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Commutative dimonoids

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Abstract. We present some congruence on the dimonoid with a commutative operation and use it to obtain a decomposition of a commutative dimonoid.

1. Introduction

Jean-Louis Loday introduced the notion of dimonoid [1]. Dimonoid is a set equipped with two associative operations satisfying some axioms (see, below). If the operations of a dimonoid coincide, then the dimonoid becomes a semigroup. The first result about dimonoids is the description of free dimonoid generated by a given set [1]. Other notion which belongs to the theory of semigroups is a notion of band of semigroups. The decompositions of semigroups are effectively described according to the construction of a band of semigroups. Tamura and Kimura [2] proved, in particular, that every commutative semigroup is a semilattice of archimedean semigroups. The decompositions of some semigroups were studied in the terms of bands in the papers [3-4].

In this work we introduce the notion of diband of dimonoids to describe the decompositions of dimonoids. In section 2 we give the necessary definitions, some properties of dimonoids with one and two commutative operations (Lemmas 1-4) and one example of a dimonoid. In section 3 we first present the least idempotent congruence on an arbitrary dimonoid with a commutative operation (Theorem 1). The main result of this paper is a generalization of the theorem by Tamura and Kimura (Theorem

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2): every commutative dimonoid is a semilattice of archimedean subdimonoids. In section 4 we construct different examples of dimonoids.

2. Preliminaries

A set D equipped with two associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$, is called a dimonoid.

A map f from dimonoid D_1 to dimonoid D_2 is homomorphism, if $(x \prec y)f = xf \prec yf$, $(x \succ y)f = xf \succ yf$ for all $x, y \in D_1$.

Define the notion of diband of dimonoids.

A dimonoid $(D, \prec, \succ)$ will be called idempotent dimonoid or diband, if $x \prec x = x = x \succ x$ for all $x \in D$.

If $\varphi : S \to T$ is a homomorphism of dimonoids, then corresponding congruence on S will be denoted by Δ_φ.

Let S be an arbitrary dimonoid, J be some idempotent dimonoid. If there exists homomorphism

$$\alpha : S \to J : x \mapsto x\alpha,$$

then every class of congruence Δ_α is a subdimonoid of the dimonoid S, and dimonoid S itself is a union of such dimonoids S_ξ, $\xi \in J$ that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \,|(x; t) \in \Delta_\alpha\},$$

$$S_\xi \prec S_\varepsilon \subseteq S_{\xi \prec \varepsilon}, \quad S_\xi \succ S_\varepsilon \subseteq S_{\xi \succ \varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \emptyset.$$

In this case we say that S is decomposable into a diband of subdimonoids (or S is a diband J of subdimonoids S_ξ $(\xi \in J)$). If J is an idempotent semigroup (band), then we say that S is a band J of subdimonoids S_ξ $(\xi \in J)$.

As usual N denotes the set of positive integers.

Let $(D, \prec, \succ)$ be a dimonoid, $a \in D$, $n \in N$. Denote by a^n (respectively, $n\, a$) the degree n of an element a concerning the operation $\prec$ (respectively, $\succ$).

Lemma 1. *Let* $(D, \prec, \succ)$ *be a dimonoid with a commutative operation* $\prec$. *For all* $b,\ c \in D$, $m \in N,\ m > 1$,

$$(b \prec c)^m = b^m \succ c^m = (b \succ c)^m.$$

Proof. For any $b,\ c \in D$ we have

$$(b \prec c)^m = b^m \prec c^m = b^m \prec c^{m-1} \prec c =$$

$$= (c \prec b^m) \prec c^{m-1} = c \prec (b^m \succ c^{m-1}) =$$

$$= (b^m \succ c^{m-1}) \prec c = b^m \succ (c^{m-1} \prec c) = b^m \succ c^m$$

according to the commutativity of $\prec$ and axioms of dimonoid.

We prove that $b^m \succ c^m = (b \succ c)^m$ using an induction on m. For $m = 2$ we have

$$b^2 \succ c^2 = (b \prec b) \succ (c \prec c) = b \succ (b \succ (c \prec c)) =$$

$$= b \succ ((b \succ c) \prec c) = b \succ (c \prec (b \succ c)) =$$

$$= (b \succ c) \prec (b \succ c) = (b \succ c)^2$$

according to the commutativity of $\prec$ and axioms of dimonoid.

Let $b^k \succ c^k = (b \succ c)^k$ for $m = k$. Then for $m = k + 1$ we obtain

$$(b \succ c)^{k+1} = (b \succ c)^k \prec (b \succ c) =$$

$$= (b^k \succ c^k) \prec (b \succ c) = b^k \succ (c^k \prec (b \succ c)) =$$

$$= b^k \succ ((b \succ c) \prec c^k) = b^k \succ (b \succ (c \prec c^k)) =$$

$$= b^k \succ (b \succ c^{k+1}) = (b^k \prec b) \succ c^{k+1} = b^{k+1} \succ c^{k+1}$$

according to the supposition, the commutativity of $\prec$ and axioms of dimonoid.

Thus, $b^m \succ c^m = (b \succ c)^m$ for $m > 1$. □

A dimonoid $(D, \prec, \succ)$ will be called commutative, if its both operations are commutative.

Lemma 2. *In commutative dimonoid* $(D, \prec, \succ)$ *the equalities*

$$(x \prec y) \prec z = x \prec (y \succ z) =$$

$$= (x \succ y) \prec z = x \succ (y \prec z) =$$

$$= (x \prec y) \succ z = x \succ (y \succ z)$$

hold for all $x, y, z \in D$.

Proof. We have for all $x, y, z \in D$

$$(x \dashv y) \vdash z = z \vdash (x \dashv y) =$$

$$= (z \vdash x) \dashv y = (x \vdash z) \dashv y =$$

$$= x \vdash (z \dashv y) = x \vdash (y \dashv z) = (x \vdash y) \dashv z,$$

$$(x \vdash y) \dashv z = z \dashv (x \vdash y) = (z \dashv x) \dashv y =$$

$$= (x \dashv y) \dashv z = x \dashv (y \vdash z)$$

according to the commutativity and axioms of dimonoid. Hence,

$$(x \dashv y) \vdash z = x \dashv (y \vdash z).$$

□

From Lemma 2 it follows that the operations $\dashv$ and $\vdash$ of a commutative dimonoid $(D, \dashv, \vdash)$ is indistinguishable for three and more multipliers and the product of these elements doesn't depend on the parenthesizing.

Lemma 3. *The operations of any commutative dimonoid $(D, \dashv, \vdash)$ with an idempotent operation $\dashv$ coincide.*

Proof. We have for all $x, y, z \in D$

$$(x \dashv y) \vdash z = (x \dashv y) \dashv z$$

according to Lemma 2. Hence, setting $x = y$, we obtain $x \dashv z = x \vdash z$ for all $x,\ z \in D$. □

Let $(D, \dashv, \vdash)$ be a dimonoid, $n \in N$. Recall that we denote by $n\,a$ the degree n of an element $a \in D$ concerning the operation $\vdash$.

Lemma 4. *Let $(D, \dashv, \vdash)$ be a dimonoid with a commutative operation $\vdash$. For all $b \in D$, $m \in N$,*

$$2\,b^m = (2m)\,b.$$

Proof. We use an induction on m. For $m = 1$, obviously, the equality is correct. For $m = 2$ we have

$$2\,b^2 = (b \dashv b) \vdash (b \dashv b) = ((b \dashv b) \vdash b) \dashv b =$$

$$= (b \vdash (b \vdash b)) \dashv b = b \vdash ((b \vdash b) \dashv b) =$$

$$= ((b \vdash b) \dashv b) \vdash b = (b \vdash b) \vdash (b \vdash b) = 2b \vdash 2b = 4\,b$$

according to the axioms of dimonoid and the commutativity of $\succ$.

Let $2\, b^k = (2k)\, b$ for $m = k$. Then for $m = k+1$ we obtain

$$\begin{aligned}
(2(k+1))\, b &= (2k+2)b = (2k)\, b \succ 2b = \\
&= 2\, b^k \succ 2b = (\, b^k \succ b) \succ (\, b^k \succ b) = \\
&= b^k \succ (b \succ (\, b^k \succ b)) = (\, b^k \prec b) \succ (\, b^k \succ b) = \\
&= b^{k+1} \succ (\, b^k \succ b) = \, b^k \succ (b \succ \, b^{k+1}) = \\
&= \, (b^k \prec b) \succ \, b^{k+1} = b^{k+1} \succ b^{k+1} = 2b^{k+1}
\end{aligned}$$

according to the supposition, axioms of dimonoid and the commutativity of $\succ$. Thus, $2\, b^m = (2m)\, b$. $\square$

Now we give an example of dimonoid.

Let X and Y be arbitrary disjoint sets, $0 \in X$, and let

$$\varphi : Y \times Y \to X, \psi : Y \times Y \to X$$

be arbitrary different maps. Define the operations $\prec$, $\succ$ on $X \bigcup Y$ by

$$x \prec y = \begin{cases} (x,y)\varphi, \ x, y \in Y, \\ \quad 0 \quad \text{otherwise}, \end{cases}$$

$$x \succ y = \begin{cases} (x,y)\psi, \ x, y \in Y, \\ \quad 0 \quad \text{otherwise} \end{cases}$$

for all $x, y \in X \bigcup Y$. It is immediate to check that $(X \bigcup Y, \prec, \ \succ)$ is a dimonoid.

3. Main results

In this section we describe the least idempotent congruence on the dimonoid with a commutative operation $\prec$ and show that every commutative dimonoid $(D, \prec, \ \succ)$ is a semilattice Y of archimedean subdimonoids $D_i, \ i \in Y$.

We will call a commutative idempotent semigroup as a semilattice.

If ρ is a congruence on the dimonoid D such that ${}^{D}/_{\rho}$ is an idempotent dimonoid, then we say that ρ is an idempotent congruence.

Let $(D, \prec, \ \succ)$ be a dimonoid with a commutative operation $\prec$ (respectively, $\succ$), $a, b \in D$. We say that a $\prec$-divide b (respectively, a $\succ$-divide b) and write $a_{\prec}|b$ (respectively, $a_{\succ}|b$), if there exists such element $x \in (D, \prec)$ (respectively, $x \in (D, \succ)$) with an identity that $a \prec x = b$ (respectively, $a \succ x = b$).

Define a relation η on the dimonoid $(D, \prec, \ \succ)$ with a commutative operation $\prec$ by

$a\eta b$ if and only if there exist positive integers
$m,\ n,\ m \neq 1, n \neq 1$ such that $a_{\prec}|b^m,\ b_{\prec}|a^n$.

Theorem 1. *The relation η on the dimonoid $(D, \prec,\ \succ)$ with a commutative operation $\prec$ is the least idempotent congruence, and $(D, \prec,\ \succ)/_{\eta}$ is a commutative idempotent dimonoid which is a semilattice.*

Proof. The fact that the relation η is a congruence on the semigroup $(D, \prec)$ has been proved by Tamura and Kimura [2]. Show that η is compatible concerning the operation $\succ$.

Let $a\eta b,\ a, b, c \in D$. Then $a \prec c\eta b \prec c$. It means that there exist $x, y \in (D, \prec),\ m, n \in N\backslash\{1\}$, for which

$$(a \prec c) \prec x = (b \prec c)^m, \tag{1}$$

$$(b \prec c) \prec y = (a \prec c)^n. \tag{2}$$

Considering both parts of the equality (1), we obtain

$$(a \prec c) \prec x = (x \prec a) \prec c =$$

$$= x \prec (a \succ c) = (a \succ c) \prec x$$

according to the commutativity of $\prec$ and an axiom of dimonoid, and $(b \prec c)^m = (b \succ c)^m$ by Lemma 1. Hence, $(a \succ c) \prec x = (b \succ c)^m$. Thus, $a \succ c_{\prec}|(b \succ c)^m$. Analogously, from the equality (2) we obtain that $b \succ c_{\prec}|(a \succ c)^n$. Together with preceding it means that $a \succ c\eta b \succ c$.

Dually, a left compatibility of the relation η concerning the operation $\succ$ can be proved. So, η is a congruence on $(D, \prec,\ \succ)$.

Obviously, $a\eta a \prec a$. Then $a \prec x_1 = (a \prec a)^{m_1}$, $(a \prec a) \prec x_2 = a^{m_2}$ for some $x_1, x_2 \in (D, \prec)$ and $m_1,\ m_2 \in N\backslash\{1\}$. From two last equalities it follows that

$$a \prec x_1 = (a \prec a)^{m_1} = (a \succ a)^{m_1},$$

$$(a \prec a) \prec x_2 = (x_2 \prec a) \prec a =$$

$$= x_2 \prec (a \succ a) = (a \succ a) \prec x_2 = a^{m_2},$$

whence $a\eta a \succ a$ and so η is an idempotent. From the commutativity of the operation $\prec$ it follows that $a \prec b\eta\, b \prec a$. Hence, $(a \prec b) \prec t_1 = (b \prec a)^{n_1}$, $(b \prec a) \prec t_2 = (a \prec b)^{n_2}$ for some $t_1, t_2 \in (D, \prec)$, $n_1, n_2 \in N\backslash\{1\}$ and

$$(a \prec b) \prec t_1 = (t_1 \prec a) \prec b = t_1 \prec (a \succ b) =$$

$$= (a \succ b) \prec t_1 = (b \prec a)^{n_1} = (b \succ a)^{n_1},$$

$$(b \prec a) \prec t_2 = (t_2 \prec b) \prec a = t_2 \prec (b \succ a) =$$
$$= (b \succ a) \prec t_2 = (a \prec b)^{n_2} = (a \succ b)^{n_2}$$

according to the commutativity of the operation $\prec$, an axiom of dimonoid and Lemma 1. That is, $a \succ b\eta\, b \succ a$. So, $(D, \prec, \ \succ)/_{\eta}$ is a commutative idempotent dimonoid. From Lemma 3 it follows that $(D, \prec, \ \succ)/_{\eta}$ is a semilattice.

The proof will be completed, if we show that η is contained in every idempotent congruence ρ on $(D, \prec, \ \succ)$.

Let $a\eta b$, $a, b \in D$. Then $a \prec z = b^k$, $b \prec d = a^l$ for some $z, d \in (D, \prec)$ and $k, l \in N\backslash\{1\}$. Since $a\rho\, a \prec a$, $b\rho\, b \prec b$ by the idempotentity of ρ, then we have $a\,\rho\, b \prec d$, $b\rho\, a \prec z$. So,

$$a\rho\ b \prec d\rho\, b \prec b \prec d\rho\, b \prec a^l \rho\, b \prec a\rho\, b^k \prec a\rho\, a \prec z \prec a\rho\, a \prec z\ \rho\, b.$$

Thus, $a\rho b$ and $\eta \subseteq \rho$. □

We say that a commutative dimonoid is archimedean, if its both semigroups are archimedean.

Theorem 2. *A semigroup $(D, \prec)$ of a commutative dimonoid $D = (D, \prec, \ \succ)$ is archimedean (respectively, regular) if and only if a semigroup $(D, \succ)$ of the commutative dimonoid D is archimedean (respectively, regular). Every commutative dimonoid D is a semilattice Y of archimedean subdimonoids D_i, $i \in Y$.*

Proof. Let $(D, \prec)$ be an archimedean semigroup, $a, b \in D$. Then $a \prec x = b^m$, $b \prec y = a^n$ for some $x, y \in D$, $m, n \in N$. Multiply both parts of the equality $a \prec x = b^m$ by b^m and the equality $b \prec y = a^n$ by a^n:

$$(a \prec x) \succ b^m = a \succ (x \prec b^m) =$$
$$= a \succ (b^m \prec x) = b^m \succ b^m = (2m)b,$$
$$(b \prec y) \succ a^n = b \succ (y \prec a^n) =$$
$$= b \succ (a^n \prec y) = a^n \succ a^n = (2n)a$$

according to Lemmas 2 and 4 and the commutativity of $\prec$. So, $a_{\succ}|(2m)b$, $b_{\succ}|(2n)a$. That is, $(D, \succ)$ is an archimedean semigroup.

Conversely, let $a \succ x = mb$, $b \succ y = na$ for some $x, y \in D$, m, $n \in N$. Take $k, p \in N\backslash\{1\}$ such that $k + m$, $p + n$ is even and multiply both parts of the equality $a \succ x = mb$ by $k\,b$ and the equality $b \succ y = na$ by $p\,a$:

$$kb \succ (a \succ x) = (kb \prec a) \succ x = x \succ (kb \prec a) =$$
$$= (x \succ kb) \prec a = a \prec (x \succ kb)$$

according to the axioms and the commutativity of dimonoid,

$$kb \succ mb = (k+m)b = 2b^{\frac{k+m}{2}} =$$
$$= b^{\frac{k+m}{2}} \succ b^{\frac{k+m}{2}} = (b \prec b)^{\frac{k+m}{2}} =$$
$$= (b^2)^{\frac{k+m}{2}} = b^{k+m}$$

by Lemmas 1 and 4. Thus, $a_{\prec}|b^{k+m}$. Analogously, $b \prec (y \succ pb) = a^{p+n}$, that is, $b_{\prec}|a^{p+n}$.

The corresponding statement about regularity of the semigroup of a dimonoid follows immediately from Lemma 2.

Now we shall prove the second part of the theorem. By Theorem 1 η is the least idempotent congruence on D, ${}^{D}\!/_{\eta}$ is a semilattice and $D \to {}^{D}\!/_{\eta} : x \mapsto [x]$ is a homomorphism ($[x]$ is a class of the congruence η, which contains x). By Tamura and Kimura [2] it follows that every class A of the congruence η is a archimedean subsemigroup of the semigroup $(D, \prec)$. According to the preceding computation A is an archimedean subsemigroup of the semigroup $(D, \succ)$. $\square$

If the operations of a commutative dimonoid coincide, then from Theorem 2 we obtain the theorem by Tamura and Kimura [2] about the decomposition of a commutative semigroup into a semilattice of archimedean subsemigroups.

4. Some examples

In this section we construct the examples of different dimonoids. First we consider the examples of dimonoids with one and two commutative operations.

a) Let $(X, \prec)$ be a zero semigroup, $(X, \succ)$ be a right zero semigroup. Then $(X, \prec, \succ)$ is dimonoid with commutative operation $\prec$. It is easy to see that the least idempotent congruence $\eta = X \times X$ on $(X, \prec, \succ)$.

b) Let $(S, \prec)$ be a zero semigroup (0 is a zero), $(S, \succ)$ be a commutative semigroup with a zero 0 such that $S \succ S \succ S = 0$. Then $(S, \prec, \succ)$ is commutative dimonoid. It is easy to see that the least idempotent congruence $\eta = S \times S$ on $(S, \prec, \succ)$.

We give an example of such dimonoid.

Let $(X, \prec)$ be a zero semigroup, that is, $x \prec y = 0$ for all $x, y \in X$. Fix elements a, b of the set X, $a \neq b$ and define on X the operation $\succ$, assuming

$$x \succ y = \begin{cases} a, & x = y = b, \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in X$. It is easy to see that $(X, \prec, \succ)$ is a commutative dimonoid.

c) Let X be an arbitrary set such that $0,\ a,\ b,\ c,\ d \in X$ and $a \neq b,\ b \neq c,\ c \neq d,\ d \neq a$. Define on the set X the operations $\prec$ and $\succ$, assuming

$$x \prec y = \begin{cases} b, & x = y = a, \\ 0 & \text{otherwise,} \end{cases}$$

$$x \succ y = \begin{cases} d, & x = y = c, \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in X$. It is immediate to check that $(X, \prec, \succ)$ is commutative dimonoid. In this case the least idempotent congruence $\eta = X \times X$.

Now we construct the examples of dimonoids which are not necessarily commutative.

d) Let S be a semigroup and let f be its idempotent endomorphism. On S define the multiplications by

$$x \prec y = x(yf),\ x \succ y = (xf)y$$

for all $x, y \in S$.

Proposition 1. *$(S, \prec, \succ)$ is dimonoid.*

Proof. For any $x, y, z \in S$ we obtain

$$(x \prec y) \prec z = x(yf) \prec z = x(yf)(zf) = x((yz)f),$$

$$x \prec (y \prec z) = x \prec y(zf) = x(y(zf))f =$$
$$= x(yf)(zf^2) = x(yf)(zf) = x((yz)f),$$
$$x \prec (y \succ z) = x \prec ((yf)z) = x((yf)z)f =$$
$$= x(yf^2)(zf) = x(yf)(zf) = x((yz)f),$$
$$(x \succ y) \succ z = (xf)y \succ z = (xf^2)(yf)z =$$
$$= (xf)(yf)z = (xy)fz,$$
$$x \succ (y \succ z) = x \succ ((yf)z) = (xf)(yf)z = (xy)fz,$$

$$(x \prec y) \succ z = x(yf) \succ z = (x(yf))fz =$$
$$= (xf)(yf^2)z = (xf)(yf)z = (xy)fz,$$
$$(x \succ y) \prec z = (xf)y \prec z = (xf)y(zf),$$
$$x \succ (y \prec z) = x \succ (y(zf)) = (xf)y(zf).$$

Comparing these expressions, we conclude that $(S, \prec, \succ)$ is dimonoid. □

e) Let S and T be semigroups, $\theta : T \to S$ is a homomorphism. On $S \times T$ define the multiplications by

$$(s, t) \prec (p, g) = (s, tg), (s, t) \succ (p, g) = ((t\theta)p, tg)$$

for all $(s, t), (p, g) \in S \times T$.

Proposition 2. *$(S \times T, \prec, \succ)$ is dimonoid.*

Proof. Obviously, the operation $\prec$ is associative. For any $(s, t), (p, g), (a, b) \in S \times T$ we have

$$((s, t) \succ (p, g)) \succ (a, b) = ((t\theta)p, tg) \succ (a, b) = ((tg)\theta a, tgb),$$
$$(s, t) \succ ((p, g) \succ (a, b)) = (s, t) \succ ((g\theta)a, gb) =$$
$$= ((t\theta)(g\theta)a, tgb) = ((tg)\theta a, tgb),$$
$$((s, t) \prec (p, g)) \succ (a, b) = (s, tg) \succ (a, b) = ((tg)\theta a, tgb),$$
$$(s, t) \prec ((p, g) \succ (a, b)) = (s, t) \prec ((g\theta)a, gb) =$$
$$= (s, tgb) = ((s, t) \prec (p, g)) \prec (a, b),$$
$$((s, t) \succ (p, g)) \prec (a, b) = ((t\theta)p, tg) \prec (a, b) = ((t\theta)p, tgb),$$
$$(s, t) \succ ((p, g) \prec (a, b)) = (s, t) \succ (p, gb) = ((t\theta)p, tgb).$$

Comparing these expressions, we conclude that $(S \times T, \prec, \succ)$ is dimonoid. □

f) Let X^* be a set of words in the alphabet X. If $w \in X^*$, then the first (respectively, the last) letter of a word w we denote by $w^{(0)}$ (respectively, $w^{(1)}$).

Assume the operations $\prec$, $\succ$ on the set X^* by

$$w \prec u = w^{(0)}w^{(1)}, \quad w \succ u = u^{(0)}u^{(1)}$$

for all $w, u \in X^*$.

Proposition 3. $(X^*, \prec, \succ)$ *is dimonoid.*

Proof. Obviously, the operations $\prec$ and $\succ$ are associative. For any $w, u, \omega \in X^*$ we obtain

$$(w \prec u) \prec \omega = w^{(0)}w^{(1)} \prec \omega = w^{(0)}w^{(1)} = w \prec (u \succ \omega),$$

$$w \succ (u \succ \omega) = w \succ \omega^{(0)}\omega^{(1)} = \omega^{(0)}\omega^{(1)} = (w \prec u) \succ \omega,$$

$$(w \succ u) \prec \omega = u^{(0)}u^{(1)} \prec \omega = u^{(0)}u^{(1)} =$$

$$= w \succ u^{(0)}u^{(1)} = w \succ (u \prec \omega).$$

□

Let $(X \times X, \prec', \succ')$ be an idempotent dimonoid with operations

$$(x, y) \prec' (a, b) = (x, y), (x, y) \succ' (a, b) = (a, b)$$

for all $(x, y), (a, b) \in X \times X$. Denote this dimonoid by $\tilde{X}$ and for all $i, j \in X$ assume

$$A_{(i,j)} = \{w \in X^* | (w^{(0)}, w^{(1)}) = (i, j)\}.$$

The next assertion describes the structure of the dimonoid $(X^*, \prec, \succ)$.

Proposition 4. *The dimonoid* $(X^*, \prec, \succ)$ *is a diband* $\tilde{X}$ *of zero semigroups* $A_{(i,j)}, (i, j) \in \tilde{X}$.

Proof. Define a map α by

$$\alpha : (X^*, \prec, \succ) \to \tilde{X} : w \mapsto (w^{(0)}, w^{(1)}).$$

The map α is a homomorphism. Indeed, if $w, u \in X^*$, then

$$(w \prec u)\alpha = (w^{(0)}w^{(1)})\alpha = (w^{(0)}, w^{(1)}) =$$

$$= (w^{(0)}, w^{(1)}) \prec' (u^{(0)}, u^{(1)}) = w\alpha \prec' u\alpha,$$

$$(w \succ u)\alpha = (u^{(0)}u^{(1)})\alpha = (u^{(0)}, u^{(1)}) =$$

$$= (w^{(0)}, w^{(1)}) \succ' (u^{(0)}, u^{(1)}) = w\alpha \succ' u\alpha.$$

It is clear that $A_{(i,j)}, (i, j) \in \tilde{X}$ is an arbitrary class of the congruence $\triangle_\alpha$. Moreover, if $w, u \in A_{(i,j)}$, then $w \prec u = w \succ u = ij$, hence $A_{(i,j)}$ is zero semigroup with the zero ij. □

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