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On some generalization of metahamiltonian groups

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Abstract. Locally step groups at which all subgroups are or normal, or have Chernikov's derived subgroup are studied.

Let G be a group and $\mathcal{L}_{\text{norm}}(G)$ be a family of all normal subgroups of G. Study of the influence of the family $\mathcal{L}_{\text{norm}}(G)$ on the structure of the group G was begun a long time ago. R. Dedekind in his classical paper [1] described the finite groups, which subgroups are normal (i.e. the family $\mathcal{L}_{\text{norm}}(G)$ consists of all subgroups). Later, R. Baer in a paper [2] extended the description of R. Dedekind on arbitrary groups. Let's note that the groups, any subgroup of which is normal, were later called Dedekind groups. O.Y. Schmidt, in his works [3] and [4], began the study of the (finite) groups, in which the family $\mathcal{L}_{\text{non-norm}}(G)$ of all non-normal subgroups satisfies certain strong enough restriction. So in the papers [3] and [4] he studied the structure of finite groups, in which either $\mathcal{L}_{\text{non-norm}}(G)$ is a conjugacy class, or is the union of two conjugacy classes. S.N. Chernikov in a paper [5] started the study of infinite groups, in which the family $\mathcal{L}_{\text{non-norm}}(G)$ satisfies certain natural finiteness condition. This work caused a large cycle of works devoted to the study of infinite groups, which family $\mathcal{L}_{\text{non-norm}}(G)$ satisfies certain natural restriction. Here we won't write in detail about these researches, they are examined in detail in the book of S.N. Chernikov, [6], and the review [7]. Here we will touch upon the works, directly related to this work. G.M. Romalis and N.F. Sesekin in a paper [8-10] started to study the groups, in which the family $\mathcal{L}_{\text{non-norm}}(G)$ consists of abelian subgroups. They

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called such groups metahamiltonian groups. The finite metahamiltonian groups have also been studied in papers [11, 12]. The full description of metahamiltonian groups have been obtained in a series of works of N.F. Kuzennyj and N.N. Semko [13-19]. The natural continuation of such researches is a consideration of a situation, when the subgroups of the family $\mathcal{L}_{\text{non-norm}}(G)$ belong to the class of groups, which are a natural extension of the class of abelian groups. Thus in papers [20, 21] there have been considered the groups, in which the subgroups of a family $\mathcal{L}_{\text{non-norm}}(G)$ have finite derived subgroups or are FC-groups. In this work, the researches in this area continue. Since the Chernikov groups are a natural extension of the finite groups, then the groups with Chernikov derived subgroups are a natural extension of the groups with finite derived subgroups. In this work, we start the study of groups, in which every subgroup is normal or has a Chernikov derived subgroup.

Lemma 1. *Let G be a group whose subgroups either are normal or have Chernikov derived subgroups. Then the following assertions hold:*

1. *If H is a subgroup of G then every subgroup of H either is normal or has Chernikov derived subgroup.*
2. *If L is a normal subgroup of G then every subgroup of G/L either is normal or has Chernikov derived subgroup.*
3. *If S is a subgroup of G such that $[S,S]$ is not Chernikov subgroup, then a factor-group G/S is Dedekind.*

All these assertion are almost obvious.

Lemma 2. *Let G be a group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that A is an abelian torsion-free subgroup of G. If the normalizer $N_G(A)$ contains an element g of finite order, then $g \in C_G(A)$.*

Proof. Suppose the contrary, let $g \notin C_G(A)$. Let m be an arbitrary positive integer and put $H_m = \langle A^m, g\rangle$. Since A^m is a characteristic subgroup of A, A^m is $\langle g\rangle$-invariant, so that $\langle A^m, g\rangle = A^m \leftthreetimes \langle g\rangle$. It implies the inclusion $[H_m, H_m] \leqslant A^m$. Assume that a subgroup $\langle g, A^m\rangle$ is abelian. Then we have $(a^g)^m = (a^m)^g = a^m$ for each element $a \in A$. It follows that $(a^g a^{-1})^m = 1$. Since A is torsion-free, we obtain that $a^g a^{-1} = 1$ or $a^g = a$. This shows that $g \in C_G(A)$, and we obtain a contradiction with our assumption. This contradiction implies the subgroups $\langle g, A^m\rangle$ are non-abelian for all $m \in N$. In turn, it follows that $[H_m, H_m]$ are non-identity for every $m \in N$. An inclusion $[H_m, H_m] \leqslant A^m$ shows that $[H_m, H_m]$ is torsion-free. In particular, it is not Chernikov. In this case a subgroup $\langle g, A^m\rangle$ is normal for each $m \in N$. Therefore $\langle g\rangle = \cap_{m\in N}\langle g, A^m\rangle$ is likewise normal. An equation $\langle g\rangle \cap A = \langle 1\rangle$ implies that $g \in C_G(A)$, what proves a result. □

Lemma 3. *Let G be a finitely generated group. If G includes a subgroup L of finite index such that $[L, L]$ is a Chernikov subgroup, then G is abelian-by-finite.*

Proof. Put $K = [L, L]$. The finiteness of $|G : L|$ implies that L is finitely generated (see, for example, [22, Theorem 1.41]). Let D be a maximal divisible subgroup (divisible part) of a Chernikov subgroup K. Then L/D includes a normal finite subgroup K/D such that L/K is abelian and finitely generated. In particular, L/D is polycyclic-by-finite. Then L/D is residually finite (see, for example, [23, Chapter 1, Theorem 1]). It follows that L/D includes a normal subgroup F/D of finite index such that $(F/D) \cap (K/D) = \langle 1\rangle$. This shows that F/D is an abelian subgroup of finite index. The finiteness of $|G : F|$ implies that F is finitely generated. Being metabelian and finitely generated, F satisfies a maximal condition on normal subgroups [24, Theorem 3]. On the other hand, if D is non-identity, it has an ascending series of G-invariant subgroups. This contradiction shows that $D = \langle 1\rangle$, i.e. $L \cong L/D$ is abelian-by-finite. The fact that $|G : L|$ is finite proves that G is abelian-by-finite. □

Lemma 4. *Let G be a group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that every subgroup of finite index has non-Chernikov derived subgroup. Let R be an intersection of all subgroup of finite index. Then either G/R is abelian, or G has a normal subgroup $V \geqslant R$ such that G/V is a quaternion group and V/R is abelian.*

Proof. Since every subgroup of finite index of G has non-Chernikov derived subgroup, then every subgroup H of finite index is normal in G. By Lemma 1 G/H is Dedekind group. Let

$$\mathfrak{M} = \{X | X \text{ is a subgroup of finite index in } G\}, \quad R = \cap\mathfrak{M}.$$

By a classical Remak theorem G/R is isomorphic to some subgroup of $\prod_{H\in\mathfrak{M}} G/H$. Suppose that G/R is non-abelian. Then G has a normal subgroup U of finite index such that G/U is a non-abelian Dedekind group. In this case $G/U = Q/U \times V/U$ where Q/U is a quaternion group, V/U is a direct product of an elementary abelian 2-subgroup and periodic abelian subgroup without the elements of order 2 (see, for example, [25, 5.3.7]). Let W be a subgroup of V having finite index. Then W has finite index in G and hence is normal in G. A factor-group G/W is non-abelian and Dedekind. If we suppose that V/W is non-abelian, then V includes a D-invariant subgroup Y such that V/Y is a quaternion group. Then G/Y is quaternion-by-quaternion. On the other hand, G/Y is Dedekind. This contradiction shows that V/W is abelian. Let

$$\mathfrak{S} = \{X|X \text{ is a subgroup of finite index in } V\}, \quad T = \cap\mathfrak{S}.$$

Clearly $\mathfrak{S}$ is a subfamily of $\mathfrak{M}$, so that $T \geqslant R$. Using again Remak's, we obtain that V/T is isomorphic to a subgroup of $\prod_{W\in\mathfrak{S}} V/W$. Since the last group is abelian, V/T is also abelian. Let $X \in \mathfrak{M}$, then $X \cap V$ has finite index in V, that is $X \cap V \in \mathfrak{S}$. It follows that $\mathfrak{S} = \{X \cap V|X \in \mathfrak{M}\}$. Then

$$R = R \cap V = (\cap_{X\in\mathfrak{M}} X) \cap V = \cap_{X\in\mathfrak{M}}(X \cap V) = T.$$

Hence V/R is abelian. □

Corollary. *Let G be a finitely generated group whose subgroups either are normal or have Chernikov derived subgroups. If G is soluble-by-finite, then G is abelian-by-finite.*

Proof. If G has a subgroup L of finite index whose derived subgroup is Chernikov, then Lemma 3 shows that G is abelian-by-finite. Therefore we can suppose that every subgroup of finite index of G has non-Chernikov derived subgroup. Denote by R the intersection of all subgroup having in G finite index. By Lemma 4 either G/R is abelian, or G includes a normal subgroup V such that $V \geqslant R$, G/V is a quaternion group and V/R is abelian. The finiteness of G/V implies that V is finitely generated. Being finitely generated and abelian-by-finite, V has the proper subgroups of finite index. It follows that $R \neq V$. Since V/R is abelian, $[V, V] = K \neq V$. Let S be a soluble radical of K. Clearly S is G-invariant. Suppose that KS. Then K/S is finite and non-soluble. A factor-group V/S is finite-by-abelian. Being finitely generated, V/S is polycyclic-by-finite. Then V/S is residually finite (see, for example, [23, Chapter 1, Theorem 1]). It follows that V/S has a normal subgroup E/S of finite index such that $(E/S) \cap (K/S) = \langle 1\rangle$. In particular, E has finite index in G, therefore it is normal in G. The relations

$$X/S \cong (X/S)/((E/S) \cap (X/S)) \cong (X/S)(E/S)/(E/S) \leqslant (G/S)/(E/S)$$

and $G/E \cong (G/S)/(E/S)$ show that G/E includes a finite non-soluble subgroup. On the other hand, we have above observed that every finite factor-group of G is Dedekind, in particular, it is soluble. This contradiction shows that $K = S$, i.e K is soluble. Suppose that K is non-identity. In this case $U = [K, K] \neq K$, and V/U is metabelian and non-abelian.

Being finitely generated, V/U is residually finite [26, Theorem 1]. It follows that a subgroup U is an intersection of some subgroups of finite index. On the other hand, R is an intersection of all subgroups having in V finite index. Therefore $R \leqslant Y$. This contradiction shows that $K = \langle 1 \rangle$, so that V is abelian. Since G/V is finite, G is abelian-by-finite, as required. □

Lemma 5. *Let G be a finitely generated group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is torsion-free and includes a normal abelian subgroup A such that G/A is a finite cyclic group. Then G is abelian.*

Proof. Let x be an element of G such that $G/A = \langle xA \rangle$. Since G is torsion-free, $1 \neq x^m \in A$ for some $m \in N$. In particular, $C = C_A(x) \neq \langle 1 \rangle$. It is not hard to prove that C is a pure subgroup of A. If $C = A$, then G is abelian, and all is proved. Therefore suppose that $C \neq A$. Clearly C is $\langle x \rangle$-invariant subgroup. The element xC has finite order in a factor-group G/C. Using Lemma 2 we obtain that $xC \in C_{G/C}(A/C)$. It follows that G/C is abelian. Since A is abelian, the choice of C shows that $C \leqslant \zeta(G)$. In particular, G is nilpotent. Being torsion-free and abelian-by-finite, G is abelian. □

Corollary 1. *Let G be a finitely generated group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is torsion-free and soluble-by-finite. Then G is abelian.*

Proof. By Corollary to Lemma 4 G has a normal abelian subgroup A such that G/A is finite. We have $G = \cup_{1 \leqslant j \leqslant k} A x_j$ for some elements $x_1, \ldots, x_k$. Lemma 5 shows that a subgroup $\langle A, x_j \rangle$ is abelian, $1 \leqslant j \leqslant k$. An equation $G = \cup_{1 \leqslant j \leqslant k} \langle A, x_j \rangle$ proves now that a group G has a center of finite index (see, for example, [27, Theorem 7.4]). In particular, G is an FC-group. We observe now that a torsion-free FC-group is abelian (see, for example, [27, Theorem 1.6]). □

Corollary 2. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is torsion-free. Then G is abelian.*

Lemma 6. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the set of all elements of G, having finite order, is a (characteristic) subgroup of G.*

Proof. Let x, y be two arbitrary elements of finite order and let $F = \langle x, y \rangle$. Suppose that a subgroup F is infinite. By Corollary to Lemma 4 F

includes a normal abelian subgroup A of finite index. Then A is also finitely generated. In this case there exists a positive integer k such that $K = A^k$ is torsion-free and non-identity. Furthermore, A/A^k is finite, so that G/K is likewise finite. Lemma 2 shows that $[K, x] = \langle 1 \rangle = [K, y]$. An equation $F = \langle x, y \rangle$ shows that $K \leqslant \zeta(F)$. In other words, the center of G has a finite index, in particular, F is an FC-group. However, the set of all elements of finite order of an FC-group is a subgroup (see, for example, [27, Theorem 1.6]). In particular, $\langle x, y \rangle$ is finite. This contradiction proves a result. □

Corollary 1. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then every finitely generated subgroup of G is central-by-finite. In particular, G is locally FC-group.*

Proof. Let F be an arbitrary finitely generated subgroup of G. Denote by T the subset of all elements of F, having finite order. By Lemma 6 T is a characteristic subgroup of F. Corollary 1 of Lemma 5 proves that F/T is abelian. In particular, the derived subgroup $[F, F]$ is finite. It follows that F is an FC-group. However a finitely generated FC-group is central-by-finite. □

Corollary 2. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the derived subgroup of G is periodic.*

Proof. Denote by T the subset of all elements of G, having finite order. Lemma 6 shows that T is a characteristic subgroup of G. A factor-group G/T is torsion-free and, by Corollary 2 of Lemma 5, G/T is abelian. Hence $[G, G] \leqslant T$. □

Having put together everything proved above, we receive the following main result of this work.

Theorem 1. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the following assertions hold:*

1. *Every finitely generated subgroup of G is central-by-finite, in particular, G is a locally FC-group.*

2. *The derived subgroup of a group G is a locally finite subgroup. In particular, if G is torsion-free, then G is abelian.*

We recall that a group G is called locally graded, if every finitely generated subgroup of G has a proper subgroup of finite index. The class of locally graded groups is very broad, it includes all locally residually finite groups and all locally (soluble-by-finite) groups.

Theorem 2. *Let G be a locally graded group whose subgroups either are normal or have Chernikov derived subgroups. Then the derived subgroup of a group G is locally finite. In particular, if G is torsion-free, then G is abelian.*

Proof. Denote If a group G includes a subgroup of finite index, having Chernikov derived subgroup, then clearly G is soluble-by-finite. Therefore we can assume, that every subgroup of finite index has non-Chernikov derived subgroup. Let

$$\mathfrak{M} = \{X|X \text{ is a subgroup having non-Chernikov derived subgroup}\},$$
$$R = \cap\mathfrak{M}.$$

Thus, if $H \in \mathfrak{M}$, then H is normal in G and G/H is a Dedekind group by Lemma 1. By a classical Remak theorem G/R is isomorphic to some subgroup of $\prod_{X\in\mathfrak{M}} G/X$. Since every Dedekind group either is abelian or nilpotent of class 2, either G/R is abelian or nilpotent of class 2. (We can noted, that it is possible that $G = R$). If H is a proper subgroup of R, then by a construction of R, $[H, H]$ is a Chernikov subgroup. Let H be an arbitrary finitely generated subgroup of R. Since G is locally graded, H includes a proper normal subgroup L of finite index. In particular, L is a proper subgroup of R and by above L has a Chernikov derived subgroup. It shows that L is soluble-by-finite. Corollary to Lemma 4 shows that L is abelian-by finite. Hence and H is abelian-by-finite. If $H = R$, then we obtain that R is abelian-by-finite, so that G is soluble-by-finite. Suppose now that R has no finite set of generators. By above proved R is locally (abelian-by-finite). By Theorem 1 a subgroup $K = [R, R]$ is locally finite. Then G/K is soluble. Using again Theorem 1 we obtain that a subgroup $P/K = [G/K, G/K]$ is locally finite. Since K and P/K is locally finite, P is locally finite. Since G/P is abelian, $[G, G] \leqslant P$, in particular, $[G, G]$ is locally finite. □

Corollary. *Let G be a locally graded group whose subgroups either are normal or have Chernikov derived subgroups. If G is periodic, then G is locally finite.*

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