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Some fixed point theorems for pseudo ordered sets

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Abstract. In this paper, it is shown that for an isotone map f on a pseudo ordered set A, the set of all fixed points of f inherits the properties of A, namely, completeness, chain-completeness and weakly chain-completeness, as in the case of posets.

1. Introduction

In 1955 Tarski [6] proved that the set of all fixed points of an order preserving map in a complete lattice constitutes a complete lattice. Later in 1976, Markowsky [2] generalized this result by proving that the set of all fixed points of an order preserving map in a chain-complete poset forms a chain-complete poset. We extend these results to generalized structures like trellises and pseudo ordered sets. Further, a counterexample is given to show that the least fixed point property does not imply weakly chain-completness even for an acyclic pseudo ordered set. This, in particular gives a negative solution to Problem 2 in [1].

2. Notations and definitions

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a non- empty set A is called a *pseudo order*. The set A together with this pseudo order $\trianglelefteq$ is called a *pseudo ordered set* or a *psoset*. For a subset of A, the notions of a lower bound, an upper bound, the greatest lower bound (or meet), the least upper

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bound (or join), the minimum (or the least) element and the maximum (or the greatest) element are defined analogous to the corresponding notions in a poset. Let B be a subset of A. Then for a subset X of B, the join of X in B is denoted by $\bigvee_B X$. For any two elements $a, b \in A$, if $a \trianglelefteq b$ and $a \neq b$, then we denote it as $a \triangleleft b$. If $a \trianglelefteq b$ does not hold, then we denote it by $a \ntrianglelefteq b$.

A *trellis* is a psoset, any two of whose elements have a join and a meet. A trellis is said to be a *complete trellis* if every subset of A has a meet and a join. An extensive investigation of the notions of psosets and related concepts can be found in H.L. Skala [3] and H. Skala [4].

A subset C of A, including $C = \phi$, is called a *chain* in A if the restriction of $\trianglelefteq$ to C is a complete order (i.e. it is a partial order on C such that every pair of elements of C are comparable). A chain C in A is said to be *well ordered* if every non-empty subset of C has the least element. A psoset A is said to be *chain-complete* if every chain in A has a join. A is said to be *weakly chain-complete* if every well ordered chain in A has a join. Eventhough the notions of chain-completeness and weakly chain-completeness coincide in posets, it is not known to date whether they are equivalent in case of psosets. The above definitions are due to Bhatta [1].

A map $f : A \to A$ is said to be *isotone* if $a \trianglelefteq b$ implies $f(a) \trianglelefteq f(b)$. An element $a \in A$ is said to be a *fixed point* for f if $f(a) = a$. If every isotone map of A into itself has a fixed point (the least fixed point), then A is said to have the *fixed point property* (*the least fixed point property*). The composition maps $f \circ f, f \circ f \circ f, \ldots$ are denoted by $f^2, f^3, \ldots$ respectively.

3. Results

The notion of an f-chain starting at a point p, comparable to its image, is well known for posets [5]. The following generalization of this definition helps us in our further discussion

Definition 3.1. *Let $< A, \trianglelefteq >$ be a psoset, $f : A \to A$ be an isotone map and B be a subset of $F_A = \{x \in A : f(x) = x\}$. For an ordinal ξ, a subset $S_\xi = \{x_\eta : \eta < \xi\}$ of A is called an f-chain on B if for any $\alpha < \xi$ we have*

$$x_\alpha = \begin{cases} \bigvee_A [B \cup \{x_\eta : \eta < \alpha\}] & \text{if } \alpha \text{ is a limit ordinal;} \\ f(x_\beta) & \text{otherwise, where } \alpha = \beta + 1. \end{cases}$$

To have more versatility later on, we shall not assume any sort of completeness on the psoset for the following lemmas.

Lemma 3.2. *Let $< A, \trianglelefteq >$ be a psoset, $f : A \to A$ be an isotone map and B be a subset of $F_A = \{x \in A : f(x) = x\}$. Then any f-chain on B is well ordered and is contained in the set of upper bounds of B in A.*

Proof. Assume the contrary. Let B^{Δ} denote the set of all upper bounds of B in A. Choose α to be the least ordinal for which either $x_\eta \ntrianglelefteq x_\alpha$ for some $\eta < \alpha$ or $x_\alpha \notin B^{\Delta}$. Clearly α is not a limit ordinal. Hence $\alpha = \beta + 1$ for some ordinal number β. Since $x_\beta \in B^{\Delta}$, it follows that $x_\alpha = f(x_\beta) \in B^{\Delta}$. Choose γ to be the least ordinal for which $x_\gamma \ntrianglelefteq x_\alpha$. Since x_α is is an upper bound for $\{x_\eta : \eta < \gamma\}$, γ cannot be a limit ordinal. Hence $\gamma = \delta + 1$ for an ordinal δ. As $x_\delta \trianglelefteq x_\beta$ and f is order preserving, we get $x_\gamma = f(x_\delta) \trianglelefteq f(x_\beta) = x_\alpha$, a contradiction. □

If $S_\alpha = \{x_\eta : \eta < \alpha\}$ and $S_\beta = \{x_\eta : \eta < \beta\}$ are two f-chains on B, then by transfinite induction it can be shown that $x_\gamma = y_\gamma$ for every $\gamma < \alpha, \beta$. Hence either both of them are equal or one should be an initial segment of the other. Thus there is a unique maximal f-chain on B.

Lemma 3.3. *Let $< A, \trianglelefteq >$ be a psoset and $f : A \to A$ be an isotone map. For a subset B of $F_A = \{x \in A : f(x) = x\}$, let S be the unique maximal f-chain on B. If $u = \bigvee_A (B \cup S)$ exists, then $u = \bigvee_{F_A} B$.*

Proof. We have $x \trianglelefteq f(x)$ for every $x \in S$. Further, $u = \bigvee_A (B \cup S) \in S$ so that $f(u) \in S$. Thus $f(u) = u$. Hence u is an upper bound for B in F_A. If y is any upper bound for B in F_A, then by transfinite induction, it follows that y is an upper bound for $B \cup S$ in A. Hence $u \trianglelefteq y$. □

The following theorems are direct consequences of Lemma 3.2 and Lemma 3.3.

Theorem 3.4. *Let $< A, \trianglelefteq >$ be a chain-complete psoset and $f : A \to A$ be an isotone map. Then $F_A = \{x \in A : f(x) = x\}$ is a chain-complete psoset in the induced order.*

Corollary 3.5. (Theorem 9, [2]) *Let P be a chain-complete poset, $f : P \to P$ isotone and $F_P = \{x \in P : f(x) = x\}$ be the set of all fixed points of f. Then*

(**i**) there is a least element $0^* \in F_P$.

(**ii**) for all $y \in P$, if $f(y) \leq y$, then $0^* \leq y$.

(**iii**) F_P is a chain-complete poset in the induced order.

Theorem 3.6. *Let $< A, \trianglelefteq >$ be a weakly chain-complete psoset and $f : A \to A$ be an isotone map. Then $F_A = \{x \in A : f(x) = x\}$ is a weakly chain-complete psoset in the induced order.*

Corollary 3.7. (Theorem, [1]) *Every weakly chain-complete psoset has the least fixed point property.*

Eventhough the following theorem follows directly from Lemma 3.3, a much shorter proof is given below.

Theorem 3.8. *Let $< A, \trianglelefteq >$ be a complete trellis and $f : A \to A$ be an isotone map. Then $F_A = \{x \in A : f(x) = x\}$ is a complete trellis in the induced order.*

Proof. Let B be a subset of F_A. Let B^Δ denote the set of all upper bounds of B in A. Then B^Δ is a complete trellis in the induced order and f is a self map on B^Δ. By Corollary 3.7 f has the least fixed point say u in B^Δ. Clearly, $u = \bigvee_{F_A} B$. Hence F_A is a complete trellis. □

Corollary 3.9.(Theorem 37, [4]) *If f is an isotone map of a complete trellis A onto itself such that $a \trianglelefteq f(a)$ for each a in A, then with respect to the same pseudo order on A, the set of all fixed points of f constitutes a complete trellis.*

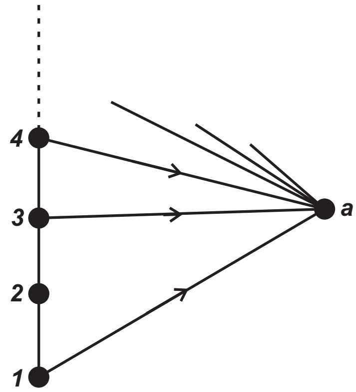
Corollary 3.10.(Theorem 1, [6]) *Let P be a complete lattice, f an isotone map of A to itself and F_P be the set of all fixed points of f. Then the set F_P is a complete lattice.*

Counterexample 3.11 The converse of Corollary 3.7 was posed as an open problem (Problem 2, [1]). A negative solution is given below to show that the converse doesn't hold even for an acyclic psoset.

Let $A = N \cup \{a\} = \{1, 2, 3, \ldots\} \cup \{a\}$. We define a pseudo-order $\trianglelefteq$ on A as follows. The elements of N are ordered by the usual natural order of the reals. Further, for any $k \in N, k \neq 2$ we have $k \triangleleft a$. This psoset is represented by the digraph in Figure 1.

Clearly, A is not weakly chain-complete as the well ordered chain $C = \{1, 2, 3, \ldots\}$ is not bounded above.

Let $f : A \to A$ be isotone. Suppose f does not have any fixed points. Since $1 \triangleleft f(1) \triangleleft f^2(1) \triangleleft \ldots$ is a chain in A, there is no $n \in N$ such that $f^n(1) = a$. For, if $f^n(1) = a$ for some $n \in N$, then we get $f(a) = a$, a contradiction. Thus $1 \triangleleft f(1) \triangleleft f^2(1) \triangleleft \ldots$ is a chain contained in N. Hence $f^2(1) \geq 3$ so that $\{f^2(1), f^3(1), \ldots\}$ is a chain contained in $\{3, 4, \ldots\}$. Since a is an upper bound for $\{f^2(1), f^3(1), \ldots\}$, $f(a)$ should be an upper bound for $\{f^3(1), f^4(1), \ldots\}$ in A. As a being the only upper bound of



$\{f^3(1), f^4(1), \ldots\}$ in A, we should have $f(a) = a$, a contradiction to the assumption that f has no fixed points. Hence f has a fixed point. Further, if $f(2) = 2$ and $f(a) = a$, then either $f(1) = 1$ or $f(1) = a$. Since $2 = f(2) \trianglelefteq f(3)$ and $f(1) \trianglelefteq f(3)$, it follows that $f(1) \neq a$. Thus A has the least fixed point property.

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