphi| = |\mathfrak{X}_\psi| = 1$, $\mathcal{Y} = \mathfrak{X}_\varphi \bigcup \mathfrak{X}_\psi$. Then, by Lemma 11, $q_A = q_B = 3$, and $\mathcal{Y} \cup \{(a_i, b_j)\} = \{(a_k, b_k) \mid k = 1, 2, 3\}$, besides, $(a_i, b_j) = (a_1, b_1)$ and $\Pi = \mathfrak{X} \bigcup \{(a_1, b_1)\}$ while $\mathfrak{X}$ is upper closed. Let $\mathfrak{X}_\varphi = \{(a_2, b_2)\}$, $\mathfrak{X}_\psi = \{(a_3, b_3)\}$.

Since $h_m(a_k, b_k) = h > 1$, $k = 1, 2, 3$, and there is a basic element of each height, then either $a_1 < a_2 < b_1 < a_3 < b_2 < b_3$ or $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$. In the first case $h_m(a_1, b_1) = h_m(a_3, b_3) = 2$ but $h_m(a_2, b_2) = 3$, and we get a contradiction.

In the second case $h_m(a_k, b_k) = 3$, $k = 1, 2, 3$. Let $\tau \in \mathsf{S}(a_3, b_1)$. Then $h(\tau\alpha_{13}) \geqslant h(\tau) + h(\alpha_{13}) \geqslant 3$, and $\tau\alpha_{13} = \sum\limits_{\xi\in\Xi} \lambda_\xi \xi$, where $\Xi = \{\xi \in \Sigma_1^1(A, B) \mid h(\xi) \geqslant 3\}$. Since each pair (a_k, b_k) is adjusted, $\mathrm{con}_{b_k}(\xi a_k) = 0$ for all $\xi \in \Xi\backslash\{\varphi, \psi\}$. Then $0 = \mathrm{con}_{b_2}(\tau\alpha_{13}a_2) = \lambda_\varphi$ while $\mathrm{con}_{b_2}(\psi a_2) = 0$. Similarly, $\lambda_\psi = 0$, in contradiction with condition $\mathrm{con}_{b_1}(\tau\alpha_{13}a_1) \neq 0$.

Therefore, $\mathsf{S}_m(a_i, b_j) = \{\varphi\} \subset \mathsf{C}(a_i, b_j)$, $h(\varphi) = h$. Since $\varphi \in \mathsf{C}(\mathfrak{X})$, there exists $(a_u, b_v) \in \mathfrak{X}$ such that $\mathsf{C}(a_u, b_v) = \{\varphi\}$, the pairs (a_i, b_j) and (a_u, b_v) are incomparable, and $i < u$, $j < v$ by Remark 6 in view of $\mathfrak{X}$ is upper closed.

Since $h > 1$, then φ is a linear combination of the summands belonging to the set

$$\Gamma = \left\{\xi\alpha,\ \beta\xi,\ \beta\xi\alpha\ \mid \alpha \in \Sigma_1^1(A, A), \beta \in \Sigma_1^1(B, B), \xi \in \Sigma_1^1(A, B)\right\} \cap \mathsf{R}^h.$$

Since $\varphi a_i \| b_j$ (resp., $\varphi a_u \| b_v$) then there is $\gamma \in \Gamma$ such that $\mathrm{con}_{b_j}(\gamma a_i) \neq 0$ (resp., $\mathrm{con}_{b_v}(\gamma a_u) \neq 0$). Denote $\Gamma' = \{\gamma \in \Gamma \mid \mathrm{con}_{b_j}(\gamma a_i) \neq 0\}$, $\Gamma'' = \{\gamma \in \Gamma \mid \mathrm{con}_{b_v}(\gamma a_u) \neq 0\}$. Then $\Gamma' = \Gamma''$. Indeed, if $\gamma = \sum\limits_{\substack{\psi\in\Sigma_1^1(A,B)\\ h(\psi)\geqslant h}} \lambda_\psi \psi \in \Gamma'$, then $\mathrm{con}_{b_j}(\psi a_i) = 0$ and $\mathrm{con}_{b_v}(\psi a_u) = 0$ for any $\psi \neq \varphi$ by the construction. Therefore $\lambda_\varphi \neq 0$, $\mathrm{con}_{b_v}(\gamma a_u) = \lambda_\varphi \mathrm{con}_{b_v}(\varphi a_u) \neq 0$ and $\gamma \in \Gamma''$. The proof of inclusion $\Gamma'' \subset \Gamma'$ is similar. Remark that $\gamma a_i \neq 0$ and $\gamma a_u \neq 0$ for any $\gamma \in \Gamma'$.

Let $\gamma \in \Gamma'$. If $\gamma = \xi\alpha$ or $\gamma = \beta\xi\alpha$, then $\alpha a_i \neq 0$ and $\alpha a_u \neq 0$. So α is joint arrow while $i < u$, and $q_A = 3$. If $\gamma = \beta\xi$ or $\gamma = \beta\xi\alpha$, then $\gamma a_i \| b_j + b'$, where $h(b') > h(b_j)$, $j \geqslant 2$, and $b_v \| b_3$. Thus $b_j, b_v \in \beta\mathsf{V}(E, B)$. But $j < v$, hence β is a joint arrow, and $q_B = 3$.

If $\gamma = \xi\alpha \in \Gamma'$, then $\alpha = \alpha_{12} = \alpha_{23}$, so $i = 1$, $u = 2$, and we have $a_1 <_\alpha a_2 <_\xi b'_j$, $a_2 <_\alpha a_3 <_\xi b'_v$. Here $h(b'_j - b_j) > h(b_j)$ and $h(b'_v - b_v) > h(b_v)$. There is $r \in \mathsf{R}(B, B)$ such that $b'_j <_r b'_v$ by Remark 5. We obtain $a_2 <_\xi b'_j <_r b'_v$, and we can find $\beta \in \Sigma_1^1(B, B)$ such that $\mathrm{con}_{b_v}(\beta\xi a_u) \neq 0$, hence $\beta\xi \in \Gamma'$ and β is a joint arrow. If $\gamma = \beta\xi$, then, similarly, α is a joint arrow.

If α, β are both joint, then $q_A = q_B = 3$, $\alpha = \alpha_{12} = \alpha_{23}$, $\beta = \beta_{12} = \beta_{23}$. This case of Lemma 9 is excluded from consideration. □

Lemma 13. *Let $\mathsf{S}_m(a_i, b_j) \subset \mathsf{C}(\mathfrak{X})$. Then one of the following conditions holds (after some careful basis change):*

1) $\mathsf{S}(a_i, b_j) = \mathsf{C}(a_i, b_j) = \{\varphi\}$, $h(\varphi) = 1$, $\varphi \in \mathsf{C}(\mathfrak{X})$, and $|\mathfrak{X}_\varphi| = 1$;

2) $q_A = q_B = 3$, $i = j = 1$, $\Pi = \mathfrak{X} \bigcup \{(a_1, b_1)\}$ and $\mathsf{C}(a_k, b_k) = \{\varphi_k\}$, $\mathfrak{X}_{\varphi_k} = \{(a_k, b_k)\}$, $k = 2, 3$, $\mathsf{C}(a_1, b_1) = \{\varphi_2, \varphi_3\}$.

Proof. By Lemma 12, $h_m(a_i, b_j) = 1$. Clearly, $\mathsf{S}_m(a_i, b_j) = \mathsf{S}(a_i, b_j)$. By Lemma 8, $\mathsf{S}(a_i, b_j) = \mathsf{C}(a_i, b_j)$. Denote $\mathcal{Y} = \bigcup\limits_{\varphi \in \mathsf{C}(a_i,b_j)} \mathfrak{X}_\varphi$. By definition 11, $|\mathcal{Y}| \geqslant |\mathsf{C}(a_i, b_j)|$, and if $(a_u, b_v) \in \mathcal{Y}$, then $h_m(a_u, b_v) = 1$. Hence, the pairs from $\mathcal{Y} \bigcup \{(a_i, b_j)\}$ are pairwise incomparable by Lemma 7, and so $|\mathcal{Y}| \leqslant 2$ by Lemma 11.

If $|\mathcal{Y}| = 1$ then $\mathsf{C}(a_i, b_j) = \{\varphi\}$ and $|\mathfrak{X}_\varphi| = 1$, which implies 1). Assume $|\mathcal{Y}| = 2$. If $\mathsf{C}(a_i, b_j) = \{\varphi\}$ then $\mathcal{Y} = \mathfrak{X}_\varphi$. Thus P_φ includes three adjusted pairs, which is impossible by Remark 4. Hence, $\mathsf{C}(a_i, b_j) = \{\varphi, \psi\}$, and $\mathcal{Y} = \mathfrak{X}_\varphi \cup \mathfrak{X}_\psi$. There are $(a', b'), (a'', b'') \in \mathfrak{X}$ such that $a' < a''$, and $\mathsf{C}(a', b') = \{\varphi\}$, $\mathsf{C}(a'', b'') = \{\psi\}$. The pairs (a_i, b_j), (a', b'), (a'', b'') are pairwise incomparable by Lemma 7, while $h(\varphi) = h(\psi) = 1$. Since $\mathfrak{X}$ is upper closed, it is possible only if $i = j = 1$, then $a_1 \underset{\varphi,\psi}{<} b_1 < a' \underset{\varphi}{<} b' < a'' \underset{\psi}{<} b''$, and $\Pi = \mathfrak{X} \bigcup \{(a_1, b_1)\}$. We obtain the case 2). $\square$

Lemma 14. *Let $\mathfrak{X} \subset \Pi$ be a canonical subset and let $(a_i, b_j) \in \Pi \backslash \mathfrak{X}$ be a boundary pair. Then there exists a careful basis change on $\Sigma_1^1(A, B)$ such that $\mathfrak{X} \cup \{(a_i, b_j)\}$ is a canonical subset in Π except of the case $q_A = q_B = 3$, $a_1 \underset{\varphi_2,\varphi_3}{<} b_1 < a_2 \underset{\varphi_2}{<} b_2 < a_3 \underset{\varphi_3}{<} b_3$, and $i = j = 1$, $\Pi = \mathfrak{X} \bigcup \{(a_1, b_1)\}$, $\mathsf{C}(a_k, b_k) = \{\varphi_k\}$, $\mathfrak{X}_{\varphi_k} = \{(a_k, b_k)\}$, $k = 2, 3$, $\mathsf{C}(a_1, b_1) = \{\varphi_2, \varphi_3\}$.*

Proof. By Lemma 12, if $\varphi \in \mathsf{C}(\mathfrak{X} \cup \{(a_i, b_j)\})$ and $h(\varphi) > 1$, then φ is single. Then the proof follows from Lemmas 10 and 13. $\square$

Combining results of Lemmas 9, 14 and 12, we obtain the following

Corollary 2. *Under conditions of Proposition 2, one of the following cases occurs:*

1) Π *is a canonical set;*

2) $\Pi \backslash \{(a_1, b_1)\}$ *is a canonical set; this is the case from Lemma 14.*

In any case, for every $a_i \in \Sigma_1^0(E, A)$ and $\varphi \in \Sigma_1^1(A, B)$, either $\varphi a_i = 0$ or $\varphi a_i \| b_j$ for some $b_j \in \Sigma_1^0(E, B)$, and the height of each joint arrow in $\Sigma_1^1(A, B)$ equals 1.

Denote by $\Pi_{A,B}$ and $\Pi_{B,A}$[1] the sets, defined by definition 10. Assume both $\Pi_{A,B}$ and $\Pi_{B,A}$ satisfy Corollary 2. Then at most one of these set may be not canonical (since $a_1 < b_1$ and $b_1 < a_1$ are not simultaneous).

[1] Defined as in Definition 10.

4.4. Multiplication rule on Σ_1^1

For any $\varphi, \psi \in \Sigma_1^1$, we say that the product $\psi\varphi$ belongs to Σ_1^1, if $\psi\varphi \parallel \xi \in \Sigma_1^1$. Further considerations are provided under assertions of Corollary 2.

Lemma 15. *Under conditions of Proposition 2, multiplication rule holds for $\mathcal{A}$.*

Proof. Due to Corollary 2, either the set $\Pi_{A,B}$ is canonical, or we have the special case of Lemma 14 and $\Pi_{A,B}\backslash\{(a_1, b_1)\}$ is a canonical set. Similarly, $\Pi_{B,A}$ or $\Pi_{B,A}\backslash\{(b_1, a_1)\}$ is a canonical set. The special case occurs at most for one of the cases $\Pi_{A,B}$ and $\Pi_{B,A}$, so we can assume $\Pi_{B,A}$ to be canonical. Due to Proposition 1, the sets $\Pi_{A,A}$ and $\Pi_{B,B}$ are canonical as well.

Since $\operatorname{Ann}_{\mathsf{R}} \mathsf{V} = 0$ then $\mathsf{C}(\Pi_{X,Y}) = \Sigma_1^1(X, Y)$ for all $X, Y \in \{A, B\}$. If X, Y or Z denotes one of the vertices A or B, then we write x_i, y_i or z_i instead of basic elements a_i or b_i respectively. Further we will omit the condition $1 \leqslant i$ for the indices.

Assume $r = \psi\varphi \neq 0$, $\varphi \in \Sigma_1^1(X, Y)$, $\psi \in \Sigma_1^1(Y, Z)$ where $X, Y, Z \in \{A, B\}$. Let

$$r = \sum_{\xi \in \Sigma_1^1(X,Z)} \lambda_\xi \xi, \ \lambda_\xi \in \Bbbk. \tag{5}$$

Denote by Σ_r the set of all ξ from decomposition (5) such that $\lambda_\xi \neq 0$. Then $h(\xi) \geqslant h(r) > 1$ for any $\xi \in \Sigma_r$, and if $X \neq Z$, ξ is single by Corollary 2.

1) If $rx_i \neq 0$ for some $i \leqslant q_X$, then there exist $j \leqslant q_Y$ and $k \leqslant q_Z$ such that $\varphi x_i \| y_j$ and $\psi y_j \| z_k$ by construction, so $rx_i \| z_k$.

2) Let $r \in \mathsf{R}(A, A)$. If $\Sigma_1^1(A, A) = \{\alpha, \alpha^2\}$, then r belongs to $\Sigma_1^1(A, A)$. Indeed, if $r = \lambda\alpha + \mu\alpha^2$, $\lambda, \mu \in \Bbbk^*$, then $ra_1 = \lambda a_2 + \mu a_3$ in contradiction with item 1), and Lemma is proved. So, if $r \in \mathsf{R}(A, A)$, we assume all the arrows from $\Sigma_1^1(A, A)$ to be single below.

3) Now it is enough to consider the situation when all the arrows from Σ_r are single. Denote $\Sigma_r = \{\tau_1, \dots, \tau_p\}$, $p > 0$, and $\mathrm{P}_{\tau_t} = \{(x_{i_t}, z_{k_t})\}$, $t = 1, \dots, p$. Then $x_{i_1}, \dots, x_{i_p}$ are pairwise different arrows and $p \leqslant 3$. Indeed, if $x_{i_u} = x_{i_v}$ then $z_{k_u} \neq z_{k_v}$ while $\Pi_{X,Z}\backslash\{(x_1, z_1)\}$ is canonical, and the sum $rx_{i_u} = \sum\limits_{v=1}^{p} \lambda_{\tau_v} \tau_v x_{i_u} = \sum\limits_{i=1}^{q_Z} \mu_i z_i$ contains at least two nonzero summands in contradiction with item 1).

4) If $\tau_1, \tau_2 \in \Sigma_r$, $\tau_1 \neq \tau_2$, then φ, ψ are joint. Indeed, since $\mathrm{P}_{\tau_t} = \{(x_{i_t}, z_{k_t})\}$, then $\varphi x_{i_t} \neq 0$ for $t = 1, 2$, and hence there exist y_{j_1}, y_{j_2} such that $(x_{i_t}, y_{j_t}) \in \mathrm{P}_\varphi$. By item 3), $x_{i_1} \neq x_{i_2}$ and hence $\mathrm{P}_\varphi = \{(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})\}$,

and φ is joint. Obviously, both (x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2}) have the same maximal height. By Lemma 7, the pairs $(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})$ are incomparable, and hence $y_{j_1} \neq y_{j_2}$ by Remark 6. Since $\psi y_{j_t} \,\|\, r x_{i_t} \,\|\, z_{k_t}$, $t = 1, 2$, then ψ is joint as well, and $\mathrm{P}_\psi = \{(y_{j_1}, z_{k_1}), (y_{j_2}, z_{k_2})\}$. The pairs $(y_{j_1}, z_{k_1}), (y_{j_2}, z_{k_2})$ are incomparable as above. In particular, $z_{k_1} \neq z_{k_2}$. Due to Remark 4, $|\Sigma_r| = p \leqslant 2$. In the case $p = 1$ the assertion of Lemma is obvious. Now assume $p = 2$.

5) The pairs (x_{i_1}, z_{k_1}), (x_{i_2}, z_{k_2}) are incomparable. Otherwise, up to renumbering, $x_{i_1} < x_{i_2} < z_{k_2} < z_{k_1}$. But $x_{i_1} < y_{j_1} < z_{k_1}$ and $x_{i_2} < y_{j_2} < z_{k_2}$. The elements y_{j_1} and y_{j_2} are comparable, hence either $x_{i_1} < y_{j_1} < y_{j_2} < z_{k_2} < z_{k_1}$, and $(y_{j_2}, z_{k_2}) \prec (y_{j_1}, z_{k_1})$, or $x_{i_1} < x_{i_2} < y_{j_2} < y_{j_1}$, and so $(x_{i_2}, y_{j_2}) \prec (x_{i_1}, y_{j_1})$. Both cases contradict to 4).

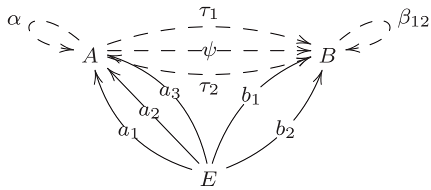
6) Let $r = \psi\varphi \in \mathsf{R}(A, B)$, $\varphi = \alpha \in \Sigma_1^1(A, A)$. Then by 4), α and $\psi \in \Sigma_1^1(A, B)$ are joint. Hence $q_A = 3$, $\Sigma_1^1(A, A) = \{\alpha, \alpha^2\}$, $\mathrm{P}_\alpha = \{(a_1, a_2), (a_2, a_3)\}$, $\mathrm{P}_\psi = \{(a_2, b_{j_1}), (a_3, b_{j_2})\}$, $j_1 < j_2$. Using 4) and 5), we obtain $\psi\alpha = \lambda_1\tau_1 + \lambda_2\tau_2$, where $\lambda_1, \lambda_2 \in \Bbbk^*$ and $\tau_1, \tau_2 \in \Sigma_1^1(A, B)$ are single, and $\mathrm{P}_{\tau_t} = \{(a_t, b_{j_t})\}$. By Corollary 2 and Definition 11, $\mathsf{C}(a_t, b_{j_t}) = \{\tau_t\}$, $\mathsf{C}(a_{t+1}, b_{j_t}) = \{\psi\}$, $t = 1, 2$. So we have the following partially ordered set:

$$\begin{tikzcd}
 & & a_3 \arrow[dr, "\psi"] & \\
 & a_2 \arrow[ur, "\alpha"] \arrow[rr, "\tau_2"] \arrow[dr, "\psi"'] & & b_{j_2} \\
a_1 \arrow[ur, "\alpha"] \arrow[rr, "\tau_1"'] & & b_{j_1} \arrow[ur, "\beta_{j_1 j_2}"'] &
\end{tikzcd}$$

where $\xrightarrow{\xi}$ means $\underset{\xi}{<}$, and $\mathsf{C}(b_{j_1}, b_{j_2}) = \{\beta_{j_1 j_2}\}$. If $|\Sigma_{\beta_{j_1 j_2}\psi}| > 1$, then $\beta_{j_1 j_2}$ is joint by 4), and the proof follows from Lemma 9. Otherwise $\beta_{j_1 j_2}\psi \| \tau_2$.

Assume that $b_{j_1} < a_3$ with $\mathsf{C}(b_{j_1}, a_3) = \{\xi\}$. Then $a_1 < a_2 <_\psi b_{j_1} <_\xi a_3 < b_{j_2}$. We obtain $\xi\psi a_2 \,\|\, a_3$ and hence $\xi\psi \,\|\, \alpha + \mu\alpha^2$, $\mu \in \Bbbk$. In this case $\xi\psi a_1 \,\|\, a_2 + \mu a_3 \neq 0$ and $\psi a_1 \neq 0$ which contradicts to Remark 4. If $a_3 < b_{j_1}$ with $\mathsf{C}(a_3, b_{j_1}) = \{\xi\}$, then $a_1 < a_2 <_\alpha a_3 <_\xi b_{j_1} < b_{j_2}$ and hence $\xi\alpha a_2 \,\|\, b_{j_1}$, and therefore $h(\psi) \geqslant h(\xi\alpha) > 1$ which contradicts to Definition 11, since ψ is joint. Thus, a_3 and b_{j_1} are incomparable. By 3) of Definition 4, $q_B = 2$. In particular, $j_1 = 1$, $j_2 = 2$.

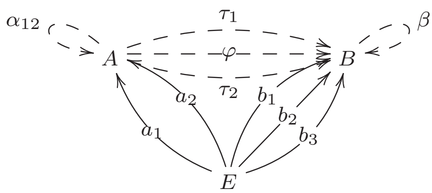
Therefore, we have the following bigraph[2]:



(a_3, b_1 are incomparable)

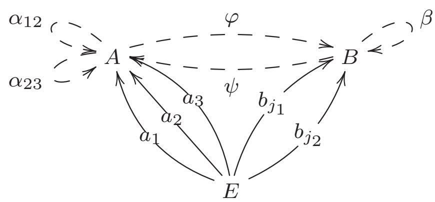
According to 2) of Definition 8 multiplication rule holds for $\psi\varphi$.

7) Let $r = \psi\varphi \in \mathsf{R}(A,B)$, $\psi = \beta \in \Sigma_1^1(B,B)$. Then similarly $q_A = 2$, $q_B = 3$, $\Sigma_1^1(B,B) = \{\beta, \beta^2\}$, $\varphi, \tau_1, \tau_2 \in \Sigma_1^1(A,B)$, $\psi = \beta$, $\beta\varphi \in \Bbbk^*\langle\tau_1, \tau_2\rangle$, $\mathrm{P}_\varphi = \{(a_1,b_1),(a_2,b_2)\}$, $\mathrm{P}_\beta = \{(b_1,b_2),(b_2,b_3)\}$, $\mathrm{P}_{\tau_t} = \{(a_t, b_{t+1})\}$, $t = 1,2$.



(a_2, b_1 are incomparable)

8) Let $r = \psi\varphi \in \mathsf{R}(A,A)$. Then $q_A = 3$, $q_B \geqslant 2$, $\varphi \in \Sigma_1^1(A,B)$, $\psi \in \Sigma_1^1(B,A)$, $\psi\varphi = \Bbbk^*\langle\alpha_{12}, \alpha_{23}\rangle$, $\alpha_{12}, \alpha_{23} \in \Sigma_1^1(A,A)$, $\alpha_{12} \neq \alpha_{23}$, and $\mathrm{P}_\varphi = \{(a_1, b_{j_1}), (a_2, b_{j_2})\}$, $\mathrm{P}_\psi = \{(b_{j_1}, a_2), (b_{j_2}, a_3)\}$, $1 \leqslant j_1 < j_2 \leqslant q_B$;



If $q_B = 3$ and $b \in \Sigma_1^1(E,B)\backslash\{b_{j_1}, b_{j_2}\}$ then either $b < a_1$ or $a_3 < b$. Besides, $\beta_{12} \neq \beta_{23}$.

Indeed, by 2), $\alpha_{12} \neq \alpha_{23}$. By 3), 4), 5) we have $a_{i_1} \underset{\varphi}{<} b_{j_1} \underset{\psi}{<} a_{k_1}$, $a_{i_2} \underset{\varphi}{<} b_{j_2} \underset{\psi}{<} a_{k_2}$, and hence $a_1 \underset{\varphi}{<} b_{j_1} \underset{\psi}{<} a_2 \underset{\varphi}{<} b_{j_2} \underset{\psi}{<} a_3$. Then $ra_t \,\|\, a_{t+1}$, $\varphi a_t \| b_{j_t}$, $\psi b_{j_t} \| a_{t+1}$, $t = 1,2$, and $r = \lambda\alpha_{12} + \mu\alpha_{23}$, $\lambda, \mu \in \Bbbk^*$.

If $q_B = 3$, then there exists $b_k \in \Sigma_1^0(E,B)\backslash\{b_{j_1}, b_{j_2}\}$. Since φ is joint, then $h(\varphi) = 1$ by Definition 11, and hence $b_k < a_1$ or $a_3 < b_k$. Assume $\beta_{12} = \beta_{23} = \beta$. If $b_1 < a_1 \underset{\varphi}{<} b_2 \underset{\psi}{<} a_2 \underset{\varphi}{<} b_3 \underset{\psi}{<} a_3$, then $\varphi\psi \| \beta + \mu\beta^2$, $\mu \in \Bbbk$, and hence $\psi b_1 \neq 0$ which is impossible by Remark 4. If $a_1 \underset{\varphi}{<} b_1 \underset{\psi}{<} a_2 \underset{\varphi}{<}$

[2]Note that the pictured subbigraphs can be non-full.

$b_2 \underset{\psi}{<} a_3 < b_3$, then $\varphi\psi \| \beta + \mu\beta^2$, $\mu \in \Bbbk$, and hence $\varphi a_3 \neq 0$ in contradiction with Remark 4. Finally, if we have the special case from Lemma 14, and $\varphi, \varphi' \in \mathsf{C}(a_1, b_1)$, then $\psi\varphi' \| \alpha_{12}$ since otherwise $\varphi' \in \mathsf{C}(a_2, b_2)$.

9) The case $r = \psi\varphi \in \mathsf{R}(B, B)$ can be obtained from 8) by swapping A and B. □

The proof of this Lemma implies the following result.

Remark 7. If $q_A \leqslant 2$, $q_B \leqslant 2$, then all the arrows from Σ_1^1 are single and the product of two basic elements is basic up to scalar multiplier.

The proof of Proposition 2 is complete.

5. General case

Remark 8. Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a faithful connected bimodule problem from $\mathcal{C}$ with nilpotent radical $\mathsf{R} = \operatorname{Rad} \mathsf{K}$ and a triangled basis Σ, and $A \in \Sigma_0^+$, $E \in \Sigma_0^-$. In this case $\mathsf{R}_{\{A,E\}} = \mathsf{R}(A, A)$. If $\mathsf{V}(E, A) \neq 0$ and $\operatorname{Ann}_{\mathsf{R}(A,A)} \mathsf{V}(E, A) \neq 0$, then $\dim_{\Bbbk} \mathsf{V}(E, A) = 1$ and $\operatorname{Ann}_{\mathsf{R}(A,A)} \mathsf{V}(E, A) = \mathsf{R}(A, A)$. Moreover, in this case $\operatorname{ord} A = 3$, and there is $E_1 \in \Sigma_0^-$ such that $\dim_{\Bbbk} \mathsf{V}(E_1, A) = 2$, $\operatorname{Ann}_{\mathsf{R}} \mathsf{V}(E_1, A) = 0$.

Proposition 3. *Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a faithful connected bimodule problem from $\mathcal{C}$ with nilpotent radical $\mathsf{R} = \operatorname{Rad} \mathsf{K}$ and a triangled basis Σ such that $|\Sigma_0^+| = 2$ (but not necessary $|\Sigma_0^-| = 1$). Then there exists a standard change $\mathfrak{C}$ of basis from Σ to quasi multiplicative one. If $\mathcal{A}^{(A)}$ for $A \in \Sigma_0^+$ is endowed with quasi multiplicative basis, then $\mathfrak{C}$ leaves these bases unchangeable.*

To prove Proposition 3, let $\Sigma_0^+ = \{A, B\}$, $\Sigma_0^- = \{E_1, \dots, E_p\}$. Since $\mathcal{A}$ is faithful, then the restrictions $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ are faithful as well by Remark 3. Due to Proposition 1, we can assume that both $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$ have quasi multiplicative bases.

Denote by $\mathcal{A}_i$ and $\mathcal{A}_i'$ the restriction and the faithful restriction of $\mathcal{A}$ to the set $\{A, B, E_i\}$, $i = 1, \dots, p$, respectively. Then $\mathsf{V}_i' = \mathsf{V}_i$, and $\mathsf{R}_i' = \mathsf{R}_i / \operatorname{Ann}_{\mathsf{R}_i} \mathsf{V}_i$. For the proof of proposition, it is enough to check, that we can choose a correct basis simultaneously for all $\mathcal{A}_i'$, $i = 1, \dots, p$.

Due to connectivity of $\mathcal{A}$, there exists i such that $\mathsf{V}(E_i, A) \neq 0$ and $\mathsf{V}(E_i, B) \neq 0$. Therefore, if $|\Sigma_0^-| > 1$, then $\dim \mathsf{V}(E_j, A) \leqslant 2$ and $\dim \mathsf{V}(E_j, B) \leqslant 2$ for all $j \neq i$. If $\mathsf{V}(E_j, A) = 0$ for some j, then denote $\widetilde{\mathcal{A}} = (\widetilde{\mathsf{K}}, \widetilde{\mathsf{V}}) = \mathcal{A}_{\{A, B, \Sigma_0^- \setminus \{E_j\}\}}$ If $\dim \mathsf{V}(E_j, B) = 1$ then $\widetilde{\mathcal{A}}$ is faithful as well as $\mathcal{A}$, and the proof for $\mathcal{A}$ follows from the proof for $\widetilde{\mathcal{A}}$. Otherwise, if $\dim \mathsf{V}(E_j, B) = 2$, then $\Sigma_1^1(B, B) = \{\beta_{12}\}$ by Proposition 1, and

$\mathrm{Ann}_{\widetilde{\mathsf{K}}}\, \widetilde{\mathsf{V}} = \langle \beta_{12} \rangle$. In this case the proof for $\mathcal{A}$ follows from the proof for the faithful part of $\widetilde{\mathcal{A}}$.

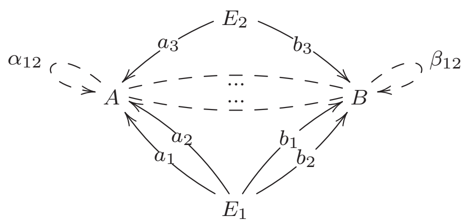
So we can assume that $\dim_{\Bbbk} \mathsf{V}(E_i, A) > 0$ and $\dim_{\Bbbk} \mathsf{V}(E_i, B) > 0$ for all $i = 1, \ldots, p$. By Definition 4, $\sum_{i=1}^{p} \dim_{\Bbbk} \mathsf{V}(E_i, A) \leqslant 3$, $\sum_{i=1}^{p} \dim_{\Bbbk} \mathsf{V}(E_i, B) \leqslant 3$. Therefore $p \leqslant 3$. If $p = 1$, then the proof follows from Proposition 2. Hence $p = 2$ or $p = 3$.

Assume there exists $E_1 \in \Sigma_0^-$ such that $\dim_{\Bbbk} \mathsf{V}(E_1, A) = 3$ (resp., $\dim_{\Bbbk} \mathsf{V}(E_1, B) = 3$); then $p = 1$. In the case $\dim_{\Bbbk} \mathsf{V}(E_1, A) = 2$ (resp., $\dim_{\Bbbk} \mathsf{V}(E_1, B) = 2$) we have $p \leqslant 2$. Finally, $p = 3$ only for the case $\dim_{\Bbbk} \mathsf{V}(E_i, A) = \dim_{\Bbbk} \mathsf{V}(E_i, B) = 1$ for $i = 1, 2, 3$.

Consider the case $\Sigma_0^- = \{E_1, E_2\}$. By Proposition 2 $\mathcal{A}_1'$ is endowed with quasi multiplicative basis $\Sigma_1^1(\mathcal{A}_1')$.

Lemma 16. *Assume* $\Sigma_0^- = \{E_1, E_2\}$, $\dim_{\Bbbk} \mathsf{V}(E_2, A) = \dim_{\Bbbk} \mathsf{V}(E_2, B) = 1$. *Then Proposition 3 holds.*

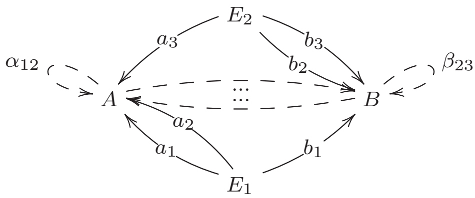
Proof. Assume $\mathcal{A}_1'$ satisfies Proposition 2. We have $1 \leqslant |\Sigma_1^0(E_1, A)| \leqslant 2$, $1 \leqslant |\Sigma_1^0(E_1, B)| \leqslant 2$. Consider the case $\Sigma_1^0(E_1, A) = \{a_1, a_2\}$, $a_1 < a_2$, $\Sigma_1^0(E_1, B) = \{b_1, b_2\}$, $b_1 < b_2$. By assumption, $\Sigma_1^0(E_2, A) = \{a_3\}$, and $\Sigma_1^0(E_2, B) = \{b_3\}$.



Note that $\mathsf{S}(a_3, b_3) = \mathsf{C}(a_3, b_3)$. Due to triangularity condition, either $\mathsf{C}(a_3, b_3) = \varnothing$ or $\mathsf{C}(b_3, a_3) = \varnothing$. Since $\mathcal{A}$ is faithful, and $h(\varphi) = 1$ for any $\varphi \in \mathsf{C}(a_3, b_3)$, then $\mathsf{C}(a_3, b_3)$ contains at least one $\varphi \notin \Sigma_1^1(\mathcal{A}_1')$. If $\varphi, \psi \in \mathsf{C}(a_3, b_3)$, $\varphi \notin \Sigma_1^1(\mathcal{A}_1')$, $\psi \in \Sigma_1^1(\mathcal{A}_1')$, then correct elementary change $\mathfrak{C}_\lambda(\varphi, \psi)$ leads to condition $\mathsf{C}(a_3, b_3) = \{\varphi\}$ without corruption of multiplication on $\Sigma_1^1(\mathcal{A}_1')$. Finally, if $\mathsf{C}(a_3, b_3) \subset \Sigma_1^1(\mathcal{A}_1')$, then $|\mathsf{C}(a_3, b_3)| \leqslant 2$ while $h_m(a_3, b_3) = 1$. If $\mathsf{C}(a_3, b_3) = \{\varphi\}$, then φ can not be joint in $\mathcal{A}_1'$ by Remark 4. Hence, φ is joint in $\mathcal{A}$. If $\mathsf{C}(a_3, b_3) = \{\varphi, \psi\}$, φ, ψ are joint parallel in $\mathcal{A}$. Obviously, multiplication rule holds in any case. The cases $|\Sigma_1^0(E_1, A)| = 1$ or $|\Sigma_1^0(E_1, B)| = 1$ are similar. $\square$

Lemma 17. *Assume* $\Sigma_0^- = \{E_1, E_2\}$, $\Sigma_1^0(E_1, A) = \{a_1, a_2\}$, $a_1 < a_2$, $\Sigma_1^0(E_2, A) = \{a_3\}$, $\Sigma_1^0(E_1, B) = \{b_1\}$, $\Sigma_1^0(E_2, B) = \{b_2, b_3\}$, $b_2 < b_3$. *Then Proposition 3 holds.*

Proof. Let us apply Proposition 2 to $\mathcal{A}_1'$. We have

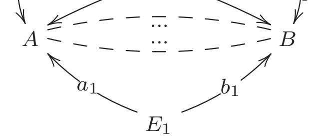


Denote $\Xi = \Sigma_1^1(\mathcal{A}_1')(A,B)$. Since $\mathcal{A}_1'$ has quasi multiplicative basis $\Sigma_1^1(\mathcal{A}_1')$, then $|\Xi \cup \Sigma_1^1(\mathcal{A}_1')(B,A)| \leqslant 2$. If $\Xi = \{\varphi, \psi\}$, then $a_1 \underset{\alpha_{12}}{<} a_2 \underset{\varphi}{<} b_1$, $a_1 \underset{\psi}{<} b_1$ and hence $h(\varphi) = 1$, $\psi \parallel \varphi\alpha_{12}$. By the associativity condition, $\psi \notin \Sigma_1^1(\mathcal{A}_2')(A,B)$. If $\Xi = \{\varphi\}$, then $h(\varphi) = 1$. Thus, if $\Xi \neq \varnothing$, then Ξ contains the unique φ of height 1. If $\varphi \notin \Sigma_1^1(\mathcal{A}_2')$ or $\Xi = \varnothing$, then there is a standard change of $\Sigma_1^1(\mathcal{A}_2')(A,B)$ such that each pair $(a_i, b_j) \in \Pi_{A,B}(\mathcal{A}_2')$ is adjusted and every arrow in $\Sigma_1^1(A,B)$ is single.

Let $\varphi \in \Xi \cap \Sigma_1^1(\mathcal{A}_2')$. Then $h(\varphi) = 1$. Consider $(a_i, b_j) \in \Pi_{A,B}(\mathcal{A}_2')$. If $\varphi, \psi \in \mathsf{S}(a_i, b_j)$, then $\mathfrak{C}_\lambda(\psi, \varphi)$ leads to condition $\varphi \notin \mathsf{S}(a_i, b_j)$. If $\varphi \notin \mathsf{S}(a_i, b_j)$, then $|\mathsf{S}(a_i, b_j)| = 1$ and $\mathsf{S}(a_i, b_j) = \mathsf{C}(a_i, b_j)$ after some standard basis change on $\Sigma_1^1(\mathcal{A}_2')(A,B)$. If $\mathsf{S}(a_i, b_j) = \{\varphi\}$, then φ is joint. In any case, each arrow from $\Sigma_1^1(A,B) \backslash \{\varphi\}$ is single. All changes above does not corrupt quasi multiplicativity of $\Sigma_1^1(\mathcal{A}_1')$. The basis on $\Sigma_1^1(\mathcal{A}_2')(B,A)$ can be chosen similarly. It is clear that multiplication rule holds on $\mathcal{A}$. $\square$

Lemma 18. *Let* $\Sigma_0^- = \{E_1, \ldots, E_p\}$, $2 \leqslant p \leqslant 3$, *and* $\Sigma_1^0(E_i, A) = \{a_i\}$, $\Sigma_1^0(E_i, B) = \{b_i\}$, $i = 1, \ldots, p$. *Then Proposition 3 holds.*

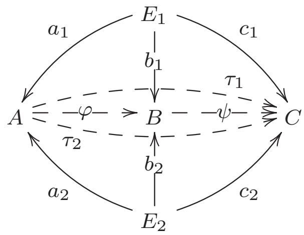
Proof. Here we have



As above, applying a standard basis change we obtain that each comparable pair is adjusted and every arrow from Σ_1^1 is either single or joint, which implies multiplication rule on $\mathcal{A}$. $\square$

The proof of Proposition 3 is completed.

Lemma 19. *Let $\mathcal{A} \in \mathcal{C}$ be a faithful bimodule problem with a triangled basis Σ, and $\Sigma_0^+ = \{A, B, C\}$. If $\varphi \in \Sigma_1^1(A, B)$, $\psi \in \Sigma_1^1(B, C)$, and $|\{A, B, C\}| = 3$, then $\psi\varphi$ belongs to Σ_1^1 except of the case:*



where $\psi\varphi \in \Bbbk^*(\tau_1, \tau_2)$, $\tau_1, \tau_2 \in \Sigma_1^1(A, C)$, and $\mathrm{P}_\varphi = \{(a_1, b_1), (a_2, b_2)\}$; $\mathrm{P}_\psi = \{(b_1, c_1), (b_2, c_2)\}$, $\mathrm{P}_{\tau_t} = \{(a_t, c_t)\}$, $t = 1, 2$.

The proof is similar to the proof of Lemma 15.

The proof of Theorem 1. Since $\mathcal{A}$ is a faithful, then the bimodule problems $\mathcal{A}^{(A)}$, $\mathcal{A}^{(A,B)}$ are faithful as well for any $A, B \in \Sigma_0^+$ by Remark 3. Applying consequently Proposition 1 to $\mathcal{A}^{(A)}$, Proposition 3 to $\mathcal{A}^{(A,B)}$, and Lemma 19 for all $A, B, C \in \Sigma_0^+$, we obtain the proof of Theorem 1. $\square$

Conclusion

The main result of the paper states the existence of quasi multiplicative basis for a faithful connected finite dimensional bimodule problem with nilpotent radical from considered class $\mathcal{C}$. The authors are going to apply the obtained results to construct the universal covering for such problem.

References

[1] *V. M. Babych, N. S. Golovashchuk.* An Application of Covering Techniques // Scientific Bulletin of Uzhgorod Univ. Series Mathematics and Informatics, **8**, 2003, 4 – 14.

[2] *V. M. Babych, N. S. Golovashchuk, S. A. Ovsienko.* Quasi multiplicative bases for bimodule problems from some class // Scientific Bulletin of Uzhgorod Univ. Series Mathematics and Informatics, **22**, 2011, 12 – 17.

[3] *R. Bautista* On algebras of strongly unbounded representation type // Comment. Math. Helv., **60**, no. 3, 1985, 392 – 399.

[4] *R. Bautista, P. Gabriel, A. V. Roiter, L. Salmeron.* Representation-finite algebras and multiplicative basis. // Invent. Math., **81**, 1985, 217 – 285.

[5] *K. Bongartz* Critical simply connected algebras // Manuscr. Math., **46**, 1984, 117 – 136.

[6] *Yu. A. Drozd.* On tame and wild matrix problems, In: Matrix Problems, Kiev, 1977, 104 – 114.

[7] *P. Gabriel, A. Roiter*. Representations of Finite Dimensional Algebras // A. I. Kostrikin, I. R. Shafarevich eds., Algebra VIII, Springer, 1991.

[8] *S. A. Ovsienko.* Bimodule and matrix problems. // Progress in Mathematics, **173**, 1999, 323 – 357.

[9] *A. V. Roiter, V. V. Sergejchuk*. Existence of a multiplicative basis for a finitely spaced module over an aggregate. // Ukrainian Math. Journal, **46**, 5, 1994, 567 – 579 (Engl. transl. 604 – 617).

Contact information

V. Babych Faculty of Mechanics and Mathematics
Taras Shevchenko National University of Kyiv
64, Volodymyrs'ka St., 01601 Kyiv, Ukraine
E-Mail: `vyacheslav.babych@gmail.com`

N. Golovashchuk Faculty of Mechanics and Mathematics
Taras Shevchenko National University of Kyiv
64, Volodymyrs'ka St., 01601 Kyiv, Ukraine
E-Mail: `golovash@gmail.com`

S. Ovsienko Faculty of Mechanics and Mathematics
Taras Shevchenko National University of Kyiv
64, Volodymyrs'ka St., 01601 Kyiv, Ukraine
E-Mail: `ovsiyenko.sergiy@gmail.com`

Received by the editors: 11.09.2011
and in final form 30.12.2011.