ON THE BEHAVIOR OF SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS AT INFINITY

The existence of limits at the infinity, generalized in the Abel sense, is established for bounded solutions of the operator-differential equation \( y'(t) = Ay(t) \) in a reflexive Banach space.

We consider a Cauchy problem

\[
y'(t) = Ay(t), \quad t \in \mathbb{R}_+, \quad y(0) = y_0,
\]

in a Banach space \( \mathcal{B} \) endowed with the norm \( \| \cdot \| \). Here, \( A \) is a linear closed operator in \( \mathcal{B} \) and \( \mathbb{R}_+ = [0, \infty) \). A function \( y(t) \) is said to be a solution of the Cauchy problem (1) if it satisfies both equalities in (1) and \( y(t) \in C^1(\mathbb{R}_+, \mathcal{B}) \).

In the present paper, we are concerned with the behavior of solutions of the Cauchy problem (1) at the infinity.

**Definition.** Let \( \alpha > 0 \) and let \( y(t) \in C(\mathbb{R}_+, \mathcal{B}) \). We define the Cesaro limit of \( y(t) \) of order \( \alpha \) as

\[
(C, \alpha) \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s) \, ds,
\]

whenever the latter exists.

**Theorem 1** [1]. Let \( A \) be a generator of a strongly continuous semigroup \( T(t), t \in \mathbb{R}_+ \). Then

a) if \( x = x_0 + x_1 \in N(A) \oplus \overline{R(A)} \), then \((C, \alpha) \lim_{t \to \infty} T(t)x = x_0\);

b) if there exists a sequence \( \{ t_j, j \in \mathbb{N} \} \), \( t_j \to \infty \), such that sequence

\[
\alpha t_j^{-\alpha} \int_0^{t_j} (t_j-s)^{\alpha-1} y(s) \, ds
\]

is weakly convergent, then \( x \in N(A) \oplus \overline{R(A)} \);

c) if \( \mathcal{B} \) is a reflexive space, then \( \mathcal{B} = N(A) \oplus \overline{R(A)} \) and the limit

\[
(C, \alpha) \lim_{t \to \infty} T(t)x
\]

exists \( \forall x \in \mathcal{B} \).

Let \( y(t) \) be a bounded solution of the Cauchy problem (1). Then statement a) of Theorem 1, generally speaking, is not true. It is shown by the following example:

**Example.** We consider a space \( \mathcal{M} \) of all bounded sequences \( \{ \beta_n \in \mathbb{C}, n \in \mathbb{N} \cup \{0\} \} \) equipped with the norm \( \| \{ \beta_n \} \| = \sup |\beta_n| \). We set \( A \{ \beta_n \} = \{ \gamma_n \} \), where \( \gamma_0 = 0, \gamma_1 = \beta_0, \gamma_n = i\beta_n/n + \beta_0, n \geq 2 \). Let \( \mathcal{M}_0 = \{ \beta_n \in \mathcal{M}, \beta_0 = 0 \} \). The restriction of \( A \) to \( \mathcal{M}_0 \) (we denote it by \( A_0 \)) generates a \( C_0 \)-semigroup of contractions \( T(t) \) [2, p. 535], and the vector \( \{ 0, 1, 1, \ldots \} \)
1, ... } ∈ N(A_0) ⊕ \overline{R}(A_0). We conclude from Theorem 1 that the (C, α)-limit of a bounded solution of the Cauchy problem (1), where \( y(0) = \{0, 1, 1, 1, \ldots\} \), does not exist. But \( y(0) = A \{1, 0, 0, \ldots\} \), concluding the example. Also it is shown that statement a) of Theorem 1 is not valid for bounded solutions of the Cauchy problem (1) when A generates an unbounded C_0-semigroup.

**Lemma 1.** Let \( y(t) \) be a bounded solution of the Cauchy problem (1). Then statement b) of Theorem 1 holds true if we substitute \( y(t) \) for \( T(s)x \) and \( y(0) \) for \( x \).

**Proof.** Since A is closed, we conclude that

\[
\alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \in D(A).
\]

By letting \( t \to \infty \), we get

\[
A \left( \alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \right) =
\]

\[
= \alpha t^{-\alpha} \int_0^{t-1} (t-s)^{\alpha-1} y'(s) \, ds + \alpha t^{-\alpha} \int_{t-1}^t (t-s)^{\alpha-1} y'(s) \, ds =
\]

\[
= \alpha t^{-\alpha} \int_0^{t-1} (t-s)^{\alpha-1} y'(s) \, ds + \alpha t^{-\alpha} y(t-1) - \alpha t^{-1} y(0) +
\]

\[
+ \alpha (\alpha - 1) t^{-\alpha} \int_0^{t-1} (t-s)^{\alpha-2} y(s) \, ds \to 0
\]

since \( \|y(t)\| \) is bounded. Since A is closed, we obtain \( x_0 \in N(A) \).

We set \( z(t) = y(t) - x_0 \). Then

\[
\alpha \int_{t_j}^{t_j+1} (t-j-s)^{\alpha-1} z(s) \, ds \to 0, \quad j \to \infty,
\]

(here, \( \to \) stands for the weak convergence in \( \mathcal{B} \)). Integrating by parts, we get

\[
z(0) = -A \left( t_{j+1}^{-\alpha} \int_0^{t_j} (t_j-s)^{\alpha-1} z(s) \, ds \right) + z(t-1) t_{j+1}^{-\alpha} -
\]

\[
- (\alpha - 1) t_j^{-\alpha} \int_0^{t_{j+1}} (t_{j+1}-s)^{\alpha-2} z(s) \, ds.
\]

When \( j \to \infty \), the last two terms on the right-hand side of the above equality tend weakly to zero. Hence, \( z(0) \in \overline{R}(A) \) and \( y(0) \in N(A) \oplus \overline{R}(A) \).

Now we are going to generalize statement b) of Theorem 1.

**Theorem 2.** Let \( \mathcal{B} \) be a reflexive Banach space. We suppose that the Cauchy problem (1) admits at most one bounded solution for any \( y_0 \in \mathcal{B} \) (i.e., if there exist few solutions for certain \( y_0 \), only one of them is bounded). If \( y(t) \) is a solution of the Cauchy problem (1) such that \( \|y(t)\| \leq M \), then \( \forall \alpha > 0 \) there exists

\[
(C, \alpha) \lim_{t \to \infty} y(t) = z, \quad z \in N(A).
\]
Proof. We denote by $\mathfrak{N}'$ the set of all $w \in \mathfrak{B}$ such that there exists a bounded solution of Cauchy problem (1) with the initial value $w$. For any $w \in \mathfrak{N}'$, we set $\|w\|_{\mathfrak{N}} = \sup \{|x(t)|, t \geq 0\}$, where $x(t)$ is the bounded solution of the Cauchy problem (1) corresponding to $w$ by the definition of $\mathfrak{N}'$. We denote by $\mathfrak{N}$ the completion of $\mathfrak{N}'$ in the norm $\|\cdot\|_{\mathfrak{N}}$. We outline that $\forall w \in \mathfrak{N}, \|w\|_{\mathfrak{N}}  \geq \|w\|$.

Without loss of generality, we assume that $\mathfrak{N}$ is dense in $\mathfrak{B}$. If this is not the case, we consider the Cauchy problem (1) in the space $\mathfrak{B}_0 := \overline{\mathfrak{N}}$ (the bar denotes the closure in $\mathfrak{B}$). In $\mathfrak{B}_0$, all the assumptions of Theorem 2 hold. So, by using the continuity and denseness of the embedding $\mathfrak{N} \subset \mathfrak{B}$, we get $\mathfrak{B}^* \subset \mathfrak{N}^*$ with the continuous embedding.

We define a semigroup of operators $T(t), t \geq 0$, on $\mathfrak{N}'$ by the relation $T(t)w = x(t), t \geq 0$, where $x(t)$ is the solution of the Cauchy problem (1) corresponding to $w$ by the definition of $\mathfrak{N}'$. It is easy to see that $T(t)$ is a semigroup of contractions; this is why $T(t)$ may be extended on $\mathfrak{N}$ by continuity.

We state that $T(t)$ is a $C_0$-semigroup on $\mathfrak{N}$. To prove this, it is sufficient to show that $T(t)$ is weakly continuous at zero ($[3], IX.1$). The latter condition holds if the functions $T(t)w$ are weakly continuous at zero $\forall w \in \mathfrak{N}'$. Otherwise,

$$\exists y_0^* \in \mathfrak{N}^* \exists \varepsilon > 0 \exists y_1 \in \mathfrak{N}' \exists \{t_n, n \in \mathbb{N}\} \ (t_n \to 0, n \to \infty) \left| y_0^*(T(t_n)y_1 - y_1) \right| > \varepsilon. \tag{2}$$

Obviously, $y_0^*$ does not belong to the closure of $\mathfrak{B}^*$ in $\mathfrak{N}^*$.

Now we are going to make some preliminary constructions. Given any $z^* \in \mathfrak{N}^*$, we define the Banach space $X := (w^* + \alpha z^*, w^* \in \mathfrak{B}^*, \alpha \in \mathbb{C})$ with the norm $\|w^* + \alpha z^*\|_X = \|w^*\|_{\mathfrak{B}^*} + |\alpha|$. Since $\mathfrak{B}^*$ and $X/\mathfrak{B}^*$ are reflexive, $X$ is reflexive, too. We consider the function $T(t)y_1$ bounded in $X$. There exist a subsequence $\{s_n, n \in \mathbb{N}\}$ of the sequence $\{t_n, n \in \mathbb{N}\}$ and $w_0 \in X$ such that $T(s_n)y_1 \rightharpoonup w_0$ in $X$ as $n \to \infty$. Since $\mathfrak{B}$ is weakly closed and $\mathfrak{B}^*$ is contained in $X$, we conclude that $w_0 \in \mathfrak{B}$. With $T(t)y_1$ being strongly continuous in $\mathfrak{B}$, we see that $w_0 = y_1$. So, we get $y_0^*(T(s_n)y_1 - y_1) \to 0, n \to \infty$. It makes a contradiction to (2). Therefore, $T(t)$ is a $C_0$-semigroup on $\mathfrak{N}$.

Let us prove that there exists a sequence $\{r_n, n \in \mathbb{N}\} \ (r_n \to 0, n \to \infty)$ such that

$$T(r_n)y_0 \rightharpoonup w \text{ in } \mathfrak{N}, n \to \infty. \tag{3}$$

Assume the contrary. We set $t_n = n$. The reflexivity of $\mathfrak{B}$ implies the existence of a subsequence $\{s_n, n \in \mathbb{N}\}$ of $\{t_n\}$ and $u \in \mathfrak{B}$ such that $T(s_n)y_0 \rightharpoonup u$ in $\mathfrak{B}$. Then there exist $y_0^* \in \mathfrak{N}^*$, a subsequence $\{r_n, n \in \mathbb{N}\}$ of $\{s_n\}$, and $\varepsilon > 0$ such that

$$\left| y_0^*(T(r_n)y_0 - u) \right| > \varepsilon. \tag{4}$$

Obviously, $y_0^*$ does not belong to the closure of $\mathfrak{B}^*$ in $\mathfrak{N}^*$. We define the space $X$ in the same way as it was done after relation (2). Repeating this argument, we arrive at the conclusion that there exist a subsequence $\{p_n, n \in \mathbb{N}\}$ of the sequence...
\{ r_n \} and \( w \in \mathbb{B} \) satisfying the relation \( y_0^\ast (T(r_n) y_0) \overset{w}{\to} u \) in \( \mathbf{X} \), \( n \to \infty \). The latter condition makes a contradiction to (4). Therefore, (4) is not true and \( \exists u \in \mathbb{N} \) such that \( T(r_n) y_0 \overset{w}{\to} u \) in \( \mathbb{N} \), \( n \to \infty \) (we point out that \( u \in \mathbb{N} \) because \( u \in N(A) \) by Lemma 1).

In view of (3), we need only to apply Theorem 1 to the semigroup \( T(t) \) and \( y_0 \). From part b) of the theorem, we deduce that \( y_0 \in N(B) \oplus \mathcal{R}(B) \) (\( B \) is a generator of \( T(t) \)). Part a) states that there exists

\[
(C, \alpha) \lim_{t \to \infty} T(t) y_0 = u.
\]

Thus, Theorem 2 is proved.

**Corollary 1.** If the open right-side halfplane \( \{ \lambda \in \mathbb{C}, \Re \lambda > 0 \} \) is not contained in the point spectrum \( \sigma_p(A) \), then the statement of Theorem 2 holds true.

Proof is immediately obtained from the proof of Theorem 23.7.1 [2].

**Theorem 3.** Let \( \mathbb{B} \) be a reflexive Banach space. We suppose that \( \exists \lambda_1, \lambda_2 \in \mathbb{C}, (\Re \lambda_1 > 0, \Re \lambda_2 > 0) \) such that there exist projection operators \( P_1 \) and \( P_2 \) onto the subspaces \( N_1 = \{ x \in \mathbb{B}, Ax = \lambda_1 x \} \), \( N_2 = \{ x \in \mathbb{B}, Ax = \lambda_2 x \} \) respectively. If \( y(t) \) is a solution of the Cauchy problem (1) such that \( \| y(t) \| \leq M, \forall \alpha > 0 \) there exists

\[
(C, \alpha) \lim_{t \to \infty} y(t) = z,
\]

and \( z \in N(A) \).

**Proof.** We set \( P_3 = I - P_2 - P_1 \), \( N_3 = P_3 \mathbb{B} \). We denote \( y_i(t) \) := \( P_i y(t) \), \( i = 1, 2, 3 \). By applying the operator \( P_2 + P_3 \) to (1), we get

\[
y_2'(t) + y_3'(t) = (P_2 + P_3) A (y_2(t) + y_3(t)).
\]

This is why the function \( y_2(t) + y_3(t) \) is a bounded solution of the equation \( z'(t) = (P_2 + P_3) A z(t), t \geq 0 \).

Since \( \lambda_1 \in \sigma_p((P_2 + P_3) A) \), we may apply Corollary 1 to the present setting. So,

\[
\exists (C, \alpha) \lim_{t \to \infty} (y_2(t) + y_3(t)) = w, \ w \in N((P_2 + P_3) A).
\]

Here, \( w \in N(A) \) because

\[
\alpha \int_0^t (t-s)^{\alpha-1} y_i(s) ds \in N_i, \ i = 1, 2, 3,
\]

and \( N((P_2 + P_3) A) = N(A) + N_1 \).

In a similar way, we can obtain

\[
\exists (C, \alpha) \lim_{t \to \infty} y_3(t) = v, \ v \in N(P_3 A),
\]

and

\[
\exists (C, \alpha) \lim_{t \to \infty} (y_1(t) + y_3(t)) = u, \ u \in N((P_1 + P_3) A)
\]

by applying to (1) the operators \( P_3 \) and \( P_1 + P_3 \), respectively. From (5), (6), and (7), we deduce the statement of Theorem 3.
Corollary 2. Let \( \mathcal{B} \) be a reflexive Banach space. If there exist \( \lambda \in \mathbb{C} \) (\( \Re \lambda > 0 \)) and a projection operator \( P \) onto the subspaces \( \mathcal{X} = \{ x \in \mathcal{B}, Ax = \lambda x \} \) such that \( A \) is invariant on \( (1 - P)\mathcal{B} \), then the statement of Theorem 3 holds true.

Corollary 3. If \( \mathcal{B} \) is a Hilbert space, then the statement of Theorem 3 remains valid.

Corollary 4. If there exist \( \lambda_1, \lambda_2 \in \mathbb{C}, \Re \lambda_1 > 0, \Re \lambda_2 > 0 \), such that the subspaces \( N_1, N_2 \), defined in Theorem 3 are finite-dimensional, then the statement of Theorem 3 holds true.

When \( A \) satisfies some additional assumptions, we can reformulate Theorem 1 in a more precise way:

Theorem 4 \([1]\). If \( A \) is a generator of a bounded analytic semigroup, then we can replace \((C, \alpha)\)-limits in Theorem 1 by the strong ones.

Theorem 5. Let the assumptions of Theorems 2 or 3 hold. If \( y(t) \) is a solution of the Cauchy problem (1), which admits a bounded analytic extension to the sector \( S_\varphi := \{ \lambda \in \mathbb{C}, |\arg \lambda| < \varphi \} \) for some \( \varphi \in (0, \pi/2) \), then there exists

\[
\lim_{t \to \infty} y(t) = z \quad \text{and} \quad z \in N(A).
\]

The proof of Theorem 5 repeats the argument used to prove Theorem 2 (or Theorem 3, respectively). We need only to redefine \( \overline{\mathfrak{H}} \) to be a set of all \( w \in \mathcal{B} \) such that there exists bounded solution \( y(t) \), analytic in \( S_\varphi \), of the Cauchy problem (1) with the initial value \( w \). Then \( \|w\|_{\overline{\mathfrak{H}}} = \{ \|x(t)\|, t \in S_\varphi \} \).

Corollary 5. If the assumptions of one of Corollaries 2–4 hold, then the statement of Theorem 5 remains valid.

Theorem 6. In the statements of Theorems 2 and 3, we can replace \((C, \alpha)\)-limit by the Abel limit (for the definition of the \( A \)-limit, see \([1, 4]\)).

Proof is an immediate consequence of Theorems 2 and 3 and the lemma in \([4, p. 92]\).

From Lemma 1, we can deduce the following corollary.

Corollary 6. Let \( \mathcal{B} \) be a reflexive Banach space. If \( N(A) \cap R(A) = \{0\} \), then

\[
\alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s) ds \to z, \quad t \to \infty, \quad z \in N(A).
\]

This fact is a generalization of Theorem 3 \([5]\).

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