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On the prime spectrum of top modules

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Abstract. In this paper we investigate some properties of top modules and consider some conditions under which the spectrum of a top module is a spectral space.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and all modules are unital. The radical of an ideal I of R is denoted by $\sqrt{I}$ and

$$\sqrt{I} = \{x \in R | x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Let M be an R-module. A submodule N of M is said to be prime if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $p = (N : M)$ is a prime ideal of R, and N is said to be p-prime (see [14]). The set of all prime submodules of M is called the spectrum of M and denoted by Spec(M). Similarly, the collection of all p-prime submodules of M for any $p \in \text{Spec}(R)$ is designated by $\text{Spec}_p(M)$. We remark that $\text{Spec}(\mathbf{0}) = \emptyset$ and that Spec(M) may be empty for some nonzero module M. For example, the $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module has no prime submodule for any prime integer p (see [16]). Such a module is said to be primeless. Throughout this paper we assume that M is a non-primeless R-module. The set of all maximal submodules of M is denoted by $Max(M)$. The Jacobson radical $Rad(M)$

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of a module M is the intersection of all its maximal submodules. A module M is called a semi-local (resp. a local) module if $Max(M)$ is a non-empty finite (resp. a singleton) set.

When $\operatorname{Spec}(M) \neq \emptyset$, the map $\psi : \operatorname{Spec}(M) \to \operatorname{Spec}(R/\operatorname{Ann}(M))$, defined by $\psi(P) = (P : M)/\operatorname{Ann}(M)$ for every $P \in \operatorname{Spec}(M)$, will be called the natural map of $\operatorname{Spec}(M)$. An R-module M is called primeful if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and has a surjective natural map (see [19]). By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M. Let p be a prime ideal of R, and $N \leq M$. By the saturation of N with respect to p, we mean the contraction of N_p in M and designate it by $S_p(N)$ (see [18]).

M is called a multiplication module if every submodule N of M is of the form IM for some ideal I of R. For any submodule N of M we define $V(N)$ to be the set of all prime submodules of M containing N. If $\zeta(M)$ denotes the collection of all subsets $V(N)$ of $X = \operatorname{Spec}(M)$, then $\zeta(M)$ contains the empty set and $\operatorname{Spec}(M)$ and it is closed under arbitrary intersections. It is said that M is a module with Zariski topology or a top module for short, if $\zeta(M)$ is closed under finite unions, i.e. for any submodules N and L of M there exists a submodule J of M such that $V(N) \cup V(L) = V(J)$ (see [20]).

Let N be a submodule of M. If $V(N)$ has at least one minimal member with respect to the inclusion, then such a minimal member is called a minimal prime submodule of N or a prime submodule minimal over N. A minimal prime submodule of $(\mathbf{0})$ is called a minimal prime submodule of M.

A non-Noetherian commutative ring R is called a quasisemilocal ring if R has only a finite number of maximal ideals. A non-Noetherian commutative ring R is called a quasilocal ring if has only one maximal ideal. Let N be a submodule of M. N is called compactly packed by prime submodules if whenever N is contained in the union of a family of prime submodules of M, N is contained in one of the prime submodules of the family. M is called compactly packed if every submodule of M is compactly packed by prime submodules (see [11]). A submodule N of M is said to be strongly irreducible if for submodules N_1 and N_2 of M, the inclusion $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Strongly irreducible submodules has been characterized in [3]. For example every prime submodule of multiplication module is strongly irreducible (see [7, p. 1142, Lemma 4.11]). A module M is called a Bezout module if every finitely generated submodule is cyclic (see [22, 23]). A module M is called distributive if the lattice of its submodules is distributive, i.e.,

$A \cap (B + C) = (A \cap B) + (A \cap C)$ and $A + (B \cap C) = (A + B) \cap (A + C)$ for all submodules A, B and C of M (see [6]). We recall that every Bezout R-module is distributive (see [22, p. 307, Corollary 2]).

Now let M be a top module. The purpose of this paper is to discuss some topological properties of Spec(M). We explore the relation between Spec(R) and Spec(M) and investigate topological space Spec(M) from the point of view of spectral spaces, topological spaces each of which is homeomorphic to Spec(S) for some ring S. In Section 2, various algebraic properties of top modules are considered. We will consider the conditions under which M is a top module. In Section 3, we will discuss some topological properties of Spec(M).

2. Top modules

Let M be an R-module. For any subset E of M, we recall that $V(E)$ is the set of all prime submodules of M containing E. Also for a submodule N of M, the radical of N defined to be the intersection of all prime submodules of M containing N and denoted by $rad_M(N)$ or briefly $rad(N)$ (see [15]). In particular $rad(0_M)$ is the intersection of all prime submodules of M. We say N is a radical submodule if $rad(N) = N$. For every subset Y of Spec(M), $\Im(Y)$ is defined to be the intersection of all prime submodules of M which belong to Y (see [18, 19]).

Let M be an R-module and $X = \text{Spec}(M)$. If N is a submodule of M generated by a set S, then $V(S) = V(N)$. We have $V(\mathbf{0}) = X$ and $V(M) = \emptyset$. If $\{N_i\}_{i\in I}$ is any family of subsets of M, then $V(\cup_{i\in I} N_i) = \cap_{i\in I} V(N_i)$. Also $V(N_1 \cap N_2) \supseteq V(N_1) \cup V(N_2)$ for any submodules N_1 and N_2 of M. Since $\sum_{i\in I} N_i$ generated by $\bigcup_{i\in I} N_i$, we have

$$V(\sum_{i\in I} N_i) = V(\bigcup_{i\in I} N_i) = \bigcap_{i\in I} V(N_i).$$

We denote $V(Rm)$ by $V(m)$.

If $\zeta(M)$ denotes the collection of all subsets $V(N)$ of $X = \text{Spec}(M)$, then $\zeta(M)$ contains the empty set and Spec(M) and it is closed under arbitrary intersections. We recall that M is a module with a Zariski topology or a top module for short, if $\zeta(M)$ is closed under finite unions, that is, for any submodules N and L of M there exists a submodule J of M such that $V(N) \cup V(L) = V(J)$. In this case $\zeta(M)$ satisfies the axioms for closed subsets of topological space (see [20]).

Theorem 2.1. *Let M be an R-module. Then M is a top module in each of the following cases.*

1. *Every prime submodule of M is strongly irreducible.*
2. *M is an R-module with the property that for any two submodules N and L of M, $(N : M)$ and $(L : M)$ are comaximal.*
3. *M is a Bezout R-module.*
4. *R is a quasisemilocal ring and M is a distributive R-module.*
5. *M is an Artinian distributive R-module.*
6. *M is a distributive R-module with the property that every submodule has only finitely many maximal submodules.*

Proof.

1. Always we have, $V(N \cap L) \supseteq V(N) \cup V(L)$ for each submodules N and L of M. Now let $P \in V(N \cap L)$, thus $N \cap L \subseteq P$. Since P is strongly irreducible, either $N \subseteq P$ or $L \subseteq P$. Therefore $P \in V(N) \cup V(L)$. Thus $\zeta(M)$ is closed under finite unions. Hence M is a top module.
2. Let P be a prime submodule of M with $N \cap L \subseteq P$. Then

   $$(N : M) \cap (L : M) \subseteq (P : M) \in \mathrm{Spec}(R).$$

   We may assume that $(N : M) \subseteq (P : M)$. Then clearly $(L : M) \not\subseteq (P : M)$ by assumption. Hence $N \subseteq P$ by [15, p. 215, Lemma 2]. Therefore P is strongly irreducible. This implies that M is a top module by part (1).
3. Let P be a prime submodule of M such that $N \cap L \subseteq P$ for submodules N and L of M. Let $N \not\subseteq P$, $a \in N \setminus P$, and $b \in L$. Then there exists $z \in M$ such that $Ra + Rb = Rz$. Thus there exists $r, s \in R$, such that $a = rz, b = sz$. Then we have that $sa \in P$, so $s \in (P : M)$. In particular $sz \in P$, whence $b \in P$. This implies that M is a top module by part (1).
4. Use [6, p. 176, Proposition 7 and p. 175, Proposition 4], and part (3).
5. Use [6, p. 176, Proposition 7], [12, p. 764, Corollary 2.9], and part (3).
6. Use [6, p. 176, Proposition 7], [12, p. 763, Theorem 2.8], and part (3). □

Remark 2.2. Let M be a top R-module. Then by [17, p. 429, Corollary 6.2 and Theorem 6.1], the natural map $\psi : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$, is injective.

Theorem 2.3. *Let M be a top R-module. Then*

1. *Every prime submodule of M is of the form $S_p(pM)$ for some $p \in V(\mathrm{Ann}(M))$.*
2. *If R satisfies ACC on prime ideals, then M satisfies ACC on prime submodules.*

Proof. 1. Let P be a prime submodule of M and $p := (P : M) \supseteq \mathrm{Ann}(M)$. Then $\mathrm{Spec}_p(M) \neq \emptyset$, so $S_p(pM)$ is a p-prime submodule of M by [18, p. 2664, Corollary 3.7]. Since M is a top module, we have $S_p(pM) = P$ by Remark 2.2.

2. Let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of prime submodules of M. This induces the following chain of prime ideals, $\psi(N_1) \subseteq \psi(N_2) \subseteq \cdots$, where ψ is the natural map $\psi : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$. Since R satisfies ACC on prime ideals, there exists a positive integer k such that for each $i \in \mathbb{N}$, $\psi(N_k) = \psi(N_{k+i})$. Now by Remark 2.2, we have $N_k = N_{k+i}$ as required.

□

Remark 2.4. Let M be an R-module and p be a prime ideal of R. For every submodule N of the R_p-module M_p, let $N \cap M$ be the inverse image of N under $M \to M_p$. Then $(N \cap M)_p = N$ (see [10, p. 68, Proposition 10]).

Theorem 2.5. *Let $(R, \underline{m})$ be a quasilocal ring and M be a nonzero top primeful R-module. Then M is a local module.*

Proof. We must show that M has exactly one maximal submodule. For each $p \in V(\mathrm{Ann}(M))$, R_p is a quasilocal ring with unique maximal ideal pR_p and M_p is a nonzero top primeful R-module by [19, p. 135, Theorem 4.1] and [20, p. 93, Lemma 3.3]. Thus there exists a prime submodule L of M_p such that $(L : M_p) = pR_p$. We claim that $L \cap M$ is a maximal submodule of M. Let N be a submodule of M such that $L \cap M \subseteq N$. Then by Remark 2.4, $L = (L \cap M)_p \subseteq N_p$. But we have $pR_p = (L : M_p) = (N_p : M_p)$. Thus N_p is a prime submodule of M_p. Therefore $N_p = L$ by Remark 2.2. This implies that

$$N \subseteq S_p(N) = N_p \cap M = L \cap M \subseteq N.$$

Hence $L \cap M = N$, so $L \cap M$ is a maximal submodule of M. This means that $((L \cap M) : M) = \underline{m}$. Now let $Q \in Max(M)$, then $(Q : M) = ((L \cap M) : M) = \underline{m}$. Therefore $Q = L \cap M$ by Remark 2.2. This completes the proof. □

For every prime ideal p of R, R_p is always a quasilocal ring. However, for an arbitrary R-module M, M_p is not necessarily a local R_p-module. But by Theorem 2.5, if M is a nonzero top primeful R-module, then M_p is a local R_p-module for each $p \in V(\mathrm{Ann}(M))$.

Proposition 2.6. *Let M be a nonzero top primeful R-module.*

1. *If M is a semi-local (resp. local) module, then $R/\mathrm{Ann}(M)$ is a quasisemilocal (resp. a quasilocal) ring.*
2. *Let M be a local module with maximal submodule P. If $(P : M) = p$, then the canonical homomorphism $M \to M_p$ is bijective.*

Proof. 1. Let M be a local module with unique maximal submodule P. Then $p := (P : M) \in Max(R)$. Now let $q \in Max(R) \cap V(\mathrm{Ann}(M))$. It is enough to prove $q = p$. To see this, we note that $S_q(qM)$ is a q-prime submodule of M by [19, p. 127, Theorem 2.1]. We show that $S_q(qM) \in Max(M)$. Let $S_q(qM) \subseteq K$ for some submodule K of M. Then we have $q = (S_q(qM) : M) = (K : M)$. Hence $S_q(qM) = K$ by Remark 2.2. This implies that $S_q(qM) = P$ and therefore $q = p$. For the semi-local case we argue similarly.

2. Use part (1) and [10, p. 87, Proposition 8].

□

3. Topological properties of Spec(M)

We recall that a topological space X is irreducible if the intersection of two non-empty open sets of X is non-empty. Every subset of a topological space consisting of a single point is irreducible and a subset Y of a topological space X is irreducible if and only if its closure $Cl(Y)$ is irreducible (see [10, §4.1]). A maximal irreducible subset Y of X is called an irreducible component of X and it is always closed. A topological space X is said to be quasi-compact if every open cover of X has a finite subcover. It is clear that every space X containing only finitely many points is quasi-compact. We begin this section by some examples.

Example 3.1. 1. Let $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ be a $\mathbb{Z}$-module, where p runs through the set of all prime numbers. Then by [8, p. 124, Theorem 3.4], Spec(M) is not an irreducible space because $rad(0_M)$ is not a prime submodule. Further, Spec(M) is not a quasi-compact space.

2. Let $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ be a $\mathbb{Z}$-module. Then by [8, p. 124, Theorem 3.4]), Spec(M) is an irreducible space because $rad(0_M) = (0) \oplus \mathbb{Z}(p^\infty)$ is a prime submodule of M.

3. Let $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ be a $\mathbb{Z}$-module. Then by [8, p. 124, Theorem 3.4], $Max(M)$ is an irreducible subset of $\operatorname{Spec}(M)$ because

$$Rad(M) = \Im(Max(M)) = \mathbb{Q} \oplus (0).$$

Proposition 3.2. *Let Y be a subset of* $\operatorname{Spec}(M)$ *for a top R-module M. If Y is irreducible, then $T = \{(P : M) \mid P \in Y\}$ is an irreducible subset of* $\operatorname{Spec}(R)$*, with respect to Zariski topology.*

Proof. $\psi(Y) = T'$ is an irreducible subset of $\operatorname{Spec}(R/\operatorname{Ann}(M))$ because ψ is continuous by [17, p. 421, Proposition 3.1]. We have

$$\Im(T') = (\Im(Y) : M)/\operatorname{Ann}(M) \in \operatorname{Spec}(R/\operatorname{Ann}(M)).$$

Therefore $\Im(T) = (\Im(Y) : M)$ is a prime ideal of R, so T is an irreducible subset of $\operatorname{Spec}(R)$ by [10, p. 102, Proposition 14]. □

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$. Note that a generic point of a closed subset Y of a topological space is unique if the topological space is a T_0-space.

Theorem 3.3. *Let M be a top R-module and $Y \subseteq \operatorname{Spec}(M)$. Then Y is an irreducible closed subset of* $\operatorname{Spec}(M)$ *if and only if $Y = V(P)$ for some $P \in \operatorname{Spec}(M)$. Thus every irreducible closed subset of* $\operatorname{Spec}(M)$ *has a generic point.*

Proof. $Y = V(P)$ is an irreducible closed subset of $\operatorname{Spec}(M)$ for any $P \in \operatorname{Spec}(M)$ by [8, p. 123, Lemma 3.3]. Conversely if Y is an irreducible closed subset of $\operatorname{Spec}(M)$, then $Y = V(N)$ for some $N \leq M$ and $\Im(Y) = \Im(V(N)) = rad(N)$ is a prime submodule by [8, p. 124, Theorem 3.4]. Hence $Y = V(N) = V(rad(N))$ as desired. □

Theorem 3.4. *Let M be a top R-module. The correspondence $V(P) \mapsto P$ is a bijection from the set of irreducible components of* $\operatorname{Spec}(M)$ *to the set of minimal prime submodules of M.*

Proof. Let Y be an irreducible component of $\operatorname{Spec}(M)$. Since each irreducible component of $\operatorname{Spec}(M)$ is a maximal element of the set $\{V(Q) \mid Q \in \operatorname{Spec}(M)\}$ by Theorem 3.3, we have $Y = V(P)$ for some $P \in \operatorname{Spec}(M)$. Obviously P is a minimal prime submodule, for if T is a prime submodule of M with $T \subseteq P$, then $V(P) \subseteq V(T)$ so that $P = T$. Now let P be a minimal prime submodule of M with $V(P) \subseteq V(Q)$ for some $Q \in \operatorname{Spec}(M)$. Then $Cl(\{P\}) = V(P) \subseteq V(Q) = Cl(\{Q\})$, hence $P = Q$. This implies that $V(P)$ is an irreducible subset of $\operatorname{Spec}(M)$ as desired. □

Example 3.5. Consider $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. By Example 3.1 and Theorem 3.4, $(0) \oplus \mathbb{Z}(p^\infty)$ is a minimal prime submodule of M.

Proposition 3.6. *Consider the following statements for a nonzero top primeful R-module M :*

1. $\operatorname{Spec}(M)$ *is an irreducible space.*
2. $\operatorname{Supp}(M)$ *is an irreducible space.*
3. $\sqrt{\operatorname{Ann}(M)}$ *is a prime ideal of R.*
4. $\operatorname{Spec}(M) = V(pM)$ *for some $p \in \operatorname{Supp}(M)$.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. *When M is a multiplication module, all the four statements are equivalent.*

Proof. $(1) \Rightarrow (2)$ By [17, p. 421, Proposition 3.1], the natural map ψ is continuous and by assumption ψ is surjective. Hence $Im(\psi) = \operatorname{Spec}(R/\operatorname{Ann}(M))$ is also irreducible. Now by [19, p. 133, Proposition 3.4] and [4, p. 13, Ex. 21], we have $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$ is homeomorphic to $\operatorname{Spec}(R/\operatorname{Ann}(M))$. This implies that $\operatorname{Supp}(M)$ is an irreducible space.
$(2) \Rightarrow (3)$ By [10, p. 102, Proposition 14], $\Im(\operatorname{Supp}(M))$ is a prime ideal of R. But we have $\Im(\operatorname{Supp}(M)) = \Im(V(\operatorname{Ann}(M))) = \sqrt{\operatorname{Ann}(M)}$.
$(3) \Rightarrow (4)$ Let $a \in \sqrt{\operatorname{Ann}(M)}$, then $a^n M = 0$ for some integer $n \in \mathbb{N}$. Hence for every prime submodule P of M, $a \in (P : M)$. Therefore $\sqrt{\operatorname{Ann}(M)} \subseteq (P : M)$, for each $P \in \operatorname{Spec}(M)$. Since M is primeful, there exists a prime submodule Q of M such that $(Q : M) = \sqrt{\operatorname{Ann}(M)}$. Hence by [17, p. 419, Result 3],

$$\begin{aligned}\operatorname{Spec}(M) &= \{P \in \operatorname{Spec}(M) \mid (P : M) \supseteq (Q : M)\} \\ &= V((Q : M)M) = V(\sqrt{\operatorname{Ann}(M)}M).\end{aligned}$$

It is clear that $p := \sqrt{\operatorname{Ann}(M)} \in \operatorname{Supp}(M)$. Therefore $\operatorname{Spec}(M) = V(pM)$.

For the last assertion, we show that (4) implies (1). Let $\operatorname{Spec}(M) = V(pM)$ for some $p \in \operatorname{Supp}(M)$. Since M is primeful, there exists $P \in \operatorname{Spec}(M)$ such that $(P : M) = p$. Since M is multiplication, we have

$$\operatorname{Spec}(M) = V(pM) = V((P : M)M) = V(P).$$

Thus $rad(0_M) = \Im(\operatorname{Spec}(M)) = \Im(V(P)) = P \in \operatorname{Spec}(M)$. This implies that $\operatorname{Spec}(M)$ is an irreducible space by [8, p. 124, Theorem 3.4]. □

Notation and Remark 3.7. For each subset S of M, we denote $\mathrm{Spec}(M)\setminus V(S)$ by $\Gamma(S)$. Further for each element $m \in M$, $\Gamma(\{m\})$ is denoted by $\Gamma(m)$. Hence

$$\Gamma(m) = \mathrm{Spec}(M) \setminus V(m) = \{P \mid P \in \mathrm{Spec}(M) \text{ and } m \notin P\}.$$

Moreover, for any family $\{N_i\}_{i\in I}$ of submodules of M, we have $\Gamma(\sum_{i\in I} N_i) = \Gamma(\bigcup_{i\in I} N_i)$.

Proposition 3.8. *Let M be a top R-module. Then the set $B = \{\Gamma(m) \mid m \in M\}$ form a basis of open sets for the Zariski topology.*

Proof. Let $\Gamma(N)$ be an open set for some submodule N of M. Let $P \in \Gamma(N)$. Hence $N \nsubseteq P$ so that there exists $m \in N \setminus P$, therefore $P \in \Gamma(m)$. Now assume that $Q \in \Gamma(m)$. It follows that $N \nsubseteq Q$ so that $\Gamma(m) \subseteq \Gamma(N)$. Thus $P \in \Gamma(m) \subseteq \Gamma(N)$. Hence B is a basis for Zariski topology on $\mathrm{Spec}(M)$ by [21, P. 80, Lemma 13.2]. □

For a submodule N of an R-module M, we use the following notation

$$\mathbb{T}(N) := \{L \mid L \subseteq N \text{ and } L \text{ is finitely generated }\}.$$

Lemma 3.9. *Let M be an R-module and N be a submodule of M. Then we have*

$$V(N) = \bigcap_{L\in\mathbb{T}(N)} V(L), \quad \Gamma(N) = \bigcup_{L\in\mathbb{T}(N)} \Gamma(L).$$

Proof. Let $P \in V(N)$. If $L \in \mathbb{T}(N)$, then $L \subseteq N \subseteq P$. Hence $P \in V(L)$, thus $V(N) \subseteq \bigcap_{L\in\mathbb{T}(N)} V(L)$. Now suppose $P \in V(L)$ for every $L \in \mathbb{T}(N)$ and $P \notin V(N)$. Since $N \nsubseteq P$, then there exists $x \in N \setminus P$. Hence $Rx \subseteq N$ and Rx is finitely generated, therefore $Rx \in \mathbb{T}(N)$. Consequently $x \in Rx \subseteq P$, a contradiction. Hence $\bigcap_{L\in\mathbb{T}(N)} V(L) \subseteq V(N)$. □

Theorem 3.10. *Let M be a top R-module. Then every quasi-compact open subset of* $\mathrm{Spec}(M)$ *is of the form $\Gamma(N)$ for some finitely generated submodule N of M. In particular if M is Bezout, then every quasi-compact open subset of* $\mathrm{Spec}(M)$ *is of the form $\Gamma(m)$ for some $m \in M$.*

Proof. Suppose $\Gamma(B) = \mathrm{Spec}(M) \setminus V(B)$ is a quasi-compact open subset of $\mathrm{Spec}(M)$. By Lemma 3.9, we have $\Gamma(B) = \bigcup_{L\in T(B)} \Gamma(L)$. Since $\Gamma(B)$ is quasi-compact, every open covering of $\Gamma(B)$ has a finite subcovering, thus

$$\Gamma(B) = \Gamma(L_1) \cup ... \cup \Gamma(L_n) = \Gamma(\sum_{i=1}^{n} L_i).$$

This completes the proof. □

Theorem 3.11. *Let R be a Noetherian ring and let M be an R-module such that for every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$. Then the open set $\Gamma(m)$ is quasi-compact for each $m \in M$.*

Proof. By [20, p. 94, Theorem 3.5], M is a top module. Since, by Proposition 3.8, the set $\{\Gamma(m) \mid m \in M\}$ forms a base for the Zariski topology on $\mathrm{Spec}(M)$, for every open cover of $\Gamma(m)$, there exists a family $\{m_\lambda \mid \lambda \in \Lambda\}$ of elements of M such that

$$\Gamma(m) \subseteq \bigcup_{\lambda\in\Lambda} \Gamma(m_\lambda) = \mathrm{Spec}(M) \setminus \bigcap_{\lambda\in\Lambda} V(m_\lambda).$$

For each $\lambda \in \Lambda$, set $V(m_\lambda) = V(J_\lambda M)$, where J_λ is an ideal of R. Then

$$\Gamma(m) \subseteq \mathrm{Spec}(M) \setminus \bigcap_{\lambda\in\Lambda} V(J_\lambda M) = \mathrm{Spec}(M) \setminus V(\sum_{\lambda\in\Lambda} J_\lambda M).$$

Therefore $\Gamma(m) \subseteq \mathrm{Spec}(M) \setminus V((\sum_{\lambda\in\Lambda} J_\lambda)M)$. Since R is a Noetherian ring, there exists a finite subset Λ' of Λ such that

$$\Gamma(m) \subseteq \mathrm{Spec}(M) \setminus V(\sum_{\lambda\in\Lambda'} J_\lambda M) = \mathrm{Spec}(M) \setminus \bigcap_{\lambda\in\Lambda'} V(m_\lambda) = \bigcup_{\lambda\in\Lambda'} \Gamma(m_\lambda).$$

□

Consider $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}$-module, where p runs through the set of all prime numbers. By [8, p. 113, Theorem 2.14], $\mathrm{Spec}(M)$ is a T_1-space because each prime submodule is a maximal element in $\mathrm{Spec}(M)$.

Proposition 3.12. *Let M be a top R-module. Then we have the following.*

1. *If $\mathrm{Spec}(R)$ is a T_1-space, then $\mathrm{Spec}(M)$ is also a T_1-space. In particular, If R is a Boolean ring, then $\mathrm{Spec}(M)$ is a T_1-space.*

2. *If $\mathrm{Spec}(M) = Max(M)$ and also M is a faithful primeful module, then $\mathrm{Spec}(R)$ is a Hausdorff space.*

Proof. 1. Suppose Q is a prime submodule of M. Then $Cl(\{Q\}) = V(Q)$. If $P \in V(Q)$, then since every prime ideal is a maximal ideal, $(Q : M) = (P : M)$ so that $Q = P$ by Remark 2.2. Therefore $Cl(\{Q\}) = \{Q\}$ and this implies that $\mathrm{Spec}(M)$ is a T_1-space.

2. Let p be a prime ideal of R. Since M is primeful, there exists a prime submodule P of M such that $(P : M) = p$. Hence p is a maximal ideal of R. This implies that $\mathrm{Spec}(R)$ is a Hausdorff space. □

A topological space X is called Noetherian if it satisfies the descending chain condition for closed sets, or equivalently X is a Noetherian space if and only if every open subset of X is quasi-compact (see [4, p. 79, Ex. 5]).

Lemma 3.13. *Let M be a top module. Then* $\mathrm{Spec}(M)$ *is a Noetherian space if and only if radical submodules of M satisfies ACC. In Particular, every top Noetherian module has Noetherian spectrum.*

Proof. Let N be a radical submodule of M. Then we have $N = \Im(V(N))$. Also, if N_1 and N_2 are two radical submodules of M with $V(N_1) = V(N_2)$, then $N_1 = N_2$. The two facts prove the lemma. □

Theorem 3.14. *Let M be a top module. Then* $\mathrm{Spec}(M)$ *is a Noetherian space in each of the following cases.*

1. *M is a compactly packed module.*
2. *R is an integral domain of dimension 1 and M a non-faithful R-module such that every closed subset of* $\mathrm{Spec}(M)$ *has finitely many irreducible components.*
3. *R is a PID and M a non-faithful R-module.*

Proof.

1. Let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of radical submodules of M and let $G := \bigcup_{i\in I} N_i$. By Lemma 3.13, it is enough to show that G is contained in N_j for some $j \in I$. To see this, we claim that $rad(G) \subseteq rad(Rx)$ for some $x \in G$. If not, then for every $x \in G$ there exists a prime submodule $P_x \in V(Rx)$ such that $G \not\subseteq P_x$. But

$$G = \bigcup_{x\in G} Rx \subseteq \bigcup_{x\in G} P_x$$

which yields a contradiction by hypothesis. Thus there exists an element $b \in G$ such that $rad(G) \subseteq rad(Rb)$. Also there exists some $j \in I$ such that $b \in N_j$. Therefore $G \subseteq rad(Rb) \subseteq N_j$, which finishes the proof.

2. Let $F = V(N)$ be a closed subset of $\mathrm{Spec}(M)$, with $N \leq M$. By assumption $V(N) = \bigcup_{i=1}^{n} Z_i$, where Z_i is irreducible component of $V(N)$. Thus M/N has finitely many minimal prime submodules $P_1^{'}, \ldots, P_n^{'}$ by Theorem 3.4. Thus there exists prime submodules $P_1, \ldots, P_n$ of M such that $P_i^{'} = P_i/N$. Let $P \in V(N)$. We show that $P = P_j$ for some j $(1 \leq j \leq n)$. By [15, p. 213, Proposition 1], $N \subseteq P_k \subseteq P$ for some k $(1 \leq k \leq n)$. Thus we have

$$\psi(P_k) \subseteq \psi(P) \Rightarrow (0) \subset \mathrm{Ann}(M) \subseteq (P_k : M) \subseteq (P : M).$$

Since M is a non-faithful top R-module and R is a one dimensional integral domain, we have $P = P_k$. Now the proof follows from Lemma 3.13.

3. By Lemma 3.13, it is enough to prove that for every submodule N of M, $|V(N)| < \infty$. Suppose that $V(N)$ contains infinitely many members. Then for each $P \in V(N)$, we have $(N : M) \subseteq (P : M)$. Note that for distinct prime submodules $P, Q \in V(N)$, we have $(P : M) \neq (Q : M)$ by Remark 2.2. This implies that $\text{Ann}(M) \subseteq (N : M) = 0$, which is a contradiction by hypothesis. This completes the proof.

□

Theorem 3.15. *Let M be a top R-module such that* Spec(M) *is a Noetherian space. Then the following statements are true.*

1. *Every ascending chain of prime submodules of M is stationary.*
2. *If M is a Bezout R-module, then M is compactly packed.*
3. *If N is a proper submodule of M, then $V(N)$ has only finitely many minimal elements.*
4. *$rad(N) = \bigcap P_i$, where the intersection is taken over the finitely many P_i of part (3).*
5. *The set of minimal prime submodules of M is finite. In particular*

$$\text{Spec}(M) = \bigcup_{i=1}^{n} V(P_i),$$

where P_i are all minimal prime submodules of M.

Proof. 1. This is clear.

2. Let N be a proper submodule of M. We claim that $rad(N) = rad(L)$ for some finitely generated submodule L of M. Suppose the claim is not true and let $x_1 \in N$ and $N_1 = rad(Rx_1)$. Then $N_1 \subset N$ because if $N_1 = N$, then

$$rad(Rx_1) = rad(rad(Rx_1)) = rad(N_1) = rad(N)$$

which is a contradiction. So there exists $x_2 \in N \setminus N_1$. Let $N_2 = rad(Rx_1 + Rx_2)$. Then $N_1 \subset N_2 \subset N$. By continuing this procedure we have an ascending chain of radical submodules

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

of M which is a contradiction by Lemma 3.13. Therefore $rad(N) = rad(L)$ for some finitely generated submodule L of M. L must be cyclic, because M is a Bezout module. Hence for each proper submodule N of M there exists $x \in N$ such that $rad(N) = rad(Rx)$.

Now let K be a proper submodule of M and let $\{P_i\}_{i\in I}$ be a family of prime submodules of M such that $K \subseteq \cup_{i\in I} P_i$. By above arguments, there exists $x \in K$ such that $K \subseteq rad(Rx) \subseteq P_j$ for some $j \in I$.

3. We have that $V(N)$ is homeomorphic to $\text{Spec}(M/N)$. Since $\text{Spec}(M)$ is Noetherian, $\text{Spec}(M/N)$ has finitely many irreducible components. Hence by Theorem 3.4, there is one-to-one correspondence between irreducible components of $\text{Spec}(M/N)$ and minimal prime submodules of M/N. Also for $P \in \text{Spec}(M)$, P/N is a minimal prime submodule of M/N if and only if P is a minimal prime submodule of N. This completes the proof.
4. This follows from part (3) and [15, p. 213, Proposition 1].
5. This follows from Theorem 3.4 and the fact that the number of irreducible components of $\text{Spec}(M)$ is finite.

□

Proposition 3.16. *Let M be a top co-semisimple R-module. Then M is a Noetherian R-module in each of the following cases.*

1. *M is compactly packed.*
2. *R is an integral domain of dimension 1 and M a non-faithful R-module such that every closed subset of* $\text{Spec}(M)$ *has finitely many irreducible components.*
3. *R is a PID and M is a non-faithful R-module.*

Proof. By Theorem 3.14, if each of the conditions (1)-(3) holds, then $\text{Spec}(M)$ is a Noetherian space. Hence M satisfies ACC on radical submodules by Lemma 3.13. But every submodule of M is a radical submodule by [2, p. 122, Ex. 14]. Therefore M is a Noetherian module. This completes the proof. □

We recall that an R-module M is called a multiplication module [12] if every submodule N of M is of the form IM for some ideal I of R and an R-module M is called a weak multiplication if every prime submodule P of M is of the form IM for some ideal I of R (see [1] and [5]).

Theorem 3.17. *Let M be a weak multiplication top primeful R-module. Then the set*

$$T = \{V(pM) \mid p \in Min(\text{Supp}(M))\}$$

is the set of all irreducible components of Spec(M).

Proof. Let Y be an irreducible component of Spec(M). Then by Theorem 3.3, $Y = V(P)$ for some $P \in \text{Spec}(M)$. Hence $Y = V(P) = V((P : M)M)$, where $p := (P : M) \in V(\text{Ann}(M)) = \text{Supp}(M)$ by [19, p. 133, Proposition 3.4]. We must show that $p \in Min(\text{Supp}(M))$. To see this let $q \in \text{Supp}(M)$ and $q \subseteq p$. Then there exists a prime submodule Q of M such that $(Q : M) = q$ because M is primeful. Thus $Y = V(P) \subseteq V(Q)$. Hence $Y = V(P) = V(Q)$. Thus by [17, p. 419, Result 1], we have that $p = q$.

Conversely let $Y \in T$. Then there exists $p \in Min(\text{Supp}(M))$ such that $Y = V(pM)$. Since M is primeful, there exists a prime submodule P of M such that $(P : M) = p$. Since M is a weak multiplication module, $Y = V(pM) = V((P : M)M) = V(P)$. Thus Y is irreducible by [8, p. 124, Theorem 3.4]. Suppose $Y = V(P) \subseteq V(Q)$, where Q is a prime submodule of M. Thus $P \in Cl(\{Q\})$. Now we have $Q \subseteq P$, so that $q := (Q : M) = p$. Therefore $Y = V(P) = V(pM) = V(qM) = V(Q)$. This completes the proof. □

Corollary 3.18. *Let M be a finitely generated multiplication R-module. Then the set*

$$T = \{V(pM) \mid p \in Min(\text{Supp}(M))\}$$

is the set of all irreducible components of Spec(M).

Following M. Hochster [13], we say that a topological space X is a spectral space in case X is homeomorphic to Spec(S), with the Zariski topology, for some ring S. Spectral spaces have been characterized by Hochster [13, p.52, Proposition 4] as the topological spaces X which satisfy the following conditions:

1. X is a T_0-space;
2. X is quasi-compact;
3. the quasi-compact open subsets of X are closed under finite intersection and form an open base;
4. each irreducible closed subset of X has a generic point.

Corollary 3.19. *Let M be a top R-module. Then* Spec(M) *is a spectral space if each of the following conditions holds.*

1. *M is compactly packed.*

2. *R is an integral domain of dimension 1 and M a non-faithful R-module such that every closed subset of* Spec(M) *has finitely many irreducible components.*

3. *R is a PID and M a non-faithful R-module.*

Proof. As we have seen in proof of Theorem 3.14, in each of the above cases M fulfils ACC on intersection of prime submodules. Hence the result follows from [9, p. 146, Theorem 3.2]. □

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