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Free rectangular dibands and free dimonoids

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Abstract. We construct a free rectangular diband and describe its structure. We also present the least rectangular diband congruence, the least (rb, rz)-congruence, the least left zero and right zero congruence, the least rectangular band congruence and the least left zero congruence on free dimonoids and use them to obtain decompositions of free dimonoids.

1. Introduction

The notion of a dialgebra was introduced by J.-L. Loday [4] and investigated in many papers (see, for example, [4], [1], [8], [3]). So, recently L.A. Bokut, Yuqun Chen and Cihua Liu [1] gave the composition-diamond lemma for dialgebras and obtained a Gröbner-Shirshov basis for dialgebras. A.P. Pozhidaev [8] studied the connection of Rota-Baxter algebras and dialgebras with associative bar-unity. Analogues of some notions of the functional analysis were defined on dialgebras in [3].

Dialgebras are linear analogues of dimonoids [4]. At the present time dimonoids have became a standard tool in the theory of Leibniz algebras. J.-L. Loday constructed a free dimonoid [4]. The least semilattice congruence on free dimonoids was described in [11]. In [12] it was constructed a free commutative dimonoid. Pirashvili [7] introduced the notion of a duplex and constructed a free duplex. Dimonoids in the sense of Loday [4] are examples of duplexes. Rectangular dimonoids (rectangular dibands) first appeared in the researches of the structure of idempotent dimonoids

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and dibands of subdimonoids (see [13], [14]). Commutative dimonoids were studied in [15].

In this paper we give new examples of dimonoids (Proposition 1 and Lemmas 2-4), construct a free rectangular dimonoid (Theorem 1) and describe its structure (Theorem 2). As a consequence of Theorem 2, we obtain the description of some least congruences on free rectangular dimonoids (Corollary 2). We also discuss the connections between rectangular dimonoids and restrictive bisemigroups which were considered earlier in some other aspects in the paper of B.M. Schein [10] (Proposition 4). Moreover, we present the least rectangular diband congruence, the least (rb, rz)-congruence, the least $(\ell z, rb)$-congruence, the least left zero and right zero congruence, the least rectangular band congruence, the least left zero congruence and the least right zero congruence on free dimonoids and use them to obtain decompositions of free dimonoids (Theorems 3-9).

The algebraic notions and notations used in this paper are standard and similar to those used in [5].

2. Preliminaries

A nonempty set D equipped with two binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{$D1$}$$

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{$D2$}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{$D3$}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{$D4$}$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z) \tag{$D5$}$$

for all $x, y, z \in D$, is called a dimonoid. If the operations of a dimonoid coincide, then the dimonoid becomes a semigroup.

Now we give a new example of a dimonoid.

Let X be an arbitrary nonempty set, $\bar{X} = \{\bar{x} \,|\, x \in X\}$ and let $R(X)$ be the set of words $x\bar{y}z$ in the alphabet $X \bigcup \bar{X}$ such that $y \in X$, $x, z \in X \bigcup \{\theta\}$, where θ is the empty word. For all $x\bar{y}z \in R(X)$ assume

$$t_{x\bar{y}z} = \begin{cases} z, & z \neq \theta, \\ y, & z = \theta, \end{cases} \quad t'_{x\bar{y}z} = \begin{cases} x, & x \neq \theta, \\ y, & x = \theta. \end{cases}$$

Define the operations $\dashv$ and $\vdash$ on $R(X)$ by

$$x\bar{y}z \dashv a\bar{b}c = x\bar{y}t_{a\bar{b}c}, \quad x\bar{y}z \vdash a\bar{b}c = t'_{x\bar{y}z}\bar{b}c$$

for all $x\bar{y}z,\ a\bar{b}c \in R(X)$.

Proposition 1. *$(R(X), \dashv, \vdash)$ is a dimonoid.*

Proof. Let $x\bar{y}z,\ a\bar{b}c,\ m\bar{n}s \in R(X)$. Then

$$(x\bar{y}z \dashv a\bar{b}c) \dashv m\bar{n}s = x\bar{y}t_{a\bar{b}c} \dashv m\bar{n}s = x\bar{y}t_{m\bar{n}s},$$

$$x\bar{y}z \dashv (a\bar{b}c \dashv m\bar{n}s) = x\bar{y}z \dashv a\bar{b}t_{m\bar{n}s} = x\bar{y}t_{a\bar{b}t_{m\bar{n}s}},$$

$$x\bar{y}z \dashv (a\bar{b}c \vdash m\bar{n}s) = x\bar{y}z \dashv t'_{a\bar{b}c}\bar{n}s = x\bar{y}t_{t'_{a\bar{b}c}\bar{n}s}.$$

If $s = \theta$, then $t_{m\bar{n}s} = n$. Hence $t_{a\bar{b}t_{m\bar{n}s}} = t_{a\bar{b}n} = n = t_{t'_{a\bar{b}c}\bar{n}s}$. If $s \neq \theta$, then $t_{m\bar{n}s} = s$. Hence $t_{a\bar{b}t_{m\bar{n}s}} = t_{a\bar{b}s} = s = t_{t'_{a\bar{b}c}\bar{n}s}$. So,

$$x\bar{y}t_{m\bar{n}s} = x\bar{y}t_{a\bar{b}t_{m\bar{n}s}} = x\bar{y}t_{t'_{a\bar{b}c}\bar{n}s}.$$

Moreover,

$$(x\bar{y}z \vdash a\bar{b}c) \vdash m\bar{n}s = t'_{x\bar{y}z}\bar{b}c \vdash m\bar{n}s = t'_{t'_{x\bar{y}z}\bar{b}c}\bar{n}s,$$
$$x\bar{y}z \vdash (a\bar{b}c \vdash m\bar{n}s) = x\bar{y}z \vdash t'_{a\bar{b}c}\bar{n}s = t'_{x\bar{y}z}\bar{n}s,$$
$$(x\bar{y}z \dashv a\bar{b}c) \vdash m\bar{n}s = x\bar{y}t_{a\bar{b}c} \vdash m\bar{n}s = t'_{x\bar{y}t_{a\bar{b}c}}\bar{n}s.$$

If $x = \theta$, then $t'_{x\bar{y}z} = y$. Hence $t'_{t'_{x\bar{y}z}\bar{b}c} = t'_{y\bar{b}c} = y = t'_{x\bar{y}t_{a\bar{b}c}}$. If $x \neq \theta$, then $t'_{x\bar{y}z} = x$. Hence $t'_{t'_{x\bar{y}z}\bar{b}c} = t'_{x\bar{b}c} = x = t'_{x\bar{y}t_{a\bar{b}c}}$. So,

$$t'_{t'_{x\bar{y}z}\bar{b}c}\bar{n}s = t'_{x\bar{y}z}\bar{n}s = t'_{x\bar{y}t_{a\bar{b}c}}\bar{n}s.$$

Finally,

$$(x\bar{y}z \vdash a\bar{b}c) \dashv m\bar{n}s = t'_{x\bar{y}z}\bar{b}c \dashv m\bar{n}s = t'_{x\bar{y}z}\bar{b}t_{m\bar{n}s},$$
$$x\bar{y}z \vdash (a\bar{b}c \dashv m\bar{n}s) = x\bar{y}z \vdash a\bar{b}t_{m\bar{n}s} = t'_{x\bar{y}z}\bar{b}t_{m\bar{n}s}.$$

Thus, $(R(X), \dashv, \vdash)$ satisfies the axioms $(D1)$-$(D5)$. □

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f. A nonempty subset T of a dimonoid $(D, \dashv, \vdash)$ is called a subdimonoid, if for any $a, b \in D$, a, $b \in T$ implies $a \dashv b,\ a \vdash\ b \in T$.

An idempotent semigroup S is called a rectangular band, if

$$xyx = x \tag{1}$$

for all $x, y \in S$. It is well-known that every rectangular band is isomorphic to the Cartesian product of a left zero semigroup and a right zero semigroup. It is clear that in any rectangular band the identity

$$xyz = xz \tag{2}$$

holds.

A dimonoid $(D,\dashv,\vdash)$ will be called a rectangular dimonoid or a rectangular diband (respectively, idempotent dimonoid or a diband), if both semigroups $(D,\dashv)$ and $(D,\vdash)$ are rectangular bands (respectively, idempotent semigroups).

Lemma 1. *([11], Lemma 1) Let $(D,\dashv,\vdash)$ be an idempotent dimonoid. Then $(D,\dashv)$ is a rectangular band if and only if $(D,\vdash)$ is a rectangular band.*

Note that the class of rectangular dimonoids is a subvariety of the variety of all dimonoids. A dimonoid which is free in the variety of rectangular dimonoids will be called a free rectangular dimonoid.

The necessary information about varieties of dimonoids can be found in [12].

The notion of a diband of subdimonoids was introduced in [15] and investigated in [14]. Recall this definition.

Let S be an arbitrary dimonoid, J be some idempotent dimonoid. Let

$$\alpha : S \to J : x \mapsto x\alpha$$

be a homomorphism. Then every class of the congruence Δ_α is a subdimonoid of the dimonoid S and the dimonoid S itself is a union of such dimonoids $S_\xi,\ \xi \in J$ that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \mid (x;t) \in \Delta_\alpha\},$$

$$S_\xi \dashv S_\varepsilon \subseteq S_{\xi\dashv\varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi\vdash\varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a diband of subdimonoids (or S is a diband J of subdimonoids $S_\xi,\ \xi \in J$). If J is a band (=idempotent semigroup), then we say that S is a band J of subdimonoids $S_\xi,\ \xi \in J$. If J is a semilattice (=commutative band), then we say that S is a semilattice J of subdimonoids $S_\xi,\ \xi \in J$. If J is a left zero semigroup (respectively, right zero semigroup), then we say that S is a left band (respectively, right band) J of subdimonoids $S_\xi,\ \xi \in J$.

Note that the notion of a diband of subdimonoids generalizes the notion of a band of semigroups [2]. The semilattice decompositions of semigroups were given in [9].

J.-L. Loday described a free dimonoid [4]. We constructed the dimonoid isomorphic to the free dimonoid in [11]. Recall this construction.

As usual N denotes the set of positive integers. Let $F[X]$ be the free semigroup in the alphabet X. We denote the length of a word $w \in F[X]$ by ℓ_w. Define the operations $\dashv$ and $\vdash$ on

$$F = \{(w, m) \in F[X] \times N \mid \ell_w \geq m\}$$

by

$$(w_1, m_1) \dashv (w_2, m_2) = (w_1 w_2, m_1),$$

$$(w_1, m_1) \vdash (w_2, m_2) = (w_1 w_2, \ell_{w_1} + m_2)$$

for all $(w_1, m_1),\ (w_2, m_2) \in F$. Denote the algebra $(F, \dashv, \vdash)$ by $\breve{F}[X]$. By Lemma 3 from [11] $\breve{F}[X]$ is isomorphic to the free dimonoid on X.

3. Free rectangular dimonoids

It is known that every idempotent dimonoid is a semilattice of rectangular subdimonoids [13] and every diband of subdimonoids of type Γ is a semilattice of subdimonoids each being a rectangular diband of subdimonoids of type Γ [14]. In this section we construct a free rectangular dimonoid.

We first give examples of rectangular dimonoids.

It is immediate to prove the following two lemmas.

Lemma 2. *Let $(D, \dashv)$ be a rectangular band and $(D, \vdash)$ be a right zero semigroup. Then $(D, \dashv, \vdash)$ is a rectangular dimonoid.*

We will call this rectangular dimonoid a (rb, rz)-dimonoid.

Lemma 3. *Let $(D, \dashv)$ be a left zero semigroup and $(D, \vdash)$ be a rectangular band. Then $(D, \dashv, \vdash)$ is a rectangular dimonoid.*

We will call this rectangular dimonoid a $(\ell z, rb)$-dimonoid.

Let $I_n = \{1, 2, ..., n\}$, $n > 1$ and let $\{X_i\}_{i \in I_n}$ be a family of arbitrary nonempty sets X_i, $i \in I_n$. Define the operations $\dashv$ and $\vdash$ on $\prod_{i \in I_n} X_i$ by

$$(x_1, ..., x_n) \dashv (y_1, ..., y_n) = (x_1, ..., x_{n-1}, y_n),$$

$$(x_1, ..., x_n) \vdash (y_1, ..., y_n) = (x_1, y_2, ..., y_n)$$

for all $(x_1, ..., x_n),\ (y_1, ..., y_n) \in \prod_{i \in I_n} X_i$.

Lemma 4. *For any $n > 1$, $(\prod_{i \in I_n} X_i, \dashv, \vdash)$ is a rectangular dimonoid.*

Proof. It is immediate to check that $(\prod_{i \in I_n} X_i, \dashv, \vdash)$ is a dimonoid. It is clear that the operations $\dashv$ and $\vdash$ are idempotent. For all $(x_1, ..., x_n)$, $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \prod_{i \in I_n} X_i$ we have

$$(x_1, ..., x_n) \dashv (y_1, ..., y_n) \dashv (x_1, ..., x_n) =$$

$$= (x_1, ..., x_{n-1}, y_n) \dashv (x_1, ..., x_n) = (x_1, ..., x_n).$$

Hence $(\prod_{i\in I_n} X_i, \dashv)$ is a rectangular band. By Lemma 1 $(\prod_{i\in I_n} X_i, \vdash)$ is a rectangular band. So, $(\prod_{i\in I_n} X_i, \dashv, \vdash)$ is a rectangular dimonoid. $\square$

Note that the operations of $(\prod_{i\in I_2} X_i, \dashv, \vdash)$ coincide and it is a rectangular band.

Let X be an arbitrary nonempty set. Then the operations of $(X^2, \dashv, \vdash)$ coincide and it is a free rectangular band [6].

Other examples of rectangular dimonoids can be found in [11] and [13].

We denote the dimonoid $(X^3, \dashv, \vdash)$ by $FRct(X)$.

The main result of this section is the following.

Theorem 1. *$FRct(X)$ is a free rectangular dimonoid.*

Proof. By Lemma 4 $FRct(X)$ is a rectangular dimonoid. Let us show that $FRct(X)$ is free.

Let $(T, \dashv', \vdash')$ be an arbitrary rectangular dimonoid, $\gamma : X \to T$ be an arbitrary map. Define a map

$$\phi : FRct(X) \to (T, \dashv', \vdash') : (x, y, z) \mapsto (x, y, z)\phi,$$

assuming

$$(x, y, z)\phi = \begin{cases} x\gamma, & x = y = z, \\ x\gamma \vdash' y\gamma \dashv' z\gamma, & x \neq y,\ y \neq z, \\ x\gamma \dashv' z\gamma, & x = y,\ y \neq z, \\ x\gamma \vdash' z\gamma, & x \neq y,\ y = z \end{cases}$$

for all $(x, y, z) \in FRct(X)$.

We show that ϕ is a homomorphism. We will use the axioms $(D1)$-$(D5)$, the idempotent property of the operations and the identities (1) and (2).

For arbitrary elements $(x, y, z),\ (a, b, c) \in FRct(X)$ we consider the following cases.

Case 1 : $x = y = z,\ \ a = b = c.$

If $x \neq a$, then

$$((x, x, x) \dashv (a, a, a))\phi = (x, x, a)\phi =$$

$$= x\gamma \dashv' a\gamma = (x, x, x)\phi \dashv' (a, a, a)\phi.$$

If $x = a$, then

$$((x, x, x) \dashv (a, a, a))\phi = (x, x, x)\phi = x\gamma =$$

$$= x\gamma \dashv' x\gamma = (x,x,x)\phi \dashv' (a,a,a)\phi.$$

Case 2 : $x \neq y,\ y \neq z,\ a \neq b,\ b \neq c.$
If $y \neq c$, then

$$((x,y,z) \dashv (a,b,c))\phi = (x,y,c)\phi =$$
$$= x\gamma \vdash' y\gamma \dashv' c\gamma = (x\gamma \vdash' y\gamma) \dashv' c\gamma =$$
$$= (x\gamma \vdash' y\gamma) \dashv' z\gamma \dashv' (a\gamma \vdash' b\gamma) \dashv' c\gamma =$$
$$= (x\gamma \vdash' y\gamma \dashv' z\gamma) \dashv' (a\gamma \vdash' b\gamma \dashv' c\gamma) =$$
$$= (x,y,z)\phi \dashv' (a,b,c)\phi.$$

If $y = c$, then

$$((x,y,z) \dashv (a,b,c))\phi = (x,y,y)\phi = x\gamma \vdash' y\gamma,$$
$$(x,y,z)\phi \dashv' (a,b,c)\phi =$$
$$= (x\gamma \vdash' y\gamma \dashv' z\gamma) \dashv' (a\gamma \vdash' b\gamma \dashv' c\gamma) =$$
$$= (x\gamma \vdash' y\gamma) \dashv' z\gamma \dashv' (a\gamma \vdash' b\gamma) \dashv' c\gamma =$$
$$= (x\gamma \vdash' y\gamma) \dashv' c\gamma = x\gamma \vdash' (y\gamma \dashv' y\gamma) =$$
$$= x\gamma \vdash' y\gamma.$$

Case 3 : $x = y = z,\ \ a = b,\ b \neq c.$
If $x \neq c$, then

$$((x,x,x) \dashv (a,a,c))\phi = (x,x,c)\phi = x\gamma \dashv' c\gamma =$$
$$= x\gamma \dashv' (a\gamma \dashv' c\gamma) = (x,x,x)\phi \dashv' (a,a,c)\phi.$$

If $x = c$, then

$$((x,x,x) \dashv (a,a,c))\phi = (x,x,x)\phi = x\gamma =$$
$$= (x\gamma \dashv' a\gamma) \dashv' x\gamma = x\gamma \dashv' (a\gamma \dashv' c\gamma) =$$
$$= (x,x,x)\phi \dashv' (a,a,c)\phi.$$

Case 4 : $x \neq y,\ \ y = z,\ \ a \neq b,\ \ b = c.$
If $y \neq b$, then

$$((x,y,y) \dashv (a,b,b))\phi = (x,y,b)\phi = x\gamma \vdash' y\gamma \dashv' b\gamma =$$
$$= (x\gamma \vdash' y\gamma) \dashv' b\gamma = (x\gamma \vdash' y\gamma) \dashv' a\gamma \dashv' b\gamma =$$
$$= (x\gamma \vdash' y\gamma) \dashv' (a\gamma \vdash' b\gamma) = (x,y,y)\phi \dashv' (a,b,b)\phi.$$

If $y = b$, then

$$((x, y, y) \dashv (a, b, b))\phi = (x, y, y)\phi = x\gamma \vdash' y\gamma,$$

$$(x, y, y)\phi \dashv' (a, b, b)\phi = (x\gamma \vdash' y\gamma) \dashv' (a\gamma \vdash' b\gamma) =$$
$$= ((x\gamma \vdash' y\gamma) \dashv' a\gamma) \dashv' b\gamma = (x\gamma \vdash' y\gamma) \dashv' b\gamma =$$
$$= x\gamma \vdash' (y\gamma \dashv' y\gamma) = x\gamma \vdash' y\gamma.$$

Case 5 : $x = y = z$, $a \neq b$, $b \neq c$.
If $x \neq c$, then

$$((x, x, x) \dashv (a, b, c))\phi = (x, x, c)\phi =$$
$$= x\gamma \dashv' c\gamma = x\gamma \dashv' (a\gamma \vdash' b\gamma) \dashv' c\gamma =$$
$$= x\gamma \dashv' (a\gamma \vdash' b\gamma \dashv' c\gamma) = (x, x, x)\phi \dashv' (a, b, c)\phi.$$

If $x = c$, then

$$((x, x, x) \dashv (a, b, c))\phi = (x, x, x)\phi =$$
$$= x\gamma = x\gamma \dashv' (a\gamma \vdash' b\gamma) \dashv' x\gamma =$$
$$= x\gamma \dashv' (a\gamma \vdash' b\gamma \dashv' c\gamma) = (x, x, x)\phi \dashv' (a, b, c)\phi.$$

Case 6 : $x \neq y,\ y \neq z,\ a \neq b,\ b = c$.
If $y \neq b$, then

$$((x, y, z) \dashv (a, b, b))\phi = (x, y, b)\phi = x\gamma \vdash' y\gamma \dashv' b\gamma =$$
$$= (x\gamma \vdash' y\gamma) \dashv' b\gamma = (x\gamma \vdash' y\gamma) \dashv' z\gamma \dashv' a\gamma \dashv' b\gamma =$$
$$= (x\gamma \vdash' y\gamma \dashv' z\gamma) \dashv' (a\gamma \vdash' b\gamma) =$$
$$= (x, y, z)\phi \dashv' (a, b, b)\phi.$$

If $y = b$, then

$$((x, y, z) \dashv (a, b, b))\phi = (x, y, y)\phi = x\gamma \vdash' y\gamma,$$

$$(x, y, z)\phi \dashv' (a, b, b)\phi = (x\gamma \vdash' y\gamma \dashv' z\gamma) \dashv' (a\gamma \vdash' b\gamma) =$$
$$= (x\gamma \vdash' y\gamma) \dashv' z\gamma \dashv' a\gamma \dashv' b\gamma = (x\gamma \vdash' y\gamma) \dashv' b\gamma =$$
$$= x\gamma \vdash' (y\gamma \dashv' y\gamma) = x\gamma \vdash' y\gamma.$$

Similarly, the remaining cases can be proved.
Thus,

$$((x, y, z) \dashv (a, b, c))\phi = (x, y, z)\phi \dashv' (a, b, c)\phi$$

for all $(x, y, z),\ (a, b, c) \in FRct(X)$. Analogously, we can prove that

$$((x, y, z) \vdash (a, b, c))\phi = (x, y, z)\phi \vdash' (a, b, c)\phi$$

for all $(x, y, z),\ (a, b, c) \in FRct(X)$. This completes the proof of Theorem 1. $\square$

Obviously, the free rectangular dimonoid $FRct(X)$ generated by a finite set X is finite. Specifically, if $|X| = n$, then $|FRct(X)| = n^3$.

4. Decompositions of $FRct(X)$

In this section we describe the structure of free rectangular dimonoids, characterize some least congruences on free rectangular dimonoids and discuss the connections between rectangular dimonoids and restrictive bisemigroups [10].

Let $X_{\ell z} = (X, \dashv)$, $X_{rz} = (X, \vdash)$, $X_{rb} = X_{\ell z} \times X_{rz}$ be a left zero semigroup, a right zero semigroup and a rectangular band, respectively. By Lemma 2 $X_{\ell z,rz} = (X, \dashv, \vdash)$ is a rectangular dimonoid. We call this dimonoid as a left zero and right zero dimonoid. We will call a left zero and right zero dimonoid also a left and right diband.

Define the operations $\dashv$ and $\vdash$ on X^2 by

$$(x, y) \dashv (a, b) = (x, b), \quad (x, y) \vdash (a, b) = (a, b)$$

for all $(x, y), (a, b) \in X^2$. It is clear that $(X^2, \dashv)$ is a rectangular band and $(X^2, \vdash)$ is a right zero semigroup. By Lemma 2 $(X^2, \dashv, \vdash)$ is a (rb, rz)-dimonoid. We denote the dimonoid obtained by $X_{rb,rz}$.

Define the operations $\dashv$ and $\vdash$ on X^2 by

$$(x, y) \dashv (a, b) = (x, y), \quad (x, y) \vdash (a, b) = (x, b)$$

for all $(x, y), (a, b) \in X^2$. It is clear that $(X^2, \dashv)$ is a left zero semigroup and $(X^2, \vdash)$ is a rectangular band. By Lemma 3 $(X^2, \dashv, \vdash)$ is a $(\ell z, rb)$-dimonoid. We denote this dimonoid by $X_{\ell z,rb}$.

For all $i, j \in X$ put

$$A_{[i,j)} = \{(x, y, z) \in FRct(X) \,|\, (y, z) = (i, j)\},$$

$$A_{(i,j]} = \{(x, y, z) \in FRct(X) \,|\, (x, y) = (i, j)\},$$

$$A_{(i]} = \{(x, y, z) \in FRct(X) \,|\, y = i\},$$

$$A_{(i,j)} = \{(x, y, z) \in FRct(X) \,|\, (x, z) = (i, j)\},$$

$$A_{(i)} = \{(x, y, z) \in FRct(X) \,|\, x = i\},$$

$$A_{[i]} = \{(x, y, z) \in FRct(X) \,|\, z = i\}.$$

In terms of dibands of subdimonoids (see section 2) we obtain the following structure theorem.

Theorem 2. *Let $FRct(X)$ be the free rectangular dimonoid. Then*

(i) $FRct(X)$ is a diband $X_{rb,rz}$ of subsemigroups $A_{[i,j)}$, $(i,j) \in X_{rb,rz}$ such that $A_{[i,j)} \cong X_{\ell z}$ for every $(i,j) \in X_{rb,rz}$;

(ii) $FRct(X)$ is a diband $X_{\ell z,rb}$ of subsemigroups $A_{(i,j]}$, $(i,j) \in X_{\ell z,rb}$ such that $A_{(i,j]} \cong X_{rz}$ for every $(i,j) \in X_{\ell z,rb}$;

(iii) $FRct(X)$ is a left and right diband $X_{\ell z,rz}$ of subsemigroups $A_{(i]}$, $i \in X_{\ell z,rz}$ such that $A_{(i]} \cong X_{rb}$ for every $i \in X_{\ell z,rz}$;

(iv) $FRct(X)$ is a rectangular band X_{rb} of subdimonoids $A_{(i,j)}$, $(i,j) \in X_{rb}$ such that $A_{(i,j)} \cong X_{\ell z,rz}$ for every $(i,j) \in X_{rb}$;

(v) $FRct(X)$ is a left band $X_{\ell z}$ of subdimonoids $A_{(i)}$, $i \in X_{\ell z}$ such that $A_{(i)} \cong X_{rb,rz}$ for every $i \in X_{\ell z}$;

(vi) $FRct(X)$ is a right band X_{rz} of subdimonoids $A_{[i]}$, $i \in X_{rz}$ such that $A_{[i]} \cong X_{\ell z,rb}$ for every $i \in X_{rz}$.

Proof. (i) By Theorem 1 the map

$$\phi_{rb,rz} : FRct(X) \to X_{rb,rz} : (x,y,z) \mapsto (x,y,z)\phi_{rb,rz} = (y,z)$$

is a homomorphism. It is clear that $A_{[i,j)}, (i,j) \in X_{rb,rz}$ is a class of $\Delta_{\phi_{rb,rz}}$ which is a subdimonoid of $FRct(X)$. If (x,y,z), $(a,b,c) \in A_{[i,j)}$, then $y = b = i$, $z = c = j$ and

$$(x,y,z) \dashv (a,b,c) = (x,y,c) = (x,i,j),$$

$$(x,y,z) \vdash (a,b,c) = (x,b,c) = (x,i,j).$$

Hence the operations of $A_{[i,j)}$ coincide and so, it is a semigroup. It is not difficult to show that for every $(i,j) \in X_{rb,rz}$ the map

$$A_{[i,j)} \to X_{\ell z} : (x,i,j) \mapsto x$$

is an isomorphism.

(ii) By Theorem 1 the map

$$\phi_{\ell z,rb} : FRct(X) \to X_{\ell z,rb} : (x,y,z) \mapsto (x,y,z)\phi_{\ell z,rb} = (x,y)$$

is a homomorphism. Similarly to (i), $A_{(i,j]}$, $(i,j) \in X_{\ell z,rb}$ is a class of $\Delta_{\phi_{\ell z,rb}}$ which is a semigroup isomorphic to X_{rz}.

(iii) By Theorem 1 the map

$$\phi_{\ell z,rz} : FRct(X) \to X_{\ell z,rz} : (x,y,z) \mapsto (x,y,z)\phi_{\ell z,rz} = y$$

is a homomorphism. Then $A_{(i]}$, $i \in X_{\ell z,rz}$ is a class of $\Delta_{\phi_{\ell z,rz}}$ which is a subdimonoid of $FRct(X)$. If (x,y,z), $(a,b,c) \in A_{(i]}$, then $y = b = i$ and

$$(x,y,z) \dashv (a,b,c) = (x,y,c) = (x,i,c),$$

$$(x, y, z) \vdash (a, b, c) = (x, b, c) = (x, i, c).$$

Hence the operations of $A_{(i]}$ coincide and so, it is a semigroup. It is immediate to check that for every $i \in X_{\ell z, rz}$ the map

$$A_{(i]} \to X_{rb} : (x, i, z) \mapsto (x, z)$$

is an isomorphism.

(iv) By Theorem 1 the map

$$\phi_{rb} : FRct(X) \to X_{rb} : (x, y, z) \mapsto (x, y, z)\phi_{rb} = (x, z)$$

is a homomorphism. It is clear that $A_{(i,j)}, (i, j) \in X_{rb}$ is a class of $\Delta_{\phi_{rb}}$ which is a subdimonoid of $FRct(X)$. It can be shown that for every $(i, j) \in X_{rb}$ the map

$$A_{(i,j)} \to X_{\ell z, rz} : (i, y, j) \mapsto y$$

is an isomorphism.

(v) By Theorem 1 the map

$$\phi_{\ell z} : FRct(X) \to X_{\ell z} : (x, y, z) \mapsto (x, y, z)\phi_{\ell z} = x$$

is a homomorphism. It is evident that $A_{(i)},\ i \in X_{\ell z}$ is a class of $\Delta_{\phi_{\ell z}}$ which is a subdimonoid of $FRct(X)$. It is easy to cheek that for every $i \in X_{\ell z}$ the map

$$A_{(i)} \to X_{rb, rz} : (i, y, z) \mapsto (y, z)$$

is an isomorphism.

(vi) By Theorem 1 the map

$$\phi_{rz} : FRct(X) \to X_{rz} : (x, y, z) \mapsto (x, y, z)\phi_{rz} = z$$

is a homomorphism. Similarly to (v), $A_{[i]}$, $i \in X_{rz}$ is a class of $\Delta_{\phi_{rz}}$ which is a dimonoid isomorphic to $X_{\ell z, rb}$. □

Note that the class of left zero and right zero dimonoids (respectively, (rb, rz)-dimonoids, $(\ell z, rb)$-dimonoids) is a subvariety of the variety of all rectangular dimonoids. A dimonoid which is free in the variety of left zero and right zero dimonoids (respectively, (rb, rz)-dimonoids, $(\ell z, rb)$-dimonoids) will be called a free left zero and right zero dimonoid (respectively, free (rb, rz)-dimonoid, free $(\ell z, rb)$-dimonoid).

Lemma 5. *Every left zero and right zero dimonoid is a free left zero and right zero dimonoid.*

Proof. Let $X_{\ell z,rz}$ and $X'_{\ell z,rz}$ be left zero and right zero dimonoids. It can be shown that any map from $X_{\ell z,rz}$ to $X'_{\ell z,rz}$ is a homomorphism. $\square$

Denote the symmetric semigroup (respectively, symmetric group) on X by $\Im(X)$ (respectively, by $\Im[X]$) and the endomorphism semigroup (respectively, automorphism group) of a dimonoid M by $End\, M$ (respectively, by $Aut\, M$).

Corollary 1. *Let $X_{\ell z,rz}$ be the left zero and right zero dimonoid. Then*
(i) $End\, X_{\ell z,rz} \cong \Im(X)$;
(ii) $Aut\, X_{\ell z,rz} \cong \Im[X]$.

Lemma 6. *$X_{rb,rz}$ is a free (rb, rz)-dimonoid.*

Proof. $X_{rb,rz}$ is a (rb, rz)-dimonoid (see above). Let us show that $X_{rb,rz}$ is free.

Let $(T, \dashv', \vdash')$ be an arbitrary (rb, rz)-dimonoid, $\gamma : X \to T$ be an arbitrary map. Define the map

$$\phi : X_{rb,rz} \to (T, \dashv', \vdash') : (x, y) \mapsto (x, y)\phi = x\gamma \dashv' y\gamma.$$

We can show that ϕ is a homomorphism. $\square$

Dually, the following lemma can be proved.

Lemma 7. *$X_{\ell z,rb}$ is a free $(\ell z, rb)$-dimonoid.*

It is immediate to prove the following two propositions.

Proposition 2. *Let $(D, \dashv, \vdash)$ be an arbitrary (rb, rz)-dimonoid. Then*
(i) $End\,(D, \dashv, \vdash) \cong End\,(D, \dashv)$;
(ii) $Aut\,(D, \dashv, \vdash) \cong Aut\,(D, \dashv)$.

Proposition 3. *Let $(D, \dashv, \vdash)$ be an arbitrary $(\ell z, rb)$-dimonoid. Then*
(i) $End\,(D, \dashv, \vdash) \cong End\,(D, \vdash)$;
(ii) $Aut\,(D, \dashv, \vdash) \cong Aut\,(D, \vdash)$.

If ρ is a congruence on the dimonoid $(D, \dashv, \vdash)$ such that the operations of $(D, \dashv, \vdash)/_{\rho}$ coincide and it is a left zero semigroup (respectively, right zero semigroup, rectangular band, semilattice), then we say that ρ is a left zero congruence (respectively, right zero congruence, rectangular band congruence, semilattice congruence). If ρ is a congruence on the dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/_{\rho}$ is a left zero and right zero dimonoid (respectively, (rb, rz)-dimonoid, $(\ell z, rb)$-dimonoid), then we say that ρ is a left zero and right zero congruence (respectively, (rb, rz)-congruence, $(\ell z, rb)$-congruence).

From Theorem 2 we obtain

Corollary 2. *Let $FRct(X)$ be the free rectangular dimonoid. Then*

(i) $\Delta_{\phi_{rb,rz}}$ is the least (rb, rz)-congruence on $FRct(X)$;

(ii) $\Delta_{\phi_{\ell z,rb}}$ is the least $(\ell z, rb)$-congruence on $FRct(X)$;

(iii) $\Delta_{\phi_{\ell z,rz}}$ is the least left zero and right zero congruence on $FRct(X)$;

(iv) $\Delta_{\phi_{rb}}$ is the least rectangular band congruence on $FRct(X)$;

(v) $\Delta_{\phi_{\ell z}}$ is the least left zero congruence on $FRct(X)$;

(vi) $\Delta_{\phi_{rz}}$ is the least right zero congruence on $FRct(X)$.

Proof. (i) By Lemma 6 $X_{rb,rz}$ is the free (rb, rz)-dimonoid. According to Theorem 2 (i) we obtain (i).

The proof of (ii) is similar.

(iii) By Lemma 5 $X_{\ell z,rz}$ is the free left zero and right zero dimonoid. According to Theorem 2 (iii) we obtain (iii).

(iv) X_{rb} is the free rectangular band (see section 3). According to Theorem 2 (iv) we obtain (iv).

(v) It is well-known that every left zero semigroup is a free left zero semigroup. By Theorem 2 (v) we obtain (v).

The proof of (vi) is similar. □

From Theorem 3.1 [13] it follows that any rectangular dimonoid is semilattice indecomposable, i.e. the least semilattice congruence on a rectangular dimonoid coincides with the universal relation on this dimonoid.

We finish this section with the discussion of connections between rectangular dimonoids and restrictive bisemigroups.

Let B be an arbitrary nonempty set and $\dashv$, $\vdash$ be binary operations on B. An ordered triplet $(B, \dashv, \vdash)$ is called a restrictive bisemigroup, if the axioms $(D1)$, $(D5)$ and

$$x \dashv x = x,$$

$$x \vdash x = x,$$

$$x \dashv y \dashv z = y \dashv x \dashv z,$$

$$x \vdash y \vdash z = x \vdash z \vdash y,$$

$$(x \dashv y) \vdash z = x \dashv (y \vdash z)$$

hold for all $x, y, z \in B$. Restrictive bisemigroups have applications in the theory of binary relations.

Let $(B, \dashv)$ be a right zero semigroup and $(B, \vdash)$ be a left zero semigroup. It is immediate to check that $(B, \dashv, \vdash)$ is a restrictive bisemigroup. We call it as a right zero and left zero bisemigroup.

Let $(A, \dashv, \vdash)$ be an algebra with two associative operations. Define new operations $\dashv'$ and $\vdash'$ on A by

$$x \dashv' y = y \dashv x, \quad x \vdash' y = y \vdash x$$

for all $x, y \in A$. The algebra $(A, \dashv', \vdash')$ will be called a dual algebra of $(A, \dashv, \vdash)$.

It is easy to prove the following statement.

Proposition 4. *Let $(B, \dashv, \vdash)$ be a right zero and left zero bisemigroup. Then the dual algebra of $(B, \dashv, \vdash)$ is a left zero and right zero dimonoid.*

5. Free dimonoids

In this section we present the least rectangular diband congruence, the least (rb, rz)-congruence, the least $(\ell z, rb)$-congruence, the least left zero and right zero congruence, the least rectangular band congruence, the least left zero congruence and the least right zero congruence on free dimonoids and use them to obtain decompositions of free dimonoids.

Let $\breve{F}[X]$ be the free dimonoid (see section 2). For every $w = x_1...x_i...x_n \in F[X]$, $x_i \in X$, $1 \leq i \leq n$ the set of all letters occurring in w will be denoted by $c(w)$. If ρ is a congruence on the dimonoid $(D, \dashv, \vdash)$ such that ${}^{(D, \dashv, \vdash)}/_{\rho}$ is a rectangular dimonoid, then we say that ρ is a rectangular diband congruence.

Take $(a, b, c) \in FRct(X)$ (see Theorem 1). Let $\Omega^{(a,b,c)}(X)$ be the set of all finite subsets Y of X such that $a, b, c \in Y$ and let $\Omega_{(a,b,c)}(X)$ be a semilattice defined on $\Omega^{(a,b,c)}(X)$ by the operation of the set theoretical union. For all $(a, b, c) \in FRct(X)$ and $Y \in \Omega_{(a,b,c)}(X)$ put

$$T_{(a,b,c)} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, (x_1, x_m, x_n) = (a, b, c)\},$$

$$T^Y_{(a,b,c)} = \{(x_1...x_i...x_n, m) \in T_{(a,b,c)} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation π on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\pi(y_1...y_j...y_s, t) \Leftrightarrow (x_1, x_m, x_n) = (y_1, y_t, y_s).$$

Theorem 3. *The relation π on the free dimonoid $\breve{F}[X]$ is the least rectangular diband congruence. The free dimonoid $\breve{F}[X]$ is a rectangular diband $FRct(X)$ of subdimonoids $T_{(a,b,c)}$, $(a, b, c) \in FRct(X)$. Every dimonoid $T_{(a,b,c)}$, $(a, b, c) \in FRct(X)$ is a semilattice $\Omega_{(a,b,c)}(X)$ of subdimonoids $T^Y_{(a,b,c)}$, $Y \in \Omega_{(a,b,c)}(X)$.*

Proof. Define a map $\mu : \breve{F}[X] \to FRct(X)$ by

$$(x_1...x_i...x_n, m) \mapsto (x_1, x_m, x_n), \;\; (x_1...x_i...x_n, m) \in \breve{F}[X].$$

For arbitrary elements $(x_1...x_i...x_n, m), (y_1...y_j...y_s, t) \in \breve{F}[X]$ we have

$$((x_1...x_i...x_n, m) \dashv (y_1...y_j...y_s, t))\mu =$$

$$= (x_1...x_ny_1...y_s, m)\mu = (x_1, x_m, y_s) =$$

$$= (x_1, x_m, x_n) \dashv (y_1, y_t, y_s) =$$

$$= (x_1...x_i...x_n, m)\mu \dashv (y_1...y_j...y_s, t)\mu,$$

$$((x_1...x_i...x_n, m) \vdash (y_1...y_j...y_s, t))\mu =$$

$$= (x_1...x_ny_1...y_s, n+t)\mu = (x_1, y_t, y_s) =$$

$$= (x_1, x_m, x_n) \vdash (y_1, y_t, y_s) =$$

$$= (x_1...x_i...x_n, m)\mu \vdash (y_1...y_j...y_s, t)\mu.$$

Thus, μ is a surjective homomorphism. By Theorem 1 $FRct(X)$ is the free rectangular dimonoid. Then Δ_μ is the least rectangular diband congruence on $\breve{F}[X]$. From the definition of μ it follows that $\Delta_\mu = \pi$. It is clear that $T_{(a,b,c)}$, $(a, b, c) \in FRct(X)$ is a class of Δ_μ which is a subdimonoid of $\breve{F}[X]$. Moreover, it is not difficult to show that for every $(a, b, c) \in FRct(X)$ the map

$$T_{(a,b,c)} \to \Omega_{(a,b,c)}(X) : (x_1...x_i...x_n, m) \mapsto c(x_1...x_i...x_n)$$

is a homomorphism. Hence $T_{(a,b,c)}$ is a semilattice $\Omega_{(a,b,c)}(X)$ of subdimonoids $T^Y_{(a,b,c)}$, $Y \in \Omega_{(a,b,c)}(X)$. □

Take $(b, c) \in X_{rb,rz}$ (see section 4). Let $\Omega^{(b,c)}(X)$ be the set of all finite subsets Y of X such that $b, c \in Y$ and let $\Omega_{(b,c)}(X)$ be a semilattice defined on $\Omega^{(b,c)}(X)$ by the operation of the set theoretical union. For all $(b, c) \in X_{rb,rz}$ and $Y \in \Omega_{(b,c)}(X)$ put

$$T_{[b,c)} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, (x_m, x_n) = (b, c)\},$$

$$T^Y_{[b,c)} = \{(x_1...x_i...x_n, m) \in T_{[b,c)} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation χ on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\chi(y_1...y_j...y_s, t) \Leftrightarrow (x_m, x_n) = (y_t, y_s).$$

Theorem 4. *The relation χ on the free dimonoid $\breve{F}[X]$ is the least (rb, rz)-congruence. The free dimonoid $\breve{F}[X]$ is a diband $X_{rb,rz}$ of subdimonoids $T_{[b,c)}$, $(b, c) \in X_{rb,rz}$. Every dimonoid $T_{[b,c)}$, $(b, c) \in X_{rb,rz}$ is a semilattice $\Omega_{(b,c)}(X)$ of subdimonoids $T^Y_{[b,c)}$, $Y \in \Omega_{(b,c)}(X)$.*

Proof. Define a map $\delta : \breve{F}[X] \to X_{rb,rz}$ by

$$(x_1...x_i...x_n, m) \mapsto (x_m, x_n), \;\; (x_1...x_i...x_n, m) \in \breve{F}[X].$$

For all $(x_1...x_i...x_n, m)$, $(y_1...y_j...y_s, t) \in \breve{F}[X]$ we have

$$((x_1...x_i...x_n, m) \dashv (y_1...y_j...y_s, t))\delta =$$

$$= (x_1...x_n y_1...y_s, m)\delta = (x_m, y_s) =$$
$$= (x_m, x_n) \dashv (y_t, y_s) =$$
$$= (x_1...x_i...x_n, m)\delta \dashv (y_1...y_j...y_s, t)\delta,$$
$$((x_1...x_i...x_n, m) \vdash (y_1...y_j...y_s, t))\delta =$$
$$= (x_1...x_n y_1...y_s, n+t)\delta = (y_t, y_s) =$$
$$= (x_m, x_n) \vdash (y_t, y_s) =$$
$$= (x_1...x_i...x_n, m)\delta \vdash (y_1...y_j...y_s, t)\delta.$$

Thus, δ is a surjective homomorphism. By Lemma 6 $X_{rb,rz}$ is the free (rb, rz)-dimonoid. Then Δ_δ is the least (rb, rz)-congruence on $\breve{F}[X]$. From the definition of δ it follows that $\Delta_\delta = \chi$. Clearly, $T_{[b,c)}, \; (b, c) \in X_{rb,rz}$ is a class of Δ_δ which is a subdimonoid of $\breve{F}[X]$. Moreover, we can show that for every $(b, c) \in X_{rb,rz}$ the map

$$T_{[b,c)} \to \Omega_{(b,c)}(X) : (x_1...x_i...x_n, m) \mapsto c(x_1...x_i...x_n)$$

is a homomorphism. Hence $T_{[b,c)}$ is a semilattice $\Omega_{(b,c)}(X)$ of subdimonoids $T^Y_{[b,c)}, \; Y \in \Omega_{(b,c)}(X)$. □

For all $(a, b) \in X_{\ell z,rb}$ (see section 4) and $Y \in \Omega_{(a,b)}(X)$ put

$$T_{(a,b]} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, (x_1, x_m) = (a, b)\},$$

$$T^Y_{(a,b]} = \{(x_1...x_i...x_n, m) \in T_{(a,b]} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation ζ on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\zeta(y_1...y_j...y_s, t) \Leftrightarrow (x_1, x_m) = (y_1, y_t).$$

Similarly to Theorem 4, the following theorem can be proved.

Theorem 5. *The relation ζ on the free dimonoid $\breve{F}[X]$ is the least $(\ell z, rb)$-congruence. The free dimonoid $\breve{F}[X]$ is a diband $X_{\ell z,rb}$ of subdimonoids $T_{(a,b]}, \; (a, b) \in X_{\ell z,rb}$. Every dimonoid $T_{(a,b]}, \; (a, b) \in X_{\ell z,rb}$ is a semilattice $\Omega_{(a,b)}(X)$ of subdimonoids $T^Y_{(a,b]}, \; Y \in \Omega_{(a,b)}(X)$.*

Take $b \in X_{\ell z,rz}$ (see section 4). Let $\Omega^b(X)$ be the set of all finite subsets Y of X such that $b \in Y$ and let $\Omega_b(X)$ be a semilattice defined on $\Omega^b(X)$ by the operation of the set theoretical union. For all $b \in X_{\ell z,rz}$ and $Y \in \Omega_b(X)$ put

$$T_{(b]} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, x_m = b\},$$

$$T_{(b]}^{Y} = \{(x_1...x_i...x_n, m) \in T_{(b]} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation ϱ on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\varrho(y_1...y_j...y_s, t) \Leftrightarrow x_m = y_t.$$

Theorem 6. *The relation ϱ on the free dimonoid $\breve{F}[X]$ is the least left zero and right zero congruence. The free dimonoid $\breve{F}[X]$ is a left and right diband $X_{\ell z,rz}$ of subdimonoids $T_{(b]}$, $b \in X_{\ell z,rz}$. Every dimonoid $T_{(b]}$, $b \in X_{\ell z,rz}$ is a semilattice $\Omega_b(X)$ of subdimonoids $T_{(b]}^{Y}$, $Y \in \Omega_b(X)$.*

Proof. Define a map $\alpha : \breve{F}[X] \to X_{\ell z,rz}$ by

$$(x_1...x_i...x_n, m) \mapsto x_m, \quad (x_1...x_i...x_n, m) \in \breve{F}[X].$$

For all $(x_1...x_i...x_n, m)$, $(y_1...y_j...y_s, t) \in \breve{F}[X]$ we have

$$((x_1...x_i...x_n, m) \dashv (y_1...y_j...y_s, t))\alpha =$$

$$= (x_1...x_n y_1...y_s, m)\alpha = x_m = x_m \dashv y_t =$$

$$= (x_1...x_i...x_n, m)\alpha \dashv (y_1...y_j...y_s, t)\alpha,$$

$$((x_1...x_i...x_n, m) \vdash (y_1...y_j...y_s, t))\alpha =$$

$$= (x_1...x_n y_1...y_s, n+t)\alpha = y_t = x_m \vdash y_t =$$

$$= (x_1...x_i...x_n, m)\alpha \vdash (y_1...y_j...y_s, t)\alpha.$$

Thus, α is a surjective homomorphism. By Lemma 5 $X_{\ell z,rz}$ is the free left zero and right zero dimonoid. Then Δ_α is the least left zero and right zero congruence on $\breve{F}[X]$. From the definition of α it follows that $\Delta_\alpha = \varrho$. Evidently, $T_{(b]}$, $b \in X_{\ell z,rz}$ is a class of Δ_α which is a subdimonoid of $\breve{F}[X]$. Moreover, it is not difficult to see that for every $b \in X_{\ell z,rz}$ the map

$$T_{(b]} \to \Omega_b(X) : (x_1...x_i...x_n, m) \mapsto c(x_1...x_i...x_n)$$

is a homomorphism. From this it follows that the last statement of the theorem holds. $\square$

For all $(a,c) \in X_{rb}$ (see section 4) and $Y \in \Omega_{(a,c)}(X)$ put

$$T_{(a,c)} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, (x_1, x_n) = (a, c)\},$$

$$T^Y_{(a,c)} = \{(x_1...x_i...x_n, m) \in T_{(a,c)} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation τ on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\tau(y_1...y_j...y_s, t) \Leftrightarrow (x_1, x_n) = (y_1, y_s).$$

Theorem 7. *The relation τ on the free dimonoid $\breve{F}[X]$ is the least rectangular band congruence. The free dimonoid $\breve{F}[X]$ is a rectangular band X_{rb} of subdimonoids $T_{(a,c)}$, $(a,c) \in X_{rb}$. Every dimonoid $T_{(a,c)}$, $(a,c) \in X_{rb}$ is a semilattice $\Omega_{(a,c)}(X)$ of subdimonoids $T^Y_{(a,c)}$, $Y \in \Omega_{(a,c)}(X)$.*

Proof. Define a map $\psi : \breve{F}[X] \to X_{rb}$ by

$$(x_1...x_i...x_n, m) \mapsto (x_1, x_n), \quad (x_1...x_i...x_n, m) \in \breve{F}[X].$$

It is easy to check that ψ is a surjective homomorphism. As X_{rb} is the free rectangular band (see section 3), then Δ_ψ is the least rectangular band congruence on $\breve{F}[X]$. From the definition of ψ it follows that $\Delta_\psi = \tau$. Obviously, $T_{(a,c)}$, $(a,c) \in X_{rb}$ is a class of Δ_ψ which is a subdimonoid of $\breve{F}[X]$.

Similarly to Theorem 4, the last statement of the theorem can be proved. □

For all $a \in X_{\ell z}$ (see section 4) and $Y \in \Omega_a(X)$ put

$$T_{(a)} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, x_1 = a\},$$

$$T^Y_{(a)} = \{(x_1...x_i...x_n, m) \in T_{(a)} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation ω on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\omega(y_1...y_j...y_s, t) \Leftrightarrow x_1 = y_1.$$

Theorem 8. *The relation ω on the free dimonoid $\breve{F}[X]$ is the least left zero congruence. The free dimonoid $\breve{F}[X]$ is a left band $X_{\ell z}$ of subdimonoids $T_{(a)}$, $a \in X_{\ell z}$. Every dimonoid $T_{(a)}$, $a \in X_{\ell z}$ is a semilattice $\Omega_a(X)$ of subdimonoids $T^Y_{(a)}$, $Y \in \Omega_a(X)$.*

Proof. Define a map $\beta : \breve{F}[X] \to X_{\ell z}$ by

$$(x_1...x_i...x_n, m) \mapsto x_1, \quad (x_1...x_i...x_n, m) \in \breve{F}[X].$$

It can be proved that β is a surjective homomorphism. As $X_{\ell z}$ is the free left zero semigroup (see Corollary 2), then Δ_β is the least left zero congruence on $\breve{F}[X]$. From the definition of β it follows that $\Delta_\beta = \omega$. It is evident that $T_{(a)},\ a \in X_{\ell z}$ is a class of Δ_β which is a subdimonoid of $\breve{F}[X]$.

Similarly to Theorem 6, the last statement of the theorem can be proved. □

For all $c \in X_{rz}$ (see section 4) and $Y \in \Omega_c(X)$ put

$$T_{[c]} = \{(x_1...x_i...x_n, m) \in \breve{F}[X] \,|\, x_n = c\},$$

$$T_{[c]}^Y = \{(x_1...x_i...x_n, m) \in T_{[c]} \,|\, c(x_1...x_i...x_n) = Y\}.$$

Define a relation σ on $\breve{F}[X]$ by

$$(x_1...x_i...x_n, m)\sigma(y_1...y_j...y_s, t) \Leftrightarrow x_n = y_s.$$

Similarly to Theorem 8, the following theorem can be proved.

Theorem 9. *The relation σ on the free dimonoid $\breve{F}[X]$ is the least right zero congruence. The free dimonoid $\breve{F}[X]$ is a right band X_{rz} of subdimonoids $T_{[c]},\ c \in X_{rz}$. Every dimonoid $T_{[c]},\ c \in X_{rz}$ is a semilattice $\Omega_c(X)$ of subdimonoids $T_{[c]}^Y,\ Y \in \Omega_c(X)$.*

Note that the least semilattice congruence on $\breve{F}[X]$ and the corresponding decomposition of the free dimonoid were described in [11].

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