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RESEARCH ARTICLE

Multi-algebras from the viewpoint of algebraic logic

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Abstract. Where $\boldsymbol{U}$ is a structure for a first-order language $\mathcal{L}^{\approx}$ with equality $\approx$, a standard construction associates with every formula f of $\mathcal{L}^{\approx}$ the set $\|f\|$ of those assignments which fulfill f in $\boldsymbol{U}$. These sets make up a (cylindric like) set algebra $Cs(\boldsymbol{U})$ that is a homomorphic image of the algebra of formulas. If $\mathcal{L}^{\approx}$ does not have predicate symbols distinct from $\approx$, i.e. $\boldsymbol{U}$ is an ordinary algebra, then $Cs(\boldsymbol{U})$ is generated by its elements $\|s \approx t\|$; thus, the function $(s,t) \mapsto \|s \approx t\|$ comprises all information on $Cs(\boldsymbol{U})$.

In the paper, we consider the analogues of such functions for multi-algebras. Instead of $\approx$, the relation ε of singular inclusion is accepted as the basic one ($s\varepsilon t$ is read as 's has a single value, which is also a value of t'). Then every multi-algebra $\boldsymbol{U}$ can be completely restored from the function $(s,t) \mapsto \|s\ \varepsilon\ t\|$. The class of such functions is given an axiomatic description.

1. Introduction

We begin, in the first subsection, with reviewing a few standard constructions used in algebraic logic. Then we outline the problem which we deal with in the paper.

1.1 Let $\mathcal{L}^{\approx}$ be a first-order language with equality over the set of variables X. For the sake of definiteness, we assume that the logical primitives of $\mathcal{L}^{\approx}$ are $\neg, \wedge, \vee, \exists$. Let, furthermore, $\boldsymbol{U} := (U, \ldots)$ be a structure

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for $\mathcal{L}^{\approx}$. For every formula f of $\mathcal{L}^{\approx}$, we denote by $\|f\|$ the set of those assignments from U^X which satisfy f in $\boldsymbol{U}$. Then

$$\|\neg f\| = -\|f\|,\ \|f \wedge g\| = \|f\| \cap \|g\|,\ \|f \vee g\| = \|f\| \cup \|g\|,$$

$$\|\exists x\, f\| = C_x\|f\|,\ \|x \approx y\| = D_{xy}.$$

Here $-$ is the set complementation, C_x is the *cylindrification* along x-axis in the "space" U^X and is defined by

$$C_x(A) := \{\varphi \in U^X\colon\ \varphi^x_u \in A \text{ for some } u \in U\} = \{\psi^x_u\colon\ \psi \in A, u \in U\}, \tag{1}$$

where φ^x_u is the assignment that assigns u to x and $\varphi(y)$ to every other variable y, and the sets

$$D_{xy} := \{\varphi \in U^X\colon\ \varphi(x) = \varphi(y)\}$$

are known as *diagonal hyperplanes* in U^X. Put $\|F\| := \{\|f\|\colon\ f \in F\}$, where F is the set of formulas of the language; the algebra

$$Cs(\boldsymbol{U}) := (\|F\|, \cup, \cap, -, C_x, D_{xy})_{x,y\in X}$$

is a version of cylindric set algebra [8, 9]. More precisely, according to Theorem 4.3.5 of [9], it is a regular and locally finite cylindric set algebra. We shall call it the *cylindric algebra of* $\boldsymbol{U}$. Two $\mathcal{L}^{\approx}$-structures have isomorphic cylindric algebras if and only if they are elementarily equivalent—this follows from Remark 4.3.68(7) in [9].

If the alphabet of $\mathcal{L}^{\approx}$ contains any operation symbols, then we may construct even a richer derived structure. Consider the term algebra $\boldsymbol{T} := (T, \ldots)$ and set

$$D_{st} := \{\varphi \in U^X\colon\ \tilde\varphi(s) = \tilde\varphi(t)\},$$

where $\tilde\varphi$ is the homomorphism $\boldsymbol{T} \to \boldsymbol{U}$ induced by φ. Now $\|s \approx t\| = D_{st}$. In terms of [2], the algebra

$$Cs_{\boldsymbol{T}}(\boldsymbol{U}) := (\|F\|, \cup, \cap, -, C_x, D_{st})_{x\in X,\ s,t\in T}$$

is a $\boldsymbol{T}$-*cylindric* set algebra, and the function $D\colon\ T\times T \to \mathcal{P}(U^X)$ defined by $D(s,t) := D_{st}$ is a $\boldsymbol{T}$-*diagonal* on it.

1.2 In the case when $\approx$ is the single predicate symbol in $\mathcal{L}^{\approx}$ and, correspondingly, $\boldsymbol{U}$ is merely an algebra, $Cs_{\boldsymbol{T}}(\boldsymbol{U})$ is generated by the "$\boldsymbol{T}$-diagonal planes" D_{st}. Hence, the $\boldsymbol{T}$-diagonal D carries then all information on $\boldsymbol{U}$ available in $Cs_{\boldsymbol{T}}(\boldsymbol{U})$, and we may concentrate on $\boldsymbol{T}$-diagonals rather than deal with whole $\boldsymbol{T}$-cylindric algebras. Actually, even more general situation was studied in [3], where $\boldsymbol{T}$ was an algebra free in some variety $\mathcal{K}$. It was shown there that every $\mathcal{K}$-algebra can be restored from its $\boldsymbol{T}$-diagonal and that homomorphisms between $\mathcal{K}$-algebras can also be characterized in terms of $\boldsymbol{T}$-diagonals. Moreover, the class of those functions $T^2 \to \mathcal{P}(U^X)$ that are $\boldsymbol{T}$-diagonals of algebras from $\mathcal{K}$ was given an axiomatic description. Axioms of $\boldsymbol{T}$-diagonals were used in [2] to introduce the concept of an abstract cylindric algebras with terms. For another approach to such algebras, involving substitutions along with diagonals, see [5].

Consequently, from the point of view of algebraic logic, algebras from $\mathcal{K}$ are well-presented by their $\boldsymbol{T}$-diagonals. Some relevant information on an algebra $\boldsymbol{U}$ may be read directly from D. For example, D_{st} may be considered as the set of solutions of the equation $s \approx t$ in $\boldsymbol{U}$, and the algebra satisfies this equation iff $D_{st} = U^X$. Given a relation $\theta \subset T \times T$, let D_θ be the intersection $\bigcap(D_{st}\colon\ (s,t) \in \theta)$. In the sense of universal algebraic geometry as it is developed in [12, 13], D_θ is essentially the algebraic variety in the space U^X described by the set of $\boldsymbol{T}$-equations θ.

1.3 Our aim in this paper is to extend the approach of [3] to multi-algebras. A minor trouble is that, for multi-algebras, there are several possible ways how to interpret the equality symbol $\approx$. Probably, the most popular one is the reading of the equation $s \approx t$ as 's and t have the same (sets of) values'. Such equations are discussed, for example, in [17]; seemingly, this interpretation of $\approx$ is suggested by tradition of complex, or powerset, algebras—see [7, 6]. On the other hand, the weak commutativity or weak distributivity laws for certain ring-like multi-algebras (see, e.g., [16]) can be written as equations, where $\approx$ expresses overlapping of values sets of both terms; then $s \approx t$ means 's and t have a common value'. A possible substituend for equality and overlapping is inclusion. In ordinary algebras all of these concepts reduce to identity of elements of the base set.

Following [14], instead of any of the above relations, we choose the relation of singular inclusion ε to be the basic one: the atomic formula $s \mathrel{\varepsilon} t$ is informally read as 'the term s has a single value, and it is also a value of t'. For partial algebras, the formula reduces to the so called existential equation $s \stackrel{\mathrm{e}}{=} t$ (see, e.g., [1]), while for ordinary algebras ε has the same meaning as $\approx$. Note that the identity relation on the base set is presented by formulas of type $s \mathrel{\varepsilon} t \wedge t \mathrel{\varepsilon} s$, and that overlapping, inclusion and equality relations for values sets of s and t are definable by formulas $\exists x(x \mathrel{\varepsilon} s \wedge x \mathrel{\varepsilon} t)$, $\forall x(x \mathrel{\varepsilon} s \to x \mathrel{\varepsilon} t)$ and $\forall x(x \mathrel{\varepsilon} s \leftrightarrow x \mathrel{\varepsilon} t)$, respectively (where x is free neither in s nor t). At last, $t \varepsilon t$ means that the term t is single-valued.

Since singular inclusion models some appropriate aspects of the set-theoretical 'element_of' relation, we consider singular inclusion as the most natural primitive for the language of multi-algebras. Inclusion has also been preferred to equality in some papers on logic of multi-algebras; see, e.g., [11, 10], where equality was shown to be a concept too weak for certain purposes. In fact, aside from inclusion, neither overlapping nor singular inclusion can be expressed in terms of equality.

2. Multi-algebras, valuations and resolvents

In this section we recall the notion of a multi-algebra and introduce the notion of an ε-resolvent of a multi-algebra, which is the ε-analogue of its $\boldsymbol{T}$-diagonal (the latter could also be termed its $\approx$-resolvent). Let Ω be some signature, and let now $\boldsymbol{T}$ be an Ω-algebra relatively free on an infinite set of variables X. We consider elements of T as "squeezed" terms.

2.1 Let us first recall some constructions and facts from [15] concerning algebras of squeezed terms. Given $Y \subset X$, we say that Y *supports* the term t if t belongs to the subalgebra of $\boldsymbol{T}$ generated by Y, and that t is *independent* of a variable x if t is supported by some Y not containing x. According to [15, Theorem 2.1], Y supports t iff $\sigma(t) = t$ for every endomorphism σ of $\boldsymbol{T}$ that coincides with the identity map on Y.

The set $\triangle t := \bigcap(Y\colon\ Y \text{ supports } t)$ of all those variables t depends on is always finite and supports t. If $\boldsymbol{T}$ is the absolutely free word algebra (as in Sect. 1), then $\triangle t$ consists just of the variables occurring in t. In any case,

$$\triangle \omega t_1 t_2 \dots t_m \subset \triangle t_1 \cup \triangle t_2 \cup \dots \cup \triangle t_m \tag{2}$$

and, if $[s/x]$ stands for the endomorphism of $\boldsymbol{T}$ that takes x into s and coincides with the identity map on $X \setminus \{x\}$, then

$$\triangle [s/x] t \subset \triangle s \cup (\triangle t \setminus \{x\}). \tag{3}$$

Note that t depends on x iff $x \in \triangle t$, and that $[s/x]t = t$ iff t is independent of x.

We further isolate, for each variable x, the subset L_x of terms *linear in* x. It is defined to be the smallest set containing x as well as all terms $\omega t_1 t_2 \dots t_m$ with $t_i \in L_x$ for some i and $x \notin \triangle t_j$ for $j \neq i$. An ordinary term is linear in x if and only if x occurs in it just once; this is the meaning in which the attribute 'linear' has been used, say, in [6].

2.2 An m-ary *multi-operation* on U is any function o of type $U^m \to \mathcal{P}(U)$. We shall identify singletons from $\mathcal{P}(U)$ with respective elements of U; therefore, any operation on U may be treated as a multi-operation. The *extension* of o is the operation $\bar{o}$ on $\mathcal{P}(U)$ defined by

$$\bar{o}(A_1, A_2, \ldots, A_m) := \bigcup (o(u_1, u_2, \ldots, u_m)\colon \\ u_1 \in A_1, u_2 \in A_2, \ldots, u_m \in A_m).$$

Definition 1. A *multi-algebra* is a system $\boldsymbol{U} := (U, \omega_{\boldsymbol{U}})_{\omega \in \Omega}$, where each $\omega_{\boldsymbol{U}}$ is a multi-operation on U whose arity is determined by ω. A mapping $\mu\colon T \to \mathcal{P}(U)$ is said to be a *valuation* in $\boldsymbol{U}$ if

$$\mu(x) \in U, \quad \mu(\omega t_1 t_2 \ldots t_m) = \bar{\omega}_{\boldsymbol{U}}(\mu(t_1), \mu(t_2), \ldots, \mu(t_m)).$$

for $x \in X$, $\omega \in \Omega$ and $t_1, t_2, \ldots, t_m \in T$.

Thus every valuation in $\boldsymbol{U}$ is an extension of some assignment from U^X, and may be regarded as a kind of multihomomorphism from $\boldsymbol{T}$ to $\boldsymbol{U}$. In particular, valuations in an ordinary algebra $\boldsymbol{U}$ are just homomorphisms from $\boldsymbol{T}$ to $\boldsymbol{U}$. Let $Val(\boldsymbol{U})$ stand for the set of all valuations in $\boldsymbol{U}$. Note that $Val(\boldsymbol{T}) = End(\boldsymbol{T})$.

A multi-algebra $\boldsymbol{U}$ is said to be $\boldsymbol{T}$*-shaped* if $Val(\boldsymbol{U})$ is maximally rich, i.e. if every assignment φ can be extended to a valuation $\tilde{\varphi}$ (necessarily unique) in U. Then elements of $\tilde{\varphi}(t)$ are thought of as *values* of the term t on φ. According to our convention on singletons, a term t has a single value on φ iff $\tilde{\varphi}(t) \in U$. We denote by $\mathcal{V}(\boldsymbol{T})$ the class of all $\boldsymbol{T}$-shaped multi-algebras. Clearly, $\mathcal{V}(\boldsymbol{T})$ includes the variety of ordinary algebras generated by $\boldsymbol{T}$, and contains all multi-algebras when $\boldsymbol{T}$ is absolutely free. Furthermore, for $\boldsymbol{U} \in \mathcal{V}(\boldsymbol{T})$,

$$\varphi|\triangle t = \psi|\triangle t \Rightarrow \tilde{\varphi}(t) = \tilde{\psi}(t) \tag{4}$$

and, if t is linear in x,

$$\tilde{\varphi}([s/x]t) = \{v\colon \exists u(v \in \widetilde{\varphi^x_u}(t) \text{ and } u \in \tilde{\varphi}(s))\}. \tag{5}$$

The routine proof of (5) is by induction on L_x, using (2) and (3).

It is easily seen that every $\boldsymbol{T}$-shaped multi-algebra is completely determined by its valuations. Indeed, assume that $\boldsymbol{U}$ and $\boldsymbol{U}'$ are two different multi-algebras with a common carrier U. Then there is an operation symbol $\omega \in \Omega$ such that $\omega_{\boldsymbol{U}}(u_1, u_2, \ldots, u_m) \neq \omega_{\boldsymbol{U}'}(u_1, u_2, \ldots, u_m)$ for some $u_1, u_2, \ldots, u_m \in U$. For sake of definiteness, suppose that $u \in \omega_{\boldsymbol{U}}(u_1, u_2, \ldots, u_m)$ and $u \notin \omega_{\boldsymbol{U}'}(u_1, u_2, \ldots, u_m)$. Furthermore, choose distinct variables $x_1, x_2, \ldots, x_m$ and a valuation μ such that $\mu(x_i) = u_i$ for all i. Now, if t is the term $\omega x_1 x_2 \cdots x_m$, then u is a value of t on μ in $\boldsymbol{U}$, but not in $\boldsymbol{U}'$. So, the sets of valuations are also distinct.

In what follows, we shall consider only $\boldsymbol{T}$-shaped multi-algebras.

2.3 Let us introduce the notion of a resolvent—the multi-algebra equivalent of a $\boldsymbol{T}$-diagonal of an ordinary algebra (see Introduction). Recall that the formula $s\ \varepsilon\ t$ can also be considered as a kind of equation, and then the resolvent provides us with solutions of these "ε-equations"; this motivates the suggested term.

Definition 2. The *ε-resolvent*, or just *resolvent of a multi-algebra* $\boldsymbol{U}$ is the function $Res(\boldsymbol{U})\colon\ T \times T \to \mathcal{P}(U^X)$ defined as follows:

$$Res(\boldsymbol{U})(s,t) := \{\varphi \in U^X\colon\ \tilde{\varphi}(s) \in \tilde{\varphi}(t)\}. \tag{6}$$

Therefore, $\|s\ \varepsilon\ t\| = Res(\boldsymbol{U})(s,t)$. Note that the set algebra

$$Cs_{\boldsymbol{T}}(\boldsymbol{U}) := (\|F\|, \cup, \cap, -, C_x, R_{st})_{x\in X,\ s,t\in T},$$

where R_{st} stands for $Res(\boldsymbol{U})(s,t)$, is an ordinary algebra generated by these elements.

A multi-algebra is completely determined even by a "half" of its resolvent, the first argument being a variable which the second one does not depend on. Namely, we can restore the operation $\omega_{\boldsymbol{U}}$ of $\boldsymbol{U}$ corresponding to an operation symbol $\omega \in \Omega$ as follows:

$$v \in \omega_{\boldsymbol{U}}(u_1, u_2, \ldots, u_m) \Leftrightarrow \varphi \in R_{yt},$$

where t is $\omega x_1 x_2, \ldots, x_m$ and $y \notin \triangle t$ for distinct variables $x_1, x_2, \ldots, x_m, y$, while φ is selected so that $\varphi(y) = v$ and $\varphi(x_i) = u_i$.

Thus, different algebras from $\mathcal{V}(\boldsymbol{T})$ have different resolvents.

By a *support* of a set $A \subset U^X$ we shall mean any subset $Y \subset X$ such that, for all $\varphi, \psi \in U^X$,

$$\varphi \in A,\ \varphi|Y = \psi|Y\ \Rightarrow\ \psi \in A.$$

This concept comes from the theory of polyadic algebras. By analogy with standard cylindric algebras (see [8, 9]), the set algebra $Cs_{\boldsymbol{T}}$ could be called *regular* if every its element A is regular in the sense that the subset $\{x \in X\colon C_x(A) \neq A\}$ is a support of A. However, apart from the note just after Theorem 2 below, we shall not concern with regularity property in this paper.

Theorem 1. *If a function* $R\colon\ T \times T \to \mathcal{P}(U^X)$ *is a resolvent of a* $\boldsymbol{T}$*-shaped multi-algebra, then it satisfies the conditions*

(R0): $R(x,y) = D_{xy}$,

(R1a): $R(r,s) \cap R(s,t) \subset R(s,r)$,

(R1b): $R(r,s) \cap R(s,t) \subset R(r,t)$,

(R2): $R(s,[r/x]t) = C_x(R(x,r) \cap R(s,t))$ *if* $t \in L_x$ *and* $x \notin \triangle r \cup \triangle s$,

(R3): *every* $R(s,t)$ *has a finite support.*

Proof. (R0) and (R1b) are obvious, while (R1a) is true because the left hand side assures that the value set of s is a singleton. We shall check only (R2) and (R3) here. By (6), (5), (4), again (6), and (1),

$$\begin{aligned}\varphi \in R(s,[r/x]t) &\Leftrightarrow \tilde{\varphi}(s) \in \tilde{\varphi}([r/x]t)\\ &\Leftrightarrow \exists u(\tilde{\varphi}(s) \in (\widetilde{\varphi^x_u})(t) \text{ and } u \in \tilde{\varphi}(r))\\ &\Leftrightarrow \exists u(\varphi^x_u(s) \in (\widetilde{\varphi^x_u})(t) \text{ and } u \in \tilde{\varphi}(r))\\ &\Leftrightarrow \exists u(\varphi^x_u \in R(s,t) \text{ and } \varphi^x_u \in R(x,r))\\ &\Leftrightarrow \varphi \in C_x(R(x,r) \cap R(s,t)),\end{aligned}$$

i.e. (R2) holds. By (2) and (4), the finite set $\triangle s \cup \triangle t$ is a support of $R(s,t)$, and (R3) also holds. □

Note that these conditions are, in fact, properties of singular inclusion written algebraically. Thus, (R1b) fixes transitivity of ε, while (R2) says that $s\varepsilon[r/x]t$ holds iff $x\varepsilon r$ and $s\varepsilon t$ hold for some value of x. We shall need only the following two particular cases of (R2):

$$R(s,r) = C_x(R(s,x) \cap R(x,r)) \tag{7}$$

with $x \notin \triangle s \cup \triangle t$, and

$$R(y,[r/x]t) = C_x(R(x,r) \cap R(y,t)) \tag{8}$$

with $t \in L_x$ and $x \neq y \notin \triangle s$, $y \notin \triangle t$. (In fact, (R2) is a consequence of them.)

Definition 3. A ***T****-resolvent on a set* U is any function $R\colon\ T \times T \to \mathcal{P}(U^X)$ satisfying the conditions (R0)–(R2). The resolvent is said to be *finitary* iff it satisfies also (R3).

According to the preceding theorem, the resolvent of any multi-algebra is a finitary resolvent in this abstract sense on its base set. The following representation theorem, which is the main result of the paper, states the converse.

Theorem 2. *Every finitary* $\boldsymbol{T}$*-resolvent is a resolvent of some multi-algebra from* $\mathcal{V}(\boldsymbol{T})$.

This theorem is a close analogue of Theorem 3 in [3] and Theorem 4.3 in [2] on superdiagonals of $\boldsymbol{T}$-cylindric algebras, with the exception that in the latter one the superdiagonal was required to be regular rather than just finitary. This difference is not essential: as all sets $\triangle t$ are finite, both conditions turn out to be equivalent in our context. The theorem will be proved in the next section.

We already observed just after Definition 2 that different algebras with the same base set still have different resolvents. So we come to a corollary which shows that, for algebraic logic, every multi-algebra $\boldsymbol{U}$ is adequately presented by some resolvent, and conversely.

Theorem 3. *The transformation Res:* $\boldsymbol{U} \mapsto Res(\boldsymbol{U})$ *provides a one-to-one correspondence between* $\boldsymbol{T}$*-shaped multi-algebras with the base set* U *and finitary* $\boldsymbol{T}$*-resolvents on* U*.*

We remind that the set algebra $Cs_{\boldsymbol{T}}(\boldsymbol{U})$, being generated by the resolvent of $\boldsymbol{U}$, is completely determined by it. Hence, Theorem 2 could serve as a basis for a representation of an appropriate class of "ε-cylindric" algebras (cf. a similar situation with $\boldsymbol{T}$-diagonals and $\boldsymbol{T}$-cylindric algebras in Sect. 4 of [2]) and, further, for an algebraic proof of completeness of a logic with multivalued terms (see [14] for such a logic).

3. Proof of Theorem 2

The proof consists of a sequence of technical lemmas.

3.1 First we derive some additional properties of $\boldsymbol{T}$-resolvents.

Lemma 4. *Suppose that* R *is a* $\boldsymbol{T}$*-resolvent on* U*. If a term* t *does not depend on the distinct variables* y *and* z*, then, for all assignments* φ *and elements* $u \in U$
(a) $\varphi \in R(y,t)$ *if and only if* $\varphi^z_u \in R(y,t)$,
(b) $\varphi^y_u \in R(y,t)$ *if and only if* $\varphi^z_u \in R(z,t)$.
If, furthermore, assignments φ *and* ψ *agree on* $\triangle t$*, and* $R(y,t)$ *has a finite support, then*
(c) $\varphi^y_u \in R(y,t)$ *if and only if* $\psi^y_u \in R(y,t)$
for all $u \in U$*.*

Proof. Assume that t, y and z are as indicated. We first note that, by (7),

$$C_z(R(y,t)) = C_z(C_z(R(y,z) \cap R(z,t))) = C_z(R(y,z) \cap R(z,t)) = R(y,t). \tag{9}$$

Now, if $\varphi \in R(y,t)$, then $\varphi^z_u \in C_z R(y,t) = R(y,t)$, but if $\varphi^z_u \in R(y,t)$, then $\varphi \in C_z R(z,t) = R(y,t)$. Therefore, (a) holds.

Once again referring to (7), and using (1), (R0), (a), we arrive at (b):

$$\begin{aligned}\varphi^y_u \in R(y,t) &\Leftrightarrow \varphi^y_u \in C_z(R(y,z)\cap R(z,t))\\ &\Leftrightarrow \exists v(\varphi^{yz}_{uv} \in R(y,z) \text{ and } \varphi^{yz}_{uv} \in R(z,t))\\ &\Leftrightarrow \exists v(u=v \text{ and } \varphi^z_v \in C_y(R(z,t)))\\ &\Leftrightarrow \varphi^z_u \in C_y(R(z,t)) = R(z,t).\end{aligned}$$

To prove (c), assume that $\varphi|\triangle t = \psi|\triangle t$. Then also $\varphi^y_u|\{y\}\cup\triangle t = \psi^y_u|\{y\}\cup\triangle t$ for any $u\in U$. If Y is a finite support of $R(y,t)$, then we do not loss generality assuming that φ and ψ agree everywhere outside Y. Hence, φ^y_u and ψ^y_u may differ only on the set $\{x_1,x_2,\dots,x_n\} := Y-(\triangle t\cup\{y\})$; we are only interested in the case $n>0$. Now let $v_i := \psi(x_i)$ for all i; then

$$\varphi^y_u \in R(y,t) \;\Leftrightarrow\; \varphi^{yx_1x_2\cdots x_n}_{uv_1v_2\cdots v_n} \in R(y,t) \;\Leftrightarrow\; \psi^y_u \in R(y,t)$$

by multiple use of (a). □

Corollary 5. *Let R be a $\boldsymbol{T}$-resolvent on U, and let $\varphi^*\colon\ T\to\mathcal{P}(U)$ be the extension of an assignment φ in U defined by the condition*

$$\varphi^*(t) := \{u\in U\colon\ \varphi^y_u \in R(y,t)\}, \tag{10}$$

where $y\notin\triangle t$. Then φ^* does not depend on the choice of y, and, if $z\notin\triangle t$,

$$R(z,t) = \{\varphi\in U^X\colon\ \varphi(z)\in\varphi^*(t)\}. \tag{11}$$

Moreover, if R is finitary, then

$$\varphi|\triangle t = \psi|\triangle t \Rightarrow \varphi^*(t) = \psi^*(t). \tag{12}$$

Proof. By (R0), $\varphi^*(x) = \varphi x$; so the function φ^* is indeed an extension of φ. The fact that φ^* does not depend on the choice of y immediately follows from Lemma 4(b), and (12) is then another form of Lemma 4(c). By (10) and Lemma 4(b),

$$\varphi(z)\in\varphi^*(t) \Leftrightarrow \varphi^y_{\varphi(z)} \in R(y,t) \Leftrightarrow \varphi^z_{\varphi(z)} \in R(z,t) \Leftrightarrow \varphi\in R(z,t);$$

so (11) also holds. □

Lemma 6. *If R is a finitary $\boldsymbol{T}$-resolvent on U, then*

$$R(s,t) = \{\varphi\in U^X\colon\ \varphi^*(s)\in\varphi^*(t)\}. \tag{13}$$

Proof. We first prove that

$$z \notin \triangle s,\ \psi \in R(s,z) \ \Rightarrow\ \psi^*(s) = \psi(z). \tag{14}$$

Suppose that $z \notin \triangle s$. If $\psi \in R(s,z)$, then $\psi \in R(z,s)$ by (R0) and (R1a), for (11) implies that $\psi \in R(z,s)$. Consequently, $\psi(z) \in \psi^*(s)$ by (11). Let, furthermore, u be any element from $\psi^*(s)$. Choose one more variable $y \notin \triangle s$; in view of (12), we may assume that $\psi(y) = u$. Then $\psi \in R(y,s)$ according to (11); so, by (R1b), $\psi \in R(y,z)$, wherefrom $u = \psi(y) = \psi(z)$—see (R0). So, $\psi^*(s)$ is a singletone and must coincide with $\psi(z)$. Now (13) follows by (7) and (1), (14) and (10), and (12):

$$\begin{aligned}
\varphi \in R(s,t) \ &\Leftrightarrow\ \exists u(\varphi^z_u \in R(s,z) \text{ and } \varphi^z_u \in R(z,t)) \\
&\Leftrightarrow\ \exists u((\varphi^z_u)^*(s) = u \text{ and } u \in \varphi^*(t)) \\
&\Leftrightarrow\ \exists u(\varphi^*(s) = u \text{ and } u \in \varphi^*(t)) \\
&\Leftrightarrow\ \varphi^*(s) \in \varphi^*(t),
\end{aligned}$$

as needed. □

In view of this lemma, it remains to show that there is a $\boldsymbol{T}$-shaped multi-algebra such that the set of all extensions φ^* turns out to be its set of valuations. This will be done in the next subsection. We need one more simple lemma.

Lemma 7. *Suppose that t is linear in x and that s does not depend on x. Then*

$$\varphi^*([s/x]t) = \bigcup((\varphi^x_v)^*(t)\colon\ v \in \varphi^*(s)). \tag{15}$$

Proof. By (10), (8) and (1), Lemma 4(a), and (11),

$$\begin{aligned}
u \in \varphi^*([s/x]t) \ &\Leftrightarrow\ \varphi^y_u \in R(y,[s/x]t) \\
&\Leftrightarrow\ \exists v(\varphi^{yx}_{uv} \in R(x,s) \text{ and } \varphi^{yx}_{uv} \in R(y,t)) \\
&\Leftrightarrow\ \exists v(\varphi^x_v \in R(x,s) \text{ and } \varphi^{yx}_{uv} \in R(y,t)) \\
&\Leftrightarrow\ \exists v(v \in \varphi^*(s) \text{ and } u \in (\varphi^x_v)^*(t)) \\
&\Leftrightarrow\ u \in \bigcup((\varphi^x_v)^*(t)\colon\ v \in \varphi^*(s)),
\end{aligned}$$

where y is appropriately chosen. □

Using the lemma repeatedly, we now obtain the following equality for every assignment φ, every term $t := \omega t_1 t_2 \cdots t_m$ and mutually distinct variables $x_1, x_2, \ldots, x_m$:

$$\varphi^*(t) = \bigcup(\psi^*(\omega x_1 x_2 \cdots x_m)\colon\ \psi \in U^X,\ \psi(x_i) \in \varphi^*(t_i) \quad (i = 1, 2, \ldots, m)). \tag{16}$$

3.2 We now claim that, for any m-ary $\omega \in \Omega$, the operation ω^R on U defined by

$$\omega^R(u_1, u_2, \ldots, u_m) := \varphi^*(t),$$

where $t := \omega x_1 x_2 \ldots x_m$ (for distinct variables x_i) and φ is an assignment in U such that $u_i = \varphi(x_i)$ for all i, does not depend on the choice of $x_1, x_2, \ldots, x_m$ and φ. Indeed, suppose that $t' = \omega y_1 y_2 \ldots y_m$ and that ψ is an assignment such that $\psi(y_i) = u_i$ for all i. If σ is any endomorphism of $\boldsymbol{T}$ that takes every x_i into y_i, then $\psi^*\sigma$ is an assignment that coincides with φ on $\{x_1, x_2, \ldots, x_m\}$. Since the later set supports t (see (2)), we may apply (12):

$$\psi^*(t') = \psi^*(\omega(\sigma x_1)(\sigma x_2) \cdots (\sigma x_m)) = \psi^*(\sigma(t)) = \varphi^*(t).$$

Note that the definition of ω^R may be rewritten in the form

$$\omega^R(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m)) = \varphi^*(t), \tag{17}$$

where now φ is arbitrary.

This way the set U can be turned into Ω-multi-algebra $(U, \omega^R)_{\omega\in\Omega}$, which we denote by $Alg(R)$. Our next claim is that every φ^* is the valuation in $Alg(R)$ induced by the assignment φ, i.e. that φ^* coincides with $\tilde{\varphi}$.

Given a term $t := \omega t_1 t_2 \ldots t_m$, select mutually distinct variables $x_1, x_2, \ldots, x_m$ outside $\triangle t$. Then, by (16) and (17) (with ψ in the role of φ) and the definition of an extended operation (viz., $\bar{\omega}^R$),

$$\begin{aligned}
\varphi^*(t) &= \bigcup((\omega x_1 x_m \cdots x_m)\colon \ \psi \in U^X,\ \psi(x_i) \in \varphi^*(t_i)) \\
&= \bigcup(\omega^R(\psi(x_1), \psi(x_2), \ldots, \psi(x_m))\colon \ \psi \in U^X,\ \psi(x_i) \in \varphi^*(t_i)) \\
&= \bar{\omega}^R(\mu(t_1), \mu(t_2), \ldots \mu(t_m)),
\end{aligned}$$

as needed.

It now follows that $Alg(R) \in \mathcal{V}(\boldsymbol{T})$. Thus, the proof of Theorem 2 is completed. Note that the transformation $Alg\colon\ R \mapsto Alg(R)$ is converse to Res mentioned in Theorem 3.

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