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On partial Galois Azumaya extensions

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Abstract. Let α be a globalizable partial action of a finite group G over a unital ring R, $A = R \star_\alpha G$ the corresponding partial skew group ring, R^α the subring of the α-invariant elements of R and $\alpha^\star$ the partial inner action of G (induced by α) on the centralizer $C_A(R)$ of R in A. In this paper we present equivalent conditions to characterize R as an α-partial Galois Azumaya extension of R^α and $C_A(R)$ as an $\alpha^\star$-partial Galois extension of the center $C(A)$ of A. In particular, we extend to the setting of partial group actions similar results due to R. Alfaro and G. Szeto [1, 2, 3].

1. Introduction

The notion of Galois Azumaya extension was introduced by Alfaro and Szeto in [3], motivated by an early work by themselves [2] about the conditions necessary and sufficient for a skew group ring to be Azumaya. Early, several other authors had considered this problem, among them DeMeyer and Janusz [6] for group rings, Szeto and Wong [19] for twisted group rings, and Ikehata [12] for skew group ring over commutative rings. In [2] Alfaro and Szeto extend the Ikehata's results to the noncommutative case. The results in [2] were considered later by Ouyang [14] for smash products, by Carvalho [5] for not twisted crossed products and by Paques and Sant'Ana [16] for partial crossed products. All these above mentioned results show in particular, each one in its respective context, that there exists an interesting and closed relation among the notions of Azumaya algebra, Galois extension and Hirata separability.

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In this paper we deal with globalizable partial actions of a finite group G over a ring R, and the respective partial skew group ring. Our aim is to extend to this context the Alfaro-Szeto's results from [1, 2]. First we prove a result analogous to [2, Theorems 1 and 2] for the partial skew group ring $A = R \star_\alpha G$, where α denotes a partial action of G on R (see Theorem 1.1). In the sequel we consider the "inner" partial action α^*, induced by α, of G on the centralizer $C_A(R)$ of R in A (see Proposition 2.8) and we also prove a result analogous to [1, Theorem 3] for the partial skew group ring $\Lambda = C_A(R) \star_{\alpha^\star} G$ (see Theorem 1.2).

Throughout, unless otherwise stated, rings and algebras are associative and unital. For any non-empty subset X of a ring R, any subring Y of R containing X and any (Y, Y)-bimodule V we will denote by $C_V(X)$ the centralizer of X in V, that is, the set of all $v \in V$ such that $xv = vx$ for all $x \in X$. If in particular $X = Y = V = R$, then $C_V(X)$ is the center of R and we will denote it simply by $C(R)$.

A *partial action* α of a group G on a ring R [7] is a pair

$$\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}),$$

where for each $g \in G$, D_g is an ideal of R and $\alpha_g : D_{g^{-1}} \to D_g$ is an isomorphism of (nonnecessarily unital) rings, satisfying the following conditions:

(i) $D_1 = R$ and α_1 is the identity automorphism I_R of R;

(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;

(iii) $\alpha_g \circ \alpha_h(r) = \alpha_{gh}(r)$, for every $r \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

If $D_g = R$ for every $g \in G$, then α is a global action of the group G on R, by automorphisms of R.

We will assume hereafter that every ideal D_g is unital, with its identity element denoted by 1_g (in particular, each 1_g is a central idempotent of R). By [7, Theorem 4.5], this condition is equivalent to say that α has a *globalization* (or an *enveloping action*), which means that there exist a (nonnecessarily unital) ring T and a global action of G on T, by automorphisms β_g $(g \in G)$, such that R can be considered an ideal of T and the following conditions hold:

(i) $T = \sum_{g\in G} \beta_g(R)$;

(ii) $D_g = R \cap \beta_g(R)$, for all $g \in G$;

(iii) $\alpha_g = \beta_g|_{D_{g^{-1}}}$.

In particular, under these conditions, we have $R = T1_R$ and

$$1_g = 1_R\beta_g(1_R), \quad \alpha_g(r1_{g^{-1}}) = \beta_g(r)1_R \quad \text{and} \quad \alpha_g(1_h1_{g^{-1}}) = 1_g1_{gh}$$

for every $g, h \in G$ and $r \in R$. Also, if G is finite then T is unital.

Following [7], the *partial skew group ring* $R \star_\alpha G$ is defined as the direct sum

$$\bigoplus_{g\in G} D_g\delta_g,$$

where the $\delta_g's$ are symbols, with the usual sum and the multiplication defined by the rule

$$(r\delta_g)(s\delta_h) = r\alpha_g(s1_{g^{-1}})\delta_{gh}$$

for all $g, h \in G, r \in D_g$ and $s \in D_h$. Since every D_g is unital by assumption, then $R \star_\alpha G$ is associative (see [7, Proposition 2.5 and Theorem 3.1]) and unital, with the identity element given by $1_R\delta_1$. Also, $R \star_\alpha G$ is a ring extension of R via the embedding $r \mapsto r\delta_1$, for all $r \in R$.

The subring of *invariants* of R under α [8] is defined as

$$R^\alpha = \{r \in R : \alpha_g(r1_{g^{-1}}) = r1_g, \text{ for all } g \in G\},$$

Note that if α is global then $R^\alpha = R^G$ as usual. We say that R is an *α-partial Galois extension* of R^α (or a *G-Galois extension* of R^G, if α is global) if G is finite and there exists a finite set $\{x_i, y_i\}_{i=1}^m$ of elements of R such that $\sum_{i=1}^m x_i\alpha_g(y_i1_{g^{-1}}) = \delta_{1,g}1_R$, for every $g \in G$. Such a set is called a *partial Galois coordinate system* of R over R^α.

A non-empty subset X of R is called *α-invariant* (or *G-invariant*, if α is global) if $\alpha_g(D_{g^{-1}} \cap X) = D_g \cap X$, for every $g \in G$. In particular, the centralizer $C_R(X)$ of any non-empty α-invariant subset X of R is also α-invariant. Since every $D_g = R1_g$ it is immediate to see that if X is an α-invariant subring of R then 1_g belongs to X for every $g \in G$. Moreover, in this case α induces by restriction a partial action on $X' = C_R(X)$ given by the pair $(\{X'1_g\}_{g\in G}, \{\alpha_g|_{X'1_{g^{-1}}}\}_{g\in G})$.

Let $S \supseteq R$ be a ring extension. We say that S is *Hirata-separable* over R [9] if $S \otimes_R S$ is isomorphic, as an S-bimodule, to a direct summand of a finite direct sum of copies of S or, equivalently, if there exist elements $x_i \in C_S(R)$ and $y_i \in C_{S\otimes_R S}(S)$, $1 \leq i \leq m$, such that $\sum_{1\leq i\leq m} x_iy_i = 1_S \otimes 1_S$ [18, Proposition 1]. The set $\{x_i, y_i \mid 1 \leq i \leq m\}$ is called an *Hirata-separable system* of S over R. Hirata-separable extensions are separable [10, Theorem 2.2]. S is called *separable* over R (see [11]) if the multiplication map $m_S : S \otimes_R S \to S$ is a splitting epimorphism of S-bimodules or, equivalently, if there exists an element $x \in C_{S\otimes_R S}(S)$

such that $m_S(x) = 1_S$. If $R \subseteq C(S)$ (resp., $R = C(S)$) we also say that S is a separable (resp., an *Azumaya*) R-algebra. A ring R is called *Azumaya* if it is an Azumaya $C(R)$-algebra. Furthermore, provided the existence of a partial action α of a finite group G on a ring R, we say that R is an *α-partial Galois Azumaya extension* of R^α if R is an α-partial Galois extension of R^α, R^α is an Azumaya ring and $C(R^\alpha) = C(R)^\alpha$.

Our main results in this paper are the following theorems.

Theorem 1.1. *Let α be a globalizable partial action of a finite group G on a ring R, and $A = R \star_\alpha G$. Then the following statements are equivalent:*

(i) *R is an α-partial Galois Azumaya extension of R^α.*

(ii) *A is Azumaya and $C(A) \subseteq R$.*

(iii) *A is Hirata-separable over R, R is a separable $C(A)$-algebra, and $C(A) = C(R)^\alpha$.*

(iv) *$C_R(R^\alpha)$ is an α-partial Galois extension of $C(A)$ and R^α is an Azumaya $C(A)$-algebra.*

Moreover, in this case, $C(A) = C(R)^\alpha = C(R^\alpha)$ and $A \simeq R^\alpha \otimes_{C(A)} End_{C(A)}(C_R(R^\alpha))$.

Theorem 1.2. *Let α be a globalizable partial action of a finite group G on a ring R, and $A = R \star_\alpha G$. Let $\alpha^\star$ be the partial inner action, induced by α, of G on $C_A(R)$, and $\Lambda = C_A(R) \star_{\alpha^\star} G$. Then the following statements are equivalent:*

(i) *$C_A(R)$ is an $\alpha^\star$-partial Galois extension of $C(A)$.*

(ii) *Λ is Hirata-separable over $C_A(R)$.*

(iii) *A is Hirata-separable over R and $C_\Lambda(C_\Lambda(C_A(R))) = C_A(R)$.*

(iv) *Λ is an Azumaya $C(A)$-algebra.*

Moreover, in this case, $C(\Lambda) = C(C_A(R))^{\alpha^\star} = C(R)^\alpha = C(A) = C_A(R)^{\alpha^\star}$.

Their proofs will be done via an explicit way going from the partial to the global case and conversely (see section 3). Some examples illustrating these results are given in the section 4.

We present in the next section the necessary preparation to prove the above theorems. Actually, the proofs of these theorems can be seen as applications of the results we will present in the next section, which also are of some independent interest.

2. Prerequisites

From now on, G will denote a finite group and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a partial action of G on a given ring R, with globalization (T, β). Also, let $A = R \star_\alpha G$ and $B = T \star_\beta G$.

Since $T = \sum_{g\in G} \beta_g(R)$, putting $G = \{g_1 = 1, g_2, \dots, g_n\}$, we have that $1_T = e_1 \oplus e_2 \oplus \cdots \oplus e_n$, where $e_1 = 1_R$ and $e_i = (1_T - 1_R)\cdots(1_T - \beta_{g_{i-1}}(1_R))\beta_{g_i}(1_R)$, for every $2 \le i \le n$, (see [8]). Let $\psi : T \to T$ be the map given by

$$\psi(x) = \sum_{i=1}^{n} \beta_{g_i}(x)e_i = \sum_{1\le l\le n} \sum_{i_1<\cdots<i_l} (-1)^{l+1}\beta_{g_{i_1}}(1_R)\cdots\beta_{g_{i_{l-1}}}(1_R)\beta_{g_l}(x),$$

for every $x \in T$. Such a map was introduced in [8], it is clearly (left and right) T^G-linear and multiplicative, and it will be useful in the sequel.

Lemma 2.1. *The following statements are equivalent:*

(i) *R is an α-partial Galois Azumaya extension of R^α.*

(ii) *T is a G-Galois Azumaya extension of T^G.*

Proof. See [15, Corollary 2.3]. □

Lemma 2.2. *The following statements are equivalent:*

(ii) *A is Azumaya and $C(A) \subseteq R$.*

(ii) *A is Hirata-separable over R, R is a separable $C(R)^\alpha$-algebra and $C(A) = C(R)^\alpha$.*

Proof. We start by observing that $C(A) \subseteq R$ if and only if $C(A) = C(R)^\alpha$. Indeed, if $r \in C(R)^\alpha$ then $(r_g\delta_g)(r\delta_1) = r_g\alpha_g(r1_{g^{-1}})\delta_g = r_g r\delta_g = (r\delta_1)(r_g\delta_g)$, for every $g \in G$ and $r_g \in D_g$. So $C(R)^\alpha \subseteq C(S)$. Conversely, if $x \in C(S) \subseteq R$ then $x \in C(R)$ and $x(1_g\delta_g) = (x\delta_1)(1_g\delta_g) = (1_g\delta_g)(x\delta_1) = \alpha_g(x1_{g^{-1}})\delta_g$, which implies $\alpha(x1_{g^{-1}}) = x1_g$, for every $g \in G$. So $x \in C(R)^\alpha$.

Now, the proof of the equivalence (i)⇔(ii) above follows by tracking the same arguments as in the global case. For the details, see the proof of (1)⇒(2) and the first paragraph of the proof of (2)⇒(3) of Theorem 1 in [2]. □

Lemma 2.3. *The following statements are equivalent:*

(i) *A is Hirata-separable over R.*

(ii) *B is Hirata-separable over T.*

Proof. (i)$\Rightarrow$ (ii) We start by observing that $C(R) \subseteq C(T)$ and $\psi(C(R)) \subseteq C(T)$. The first assertion is immediate since for any $r \in C(R)$ and any $t \in T$ we have $rt = r(1_R t) = (1_R t)r = tr$. For the second assertion, taking $r \in C(R)$ and putting $x = \psi(r) \in T$, it is clear that $\psi(s)x = x\psi(s)$ for every $s \in R$ and thus $\sum_i \beta_{g_i}(s)e_i x = \sum_i x\beta_{g_i}(s)e_i$, which implies $\beta_{g_i}(s)e_i x = x\beta_{g_i}(s)e_i$, for all $1 \leq i \leq n$. Since $T = \bigoplus_i \beta_{g_i}(R)e_i$ the result follows.

We also have that $C_A(R) \subseteq C_B(T)$. Indeed, recall from [16, Lemma 2.1] that $C_A(R) = \sum_{g\in G} \phi_g(R)\delta_g$, where $\phi_g(R)$ is the set of all $r \in D_g$ such that $r\alpha_g(s1_{g^{-1}}) = sr$, for all $s \in R$. So, it is enough to show that $\phi_g(R) \subseteq \phi_g(T)$ for all $g \in G$, which is immediate since given $r \in \phi_g(R)$ we have $r\beta_g(t) = r1_g\beta_g(t) = r1_R\beta_g(1_R t) = r\alpha_g(1_R t1_{g^{-1}}) = 1_R tr = tr$, for all $t \in T$.

Now suppose that A is Hirata-separable over R and take $x_i \in C_A(R)$ and $y_i \in C_{A\otimes_R A}(A)$, $1 \leq i \leq m$, such that $\sum_i x_i y_i = 1_A \otimes 1_A$. Thus, each $x_i = \sum_{g\in G} r_{i,g}\delta_g$, with $r_{i,g} \in \phi_g(R)$ for all $g \in G$. In particular $r_{i,1} \in C(R)$. Also, by [16, Lemma 2.1(ii)-(iv)] we can take each $y_i = \sum_{h\in G} \alpha_h(c_i 1_{h^{-1}})\delta_h \otimes 1_{h^{-1}}\delta_{h^{-1}}$, with $c_i \in C(R)$. From $\sum_i x_i y_i = 1_A \otimes 1_A$ we get $\sum_i r_{i,1}c_i = 1_R$ and $\sum_i r_{i,g}\alpha_{gh}(c_i 1_{(gh)^{-1}}) = 0$ if either $g \neq 1$ or $h \neq 1$.

Then, denoting $r_{i,l} := r_{i,g_l}$ we have

$$\begin{aligned}
\textstyle\sum_i r_{i,l}\beta_{g_l g_j}(c_i)\beta_{g_l}(1_T - e_j) &= \textstyle\sum_i r_{i,l}1_R\beta_{g_l g_j}(c_i)\beta_{g_l}(1_T - e_j) \\
&= (\textstyle\sum_i r_{i,l}\alpha_{g_l g_j}(c_i 1_{(g_l g_j)^{-1}}))\beta_{g_l}(1_T - e_j) \\
&= 0
\end{aligned}$$

for all $2 \leq j, l \leq n$,

$$\begin{aligned}
\textstyle\sum_i \psi(r_{i,1})\beta_{g_j}(c_i)(1_T - e_j) &= \textstyle\sum_{i,k} \beta_{g_k}(r_{i,1})e_k\beta_{g_j}(c_i)(1_T - e_j) \\
&= \textstyle\sum_k \beta_{g_k}(\sum_i r_{i,1}\beta_{g_k^{-1}g_j}(c_i))e_k(1_T - e_j) \\
&= \textstyle\sum_k \beta_{g_k}(\sum_i r_{i,1}\alpha_{g_k^{-1}g_j}(c_i 1_{g_j^{-1}g_k}))e_k(1_T - e_j) \\
&= \beta_{g_j}(1_R)e_j(1_T - e_j) \\
&= 0
\end{aligned}$$

for all $j \neq 1$, and

$$\begin{aligned}
\textstyle\sum_i r_{i,l}\beta_{g_l}(\psi(c_i)) &= \textstyle\sum_i r_{i,l}1_{g_l}\beta_{g_l}(\psi(c_i)) \\
&= \textstyle\sum_i r_{i,l}1_R\beta_{g_l}(\psi(c_i)1_R) \\
&= \textstyle\sum_i r_{i,l}1_R\beta_{g_l}(c_i) \\
&= \textstyle\sum_i r_{i,l}\alpha_{g_l}(c_i 1_{g_l^{-1}}) \\
&= 0
\end{aligned}$$

for all $l \neq 1$.

Taking $u_i = \psi(r_{i,1})\delta_1 + \sum_{2\leq l\leq n} r_{i,l}\delta_{g_l}$ and $v_i = \psi(c_i)\delta_1 \otimes 1_T\delta_1 + \sum_{2\leq j\leq n} \beta_{g_j}(c_i)(1_T - e_j)\delta_{g_j} \otimes 1_T\delta_{g_j^{-1}}$, we have from the above that $u_i \in C_B(T), v_i \in C_{B\otimes_T B}(B)$ for each $1 \leq i \leq m$, and $\sum_i u_iv_i = \psi(\sum_i r_{i,1}c_i)\delta_1 \otimes 1_T\delta_1 = \psi(1_R)\delta_1 \otimes 1_T\delta_1 = 1_T\delta_1 \otimes 1_T\delta_1 = 1_B \otimes 1_B$. Hence, B is Hirata-separable over T.

(ii)⇒ (i) Let $u_i \in C_B(T)$ and $v_i \in C_{B\otimes_T B}(B)$, $1 \leq i \leq m$, be a Hirata-separable system of B over T. Then by [16, Lemma 2.1] $u_i = \sum_{g\in G} t_{i,g}\delta_g$ with $t_{i,g} \in \phi_g(T)$, and we can take $v_i = \sum_{g\in G} \beta_g(d_i)\delta_g \otimes 1_T\delta_{g^{-1}}$ with $d_i \in C(T)$, for every $1 \leq i \leq m$. It is immediate that $c_i = d_i1_R \in C(R)$ for each $1 \leq i \leq m$. Also, every $r_{i,g} = t_{i,g}1_g \in \phi_g(R)$ since $r_{i,g} \in D_g$ and $r_{i,g}\alpha_g(r1_{g^{-1}}) = r_{i,g}\beta_g(r) = 1_gt_{i,g}\beta_g(r) = 1_grt_{i,g} = rr_{i,g}$ for all $r \in R$. Again by [16, Lemma 2.1] we have $x_i = \sum_{g\in G} r_{i,g}\delta_g \in C_A(R)$ and $y_i = \sum_{g\in G} \alpha_g(c_i1_{g^{-1}})\delta_g \otimes 1_{g^{-1}}\delta_{g^{-1}} \in C_{A\otimes_R A}(A)$.

Notice that the bi-additive map $B \times B \to A \otimes_R A$, given by $(b, b') \mapsto 1_Ab1_A \otimes 1_Ab'1_A$ is T-balanced, and so induces a well-defined left T-linear map $\theta : B \otimes_T B \to A \otimes_R A$.

Therefore,

$$\begin{aligned}
\sum_i x_iy_i &= \sum_i \sum_{g,h\in G}(r_{i,g}\delta_g)(\alpha_h(c_i1_{h^{-1}})\delta_h) \otimes 1_{h^{-1}}\delta_{h^{-1}}\\
&= \sum_i \sum_{g,h\in G} r_{i,g}\alpha_g(\alpha_h(c_i1_{h^{-1}})1_{g^{-1}})\delta_{gh} \otimes 1_{h^{-1}}\delta_{h^{-1}}\\
&= \sum_i \sum_{g,h\in G} r_{i,g}\alpha_{gh}(c_i1_{(gh)^{-1}})\delta_{gh} \otimes 1_{h^{-1}}\delta_{h^{-1}}\\
&= \sum_i \sum_{g,h\in G} t_{i,g}1_g\alpha_{gh}((d_i1_R)1_{(gh)^{-1}})\delta_{gh} \otimes 1_{h^{-1}}\delta_{h^{-1}}\\
&= \sum_i \sum_{g,h\in G} t_{i,g}\alpha_{gh}((d_i1_R)1_{(gh)^{-1}})\delta_{gh} \otimes 1_{h^{-1}}\delta_{h^{-1}}\\
&= \sum_i \sum_{g,h\in G} t_{i,g}1_R\beta_{gh}(d_i1_R)\delta_{gh} \otimes 1_{h^{-1}}\delta_{h^{-1}}\\
&= \sum_i \sum_{g,h\in G} 1_A(t_{i,g}\beta_{gh}(d_i)\delta_{gh})1_A \otimes 1_A(1_T\delta_{h^{-1}})1_A\\
&= \theta(\sum_i \sum_{g,h\in G} t_{i,g}\beta_{gh}(d_i)\delta_{gh} \otimes 1_T\delta_{h^{-1}})\\
&= \theta(1_B \otimes 1_B)\\
&= 1_A1_B1_A \otimes 1_A1_B1_A\\
&= 1_A \otimes 1_A
\end{aligned}$$

and A is Hirata-separable over R. □

Lemma 2.4. *The following statements are equivalent:*

(i) *R is a separable $C(R)^\alpha$-algebra.*

(ii) *T is a separable $C(T)^G$-algebra.*

Proof. It follows from [15, Lemma 2.1 (i), (iii) and (ix)]. □

Corollary 2.5. *The following statements are equivalent:*

(i) *A is Azumaya and $C(A) \subseteq R$.*

(ii) *B is Azumaya and $C(B) \subseteq T$.*

Proof. It follows from Lemmas 2.2, 2.3 and 2.4, and [2, Theorem 1]. $\square$

Lemma 2.6. *The following statements are equivalent:*

(i) *R^α is Azumaya.*

(ii) *T^G is Azumaya.*

Proof. It follows from [15, Lemma 2.1 (iii) and (viii)]. $\square$

Lemma 2.7. *The following statements are equivalent:*

(i) *$C_R(R^\alpha)$ is an α-partial Galois extension of $C(R^\alpha)$.*

(ii) *$C_T(T^G)$ is a G-Galois extension of $C(T^G)$.*

Proof. It follows from [15, Lemma 2.1(ii)-(iii) and Lemma 2.4] that $(C_T(T^G), \beta')$, with $\beta'_g = \beta_g|_{C_T(T^G)}$ for all $g \in G$, is a globalization of $(C_R(R^\alpha), \alpha')$, with $\alpha'_g = \alpha_g|_{C_R(R^\alpha)}$ for all $g \in G$. Now the result follows from [8, Theorem 3.3]. $\square$

These above listed results are sufficient to prove Theorem 1.1 (see section 3). For the proof of Theorem 1.2 we also need to introduce the notion of an “inner" partial action, denoted $\alpha^\star$, of G on $C_A(R)$, induced by the partial action α on R. In the particular case that $C_A(R) \subseteq R$, the partial action $\alpha^\star$ coincides with the restriction of α to $C_A(R)$.

Proposition 2.8. *Under the conditions above assumed, we have that:*

(i) *β induces an action $\beta^\star$ of G on B by inner automorphisms $\beta^\star_g$, given by $\beta^\star_g(t\delta_h) = (1_T\delta_g)(t\delta_h)(1_T\delta_{g^{-1}}) = \beta_g(t)\delta_{ghg^{-1}}$, for all $g, h \in G$ and $t \in T$,*

(ii) *$\beta^\star_g(1_{g^{-1}}) = 1_g$, $\beta^\star_g(1_{g^{-1}}1_h) = 1_g 1_{gh}$, and $1_R\beta^\star_g(r) = \alpha_g(r1_{g^{-1}})$, for $g, h \in G$ and $r \in R$,*

(iii) *$C_B(T)$ is $\beta^\star$-invariant, and therefore $\beta^\star$ induces, by restriction, an action of G on $C_B(T)$,*

(iv) *$C_A(R) = C_B(T)1_R = C_B(T)1_A$,*

(v) *$\alpha^\star := (\{C_A(R)_g = C_A(R)1_g\}_{g\in G}, \{\alpha^\star_g = \beta^\star_g|_{C_A(R)_{g^{-1}}}\}_{g\in G})$ is a partial action of G on $C_A(R)$, and $\alpha^\star_g(a) = (1_g\delta_g)a(1_{g^{-1}}\delta_{g^{-1}})$, for all $g \in G$ and $a \in C_A(R)_{g^{-1}}$,*

(vi) $(C_B(T),\ \beta^\star|_{C_B(T)})$ *is a globalization for* $(C_A(R),\ \alpha^\star)$,

(vii) $C_A(R)^{\alpha^\star} := \{a \in C_A(R) \mid \alpha^\star_g(a1_{g^{-1}}) = a1_g,$ *for all* $g \in G\} = C(A)$.

Proof. (i) It is enough to see that the map $\beta^\star : G \to \mathrm{Aut}(B)$, $g \mapsto \beta^\star_g$, is a homomorphism of groups. Clearly, $\beta^\star$ is well-defined, and for all $g, h, l \in G$ and $t \in T$ we have

$$\begin{aligned} \beta^\star(gh)(t\delta_l) &= \beta^\star_{gh}(t\delta_l) = \beta_{gh}(t)\delta_{ghl(gh)^{-1}} = \beta_g(\beta_h(t))\delta_{g(hlh^{-1})g^{-1}} \\ &= \beta^\star_g(\beta_h(t)\delta_{hlh^{-1}}) = \beta^\star_g(\beta^\star_h(t\delta_l)) = \beta^\star(g) \circ \beta^\star(h)(t\delta_l) \end{aligned}$$

(ii) For every $r \in R$ and $g, h \in G$ we have

$$\begin{aligned} \beta^\star_g(1_{g^{-1}}) &= \beta^\star_g(1_{g^{-1}}\delta_1) = \beta_g(1_R\beta_{g^{-1}}(1_R))\delta_1 \\ &= \beta_g(1_R)1_R\delta_1 = 1_g\delta_1 = 1_g. \end{aligned}$$

$$\begin{aligned} \beta^\star_g(1_{g^{-1}}1_h) &= \beta^\star_g(1_R\beta_{g^{-1}}(1_R)\beta_h(1_R)\delta_1) = \beta_g(1_R\beta_{g^{-1}}(1_R)\beta_h(1_R))\delta_1 \\ &= \beta_g(1_R)1_R\beta_{gh}(1_R)\delta_1) = 1_g1_{gh}\delta_1 = 1_g1_{gh}. \end{aligned}$$

$$1_R\beta^\star_g(r) = 1_R\beta^\star_g(r\delta_1) = 1_R\beta_g(r)\delta_1 = \alpha_g(r1_{g^{-1}})\delta_1 = \alpha_g(r1_{g^{-1}}).$$

(iii) For every $b \in C_B(T)$, $g \in G$ and $t \in T$ we have

$$\begin{aligned} \beta^\star_g(b)t &= (1_T\delta_g)b(1_T\delta_{g^{-1}})(t\delta_1) = (1_T\delta_g)b(\beta_{g^{-1}}(t)\delta_{g^{-1}}) \\ &= (1_T\delta_g)b(\beta_{g^{-1}}(t)\delta_1)(1_T\delta_{g^{-1}}) = (1_T\delta_g)(\beta_{g^{-1}}(t)\delta_1)b(1_T\delta_{g^{-1}}) \\ &= \beta_g(\beta_{g^{-1}}(t))(1_T\delta_g)b(1_T\delta_{g^{-1}}) = t\beta^\star_g(b), \end{aligned}$$

so $\beta^\star_g(b) \in C_B(T)$.

(iv) First note that

$$\begin{aligned} 1_RB1_R &= 1_R(\bigoplus_{g\in G} T\delta_g)1_R = \bigoplus_{g\in G}(T1_R\delta_g)(1_R\delta_1) \\ &= \bigoplus_{g\in G} T1_R\beta_g(1_R)\delta_g = \bigoplus_{g\in G} T1_g\delta_g = \bigoplus_{g\in G} D_g\delta_g = A. \end{aligned}$$

Also, 1_R is clearly a central idempotent in $C_B(T)$. Hence, $C_B(T)1_R = 1_RC_B(T)1_R = C_{1_RB1_R}(T) = C_A(T) \subseteq C_A(R)$. For the reverse inclusion, observe that given $a \in C_A(R)$ there exists $b \in B$ such that $a = 1_Rb1_R$ and so $at = (a1_R)t = a(1_Rt) = (1_Rt)a = t(1_Ra) = ta$, for every $t \in T$.

(v) Clearly, each $C_A(R)_g$ is an ideal of $C_A(R)$ and

$$\alpha^\star_g(C_A(R)_{g^{-1}}) = \beta^\star_g(C_A(R)1_{g^{-1}}) = \beta^\star_g(C_B(T)1_{g^{-1}})$$

$$\begin{aligned} &= \beta^\star_g(C_B(T))\beta^\star_g(1_{g^{-1}}) = C_B(T)1_g \\ &= C_A(R)1_g = C_A(R)_g. \end{aligned}$$

Thus, each $\alpha^\star_g : C_A(R)_{g^{-1}} \to C_A(R)_g$ is an isomorphism of rings. It is straightforward to check that the three conditions for $\alpha^\star$ to be a partial action are also satisfied. And

$$\begin{aligned} (1_g\delta_g)a(1_{g^{-1}}\delta_{g^{-1}}) &= (1_T\alpha_g(1_{g^{-1}})\delta_g)a(1_{g^{-1}}1_T\delta_{g^{-1}}) \\ &= (1_T\delta_g)(1_{g^{-1}}a1_{g^{-1}})(1_T\delta_{g^{-1}}) = \beta^\star_g(a) = \alpha^\star_g(a), \end{aligned}$$

for all $g \in G$ and $a \in C_A(R)_{g^{-1}}$.

(vi) Let $G = \{g_1 = 1, g_2, \ldots, g_n\}$. It follows from (iv) that $\beta^\star_{g_i}(C_A(R))$ is an ideal of $C_B(T)$ for all $1 \le i \le n$. Since $1_A = 1_R\delta_1 \in C_A(R)$, we have $1_B = 1_T\delta_1 = \sum_i \beta_{g_i}(1_R)e_i\delta_1 = \sum_i(\beta_{g_i}(1_R)\delta_1)(e_i\delta_1) = \sum_i \beta^\star_{g_i}(1_A)e_i\delta_1 \in \sum_i \beta^\star_{g_i}(C_A(R))$, and consequently $C_B(T) = \sum_i \beta^\star_{g_i}(C_A(R))$. Finally,

$$\begin{aligned} C_A(R) \cap \beta^\star_g(C_A(R)) &= C_B(T)1_R \cap \beta^\star_g(C_B(T)1_R) \\ &= C_B(T)1_R \cap \beta^\star_g(C_B(T))\beta^\star_g(1_R) \\ &= C_B(T)1_R \cap C_B(T)\beta^\star_g(1_R) = C_B(T)1_R\beta^\star_g(1_R) \\ &= C_B(T)1_g = C_A(R)1_g = C_A(R)_g. \end{aligned}$$

(vii) It is immediate, from the definitions of centralizer and subring of $\alpha^\star$-invariants, that $x \in C_A(R)^{\alpha^\star}$ if and only if $(r\delta_g)x = x(r\delta_g)$ for all $g \in G$ and $r \in D_g$ if and only if $x \in C(A)$. □

Lemma 2.9. *Let* $\Gamma = C_B(T) \star_{\beta^\star} G$, $\Lambda = C_A(R) \star_{\alpha^\star} G$, $C_0(\Gamma) = C_B(T)$ *(resp.,* $C_0(\Lambda) = C_A(R)$*) and* $C_i(\Gamma) = C_\Gamma(C_{i-1}(\Gamma))$ *(resp.,* $C_i(\Lambda) = C_\Lambda(C_{i-1}(\Lambda))$*) for all* $i \ge 1$*. Then,*

(i) $1_R\Gamma 1_R = \Lambda$,

(ii) 1_R *is a central idempotent in* $C_i(\Gamma)$ *and*

(iii) $C_i(\Gamma)1_R = C_i(\Lambda)$ *for all* $i \ge 0$.

Proof. It follows by induction via the same arguments used in the proof of Proposition 2.8(iv). □

Remark 2.10. Note that $\alpha^\star$ induces an inner partial action, denoted $\alpha^{\star\star}$, of G on $C_\Lambda(C_A(R))$ in the same fashion that α induces $\alpha^\star$. Therefore, Proposition 2.8 also applies in this similar situation. Furthermore, the restriction of $\alpha^{\star\star}$ to $C_A(R)$ (resp. R) coincides with $\alpha^\star$ (resp. α).

3. The Proofs

Proof of Theorem 1.1:

(i)⇒(ii) It follows from Lemma 2.1, [3, Theorem 1, (3)⇒(1)], and Corollary 2.5.

(ii)⇒(iii) It follows from Lemma 2.2.

(iii)⇒(i) If follows from Lemmas 2.3 and 2.4, [2, Theorem 1, (2)⇒(3)], and Lemma 2.1.

(i)⇒(iv) It follows from Lemma 2.1, [2, Theorem 2(3)], and Lemmas 2.6 and 2.7.

(iv)⇒(i) It is enough to notice that any partial Galois coordinate system of $C_R(R^\alpha)$ over $C(A)$ is also a partial Galois coordinate system of R over R^α.

For the last assertion we observe that by the same arguments used in the proof of [2, Theorem 2(1)] we have $R\star_\alpha G \simeq R^\alpha \otimes_{C(A)} (C_R(R^\alpha)\star_\alpha G)$. Since $C_R(R^\alpha)$ is an α-partial Galois extension of $C(A)$, then $C_R(R^\alpha) \star_\alpha G \simeq \mathrm{End}_{C(A)}(C_R(R^\alpha))$ by [8, Theorem 3.3], and the result follows. □

Proof of Theorem 1.2:

(i)⇔(ii) By Proposition 2.8(vi) $(C_B(T), \beta^\star)$ is a globalization of $(C_A(R), \alpha^\star)$ and by [8, Theorem 3.3], $C_B(T)$ is a Galois extension of $C_B(T)^{\beta^\star} (= C(B))$ if and only if $C_A(R)$ is an $\alpha^\star$-partial Galois extension of $C_A(R)^{\alpha^\star} (= C(A))$. Then, the result follows from [8, Theorem 3.3], [1, Theorem 2], and Lemma 2.3.

(ii)⇒(iii) It follows from Lemma 2.3, [1, Theorem 3] and Lemma 2.9.

(iii)⇒(iv) It follows by some arguments similar to the corresponding ones used in the proof of [1, Proposition 3], as we will see. From [10, Theorem 2.2] we have that A is a separable extension of R and so, by [4, Proposition 3.1] there exists $c \in C(R)$ such that $t_\alpha(c) := \sum_{g\in G} \alpha_g(c1_{g^{-1}}) = 1_R$. We also have from [17, Proposition 1.2] that $C_A(C_A(R)) = R$, for R is a direct summand of A as a left R-module. Hence,

$$C(A) = C(R)^\alpha$$

(since $C(A) \subseteq C_A(C_A(R))$ (see the proof of Lemma 2.2)),

$$C(R) = R \cap C_A(R) = C_A(C_A(R)) \cap C_A(R) = C(C_A(R))$$

and

$$tr_{\alpha^\star}(c\delta_1) \quad := \quad \sum_{g\in G} \alpha_g^\star((c\delta_1)1_{g^{-1}}) = \sum_{g\in G} \beta_g^\star(c1_{g^{-1}}\delta_1)$$

$$= \sum_{g\in G} \beta_g(c1_{g^{-1}})\delta_1 = \sum_{g\in G} \alpha_g(c1_{g^{-1}})\delta_1$$
$$= tr_\alpha(c)\delta_1 = 1_R\delta_1 = 1_A,$$

which implies, again by [4, Proposition 3.1], that Λ is separable over $C_A(R)$. Since $C_A(R)$ is separable over $C(A)$ by [13, Lemma 1], then Λ is also separable over $C(A)$ by the transitivity of the separability. Thus, it remains to prove that $C(\Lambda) = C(A)$. Indeed, setting $L = C_\Lambda(C_A(R))$, it follows by Proposition 2.8 that $C(\Lambda) = L^{\alpha^{\star\star}}$. Furthermore, it is immediate to see that $L^{\alpha^{\star\star}} = C(L)^{\alpha^{\star\star}}$ and then

$$\begin{aligned} C(\Lambda) &= C(L)^{\alpha^{\star\star}} = (C_\Lambda(L)\cap L)^{\alpha^{\star\star}} = (C_\Lambda(C_\Lambda(C_A(R)))\cap L)^{\alpha^{\star\star}} \\ &= (C_A(R)\cap L)^{\alpha^\star} = (C_A(R)\cap C_\Lambda(C_A(R)))^{\alpha^\star} = C(C_A(R))^{\alpha^\star} \\ &= C(R)^\alpha = C(A) \end{aligned}$$

(iv)$\Rightarrow$ (i) It follows from Corollary 2.5, [1, Proposition 3] and [8, Theorem 3.3].

The last assertion is immediate from the above. □

Remark 3.1. The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv) of Theorem 1.2 can also be seen as a corollary of Theorem 1.1 since $C_A(R)$, under the condition (i), is an $\alpha^\star$-partial Galois Azumaya extension of $C_A(R)^{\alpha^\star}$ indeed.

4. Some Examples

Example 4.1. Let R be an Azumaya ring, G a finite group and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ the trivial partial action of G on R, that is, $D_1 = R$, $\alpha_1 = I_R$ and $D_g = 0 = \alpha_g$ for all $g \neq 1$. In this case we have that $A = R\star_\alpha G = R = R^\alpha$, $C(A) = C(R) = C(R)^\alpha = C(R^\alpha)$ and R is an α-partial Galois Azumaya extension of R^α. Furthermore, $C_A(R) = C(R)$, $\alpha^\star$ coincides with the restriction of α to $C(R)$, $\Lambda = C_A(R)\star_{\alpha^\star} G = C(R)$, and each one of the equivalent assertions of Theorems 1.1 and 1.2 is trivially satisfied.

Example 4.2. Let S be an Azumaya ring and set $T = \prod_{1\le i\le 4} S_i = \bigoplus_{1\le i\le 4} S_ie_i$, where $S_i = S$ for every $1 \le i \le 4$, and each e_i is the quadruple whose j^{th}-coordinate is 1_S if $j = i$ and zero otherwise.

Consider G a cyclic group of order 4 generated by g, and $\beta : G \to \mathrm{Aut}(T)$ a global action of G on T by automorphisms β_{g^i} given by $\sum_{1\le j\le 4} a_je_j \mapsto \sum_{1\le j\le 4} a_je_{j+i(\mathrm{mod}\, 4)}$. Take $R = Se_1 \oplus Se_3 \oplus Se_4$ and α the partial action of G on R obtained from β by restriction to R, that is, $\alpha = (\{D_{g^i}\}_{1\le i\le 4}, \{\alpha_{g^i}\}_{1\le i\le 4})$ where

$D_1 = D_{g^0} = R$, $D_g = Se_1$, $D_{g^2} = Se_1 \oplus Se_3$, $D_{g^3} = Se_4$,
$\alpha_1 = \alpha_{g^0} = I_R$,
$\alpha_g : D_{g^3} \to D_g,\ se_4 \mapsto se_1$,
$\alpha_{g^2} : D_{g^2} \to D_{g^2},\ se_1 + s'e_3 \mapsto s'e_1 + se_3$,
$\alpha_{g^3} : D_g \to D_{g^3},\ \ se_1 \mapsto se_4$

Clearly (T, β) is a globalization of (R, α) and it is straightforward to verify that $R^\alpha = S(e_1 + e_3 + e_4) \simeq S$ and $C(R^\alpha) = C(R)^\alpha = C(S)(e_1 + e_3 + e_4) \simeq C(S)$. It is also clear that R is an α-partial Galois extension of R^α. In this case a partial Galois coordinate system is given by $x_1 = y_1 = e_1, x_2 = y_2 = e_3$ and $x_3 = y_3 = e_4$. Hence, R is an α-partial Galois Azumaya extension of R^α. Consequently, $A = R \star_\alpha G$ is Azumaya with $C(A) = C(R)^\alpha$ by Theorem 1.1. Moreover, it is easy to check that $C_A(R) = C(R)$ and so $\alpha^\star$ is the restriction of α to $C(R)$. Since each $e_i \in C(R)$, $i = 1, 3, 4$, then $C_A(R)$ is also an α-partial Galois Azumaya extension of $C(R)^\alpha$.

Example 4.3. (see [8, Example 6.3]) Let S be a commutative ring and G is a cyclic group of order 6, generated by g. Assume that S is a (global) Galois extension of S^G and set $R = \prod_{1\le i\le 5} S_i = \bigoplus_{1\le i\le 5} S_i e_i$, where $S_i = S$ for every $1 \le i \le 5$, and each e_i is the quintuple whose j^{th}-coordinate is 1_S if $j = i$ and zero otherwise. Taking $D_{g^0} = R$, $D_{g^i} = Se_{6-i} = Re_{6-i}$, $\alpha_{g^0} = I_R$ and $\alpha_{g^i}(se_i) = g^i(s)e_{6-i}$, $1 \le i \le 5$, it is straightforward to check that $\alpha = (\{D_{g^i}\}_{i=0}^5, \{\alpha_{g^i}\}_{i=0}^5)$ is a partial action of G on R and $R^\alpha = \{s_1e_1 + s_2e_2 + s_3e_3 + \sigma^2(s_2)e_4 + \sigma(s_1)e_5 \,|\, s_1, s_2 \in S, s_3 \in S^{\sigma^3}\}$. Let $s_i, t_i \in S, 1 \le i \le m$, be a Galois coordinate system for S over S^G and consider the elements $x_j = y_j = e_j, j = 1, 2, 4, 5$ together with the elements $x_{i3} = s_ie_3, y_{i3} = t_ie_3$. It is easy to see that this gives a partial Galois coordinate system for R over R^α. Since, in addition, R is commutative, then R is an α-partial Galois Azumaya extension of R^α. By Theorem 1.1, $A = R \star_\alpha G$ is Azumaya and $C(A) = R^\alpha$. Furthermore, $C_A(R) = R\delta_{g^0} \oplus Se_3\delta_{g^3}$ and it is an $\alpha^\star$-partial Galois extension of $C_A(R)^{\alpha^\star} = C(A)$ with the partial Galois coordinate system given by the elements $u_j = v_j = e_j\delta_{g^0}$, $j = 1, 2, 4, 5$ and $u_{i3} = s_ie_3\delta_{g^0}, v_{i3} = t_ie_3\delta_{g^0}$, $1 \le i \le m$.

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