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RESEARCH ARTICLE

Odd-quadratic Malcev superalgebras

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Abstract. We present some properties of odd-quadratic Malcev superalgebras. The aim of this paper is to present the notions of double extension and generalized double extension of odd-quadratic Malcev superalgebras, culminating in the inductive description of odd-quadratic Malcev superalgebras such that the even part is a reductive Malcev algebra.

Introduction

We consider finite dimensional Malcev superalgebras over an algebraically closed commutative field $\mathbb{K}$ of characteristic zero. A quadratic (respectively, odd-quadratic) Malcev superalgebra is a Malcev superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with a non-degenerate, supersymmetric, even (respectively, odd), and M-invariant bilinear form. Malcev superalgebras are naturally a generalization of Lie superalgebras and interesting descriptions of quadratic, and odd-quadratic, Lie superalgebras are done (see [2, 3, 9, 10]). A next obvious problem is whether any description of quadratic, and odd-quadratic, Malcev superalgebras do exist. An inductive description of quadratic Malcev superalgebras is already established. H. Albuquerque and S. Benayadi [5], transferring the procedure used in Lie case [10] to the Malcev superalgebras, gave the inductive description of quadratic Malcev superalgebras with reductive even part and completely reducible action of the even part on the odd part. Motivated by the inductive description of quadratic Lie superalgebras with reductive even part [2] and by the work of [5], in a previous paper [4] it was established the description of

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quadratic Malcev superalgebras with reductive even part. Now, naturally raises the question of presenting a description of odd-quadratic Malcev superalgebras. The aim of this paper is to provide a description of this class of Malcev superalgebras. We use the notion of double extension of Malcev superalgebras given in [5] and the notion of generalized double extension of Malcev superalgebras introduced in [4] in order to state the inductive description of odd-quadratic Malcev superalgebras such that the even part is a reductive Malcev algebra.

For more references to Malcev superalgebras and representation theory, the reader can consult [1, 6, 7, 13, 15, 18, 20].

1. Preliminaries and basic properties

Definition 1.1. *A superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ (meaning a $\mathbb{Z}_2$-graded algebra) is a Malcev superalgebra if the following assertions are verified:* $\forall_{X\in M_x, Y\in M_y, Z\in M_z, T\in M_t}$,

(i) $XY = -(-1)^{xy} YX$,
(ii) $(-1)^{yz}(XZ)(YT) = ((XY)Z)T + (-1)^{x(y+z+t)}((YZ)T)X$
$+(-1)^{(x+y)(z+t)}((ZT)X)Y + (-1)^{t(x+y+z)}((TX)Y)Z.$

We shall write $X \in M_x$ to mean that X is a homogeneous element of the Malcev superalgebra M of degree x, with $x \in \mathbb{Z}_2$.

Given a Malcev superalgebra M, its *center* $\mathfrak{z}(M)$ is the ideal of M defined by: $\mathfrak{z}(M) = \{X \in M : XY = 0, \ \forall_{Y\in M}\}$.

Definition 1.2. *Let M be a Malcev superalgebra. A bilinear form B on M is called*

(i) supersymmetric if $B(X,Y) = (-1)^{xy}B(Y,X)$, $\forall_{X\in M_x, Y\in M_y}$;

(ii) non-degenerate if $X \in M$ satisfies $B(X,Y) = 0$, $\forall_{Y\in M}$, then $X = 0$;

(iii) invariant if $B(XY,Z) = B(X,YZ)$, $\forall_{X,Y,Z\in M}$;

(iv) odd if $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}) = B(\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}) = \{0\}$.

Definition 1.3. *An odd-quadratic Malcev superalgebra (M,B) is a Malcev superalgebra M where is defined an odd supersymmetric non-degenerate invariant bilinear form B. In this case, B is called an odd-invariant scalar product on M.*

Definition 1.4. *Let (M,B) and (N,T) be two odd-quadratic Malcev superalgebras. It is easy to see that $(M\oplus N,\gamma)$ is an odd-quadratic Malcev superalgebra where the odd-invariant scalar product on $M\oplus N$ is defined by*

$$\gamma\mid_{(M\times M)}=B,\quad \gamma\mid_{(N\times N)}=T,\quad \gamma\mid_{(M\times N)}=0.$$

It is called the orthogonal direct sum of (M,B) and (N,T).

Definition 1.5. *Let (M,B) be an odd-quadratic Malcev superalgebra.*

- *(i) A graded ideal I of M is called non-degenerate if the restriction of B to $I\times I$ is a non-degenerate bilinear form on I. Otherwise, we said that I is degenerate.*
- *(ii) M is called B-irreducible if M contains no non-zero non-degenerate graded ideals.*
- *(iii) A graded ideal I of M is B-irreducible if I is non-degenerate and I contains no non-zero non-degenerate graded ideals of M.*

Definition 1.6. *Let M be a Malcev superalgebra.*

- *(i) Let V be a $\mathbb{Z}_2$-graded vector space. An even linear map $\psi: M\longrightarrow End(V)$ is a Malcev representation of M in V if*

$$\begin{aligned}\psi((XY)Z) &= \psi(X)\psi(Y)\psi(Z)-(-1)^{x(y+z)}\psi(YZ)\psi(X)\\ &\quad -(-1)^{z(x+y)}\psi(Z)\psi(X)\psi(Y)\\ &\quad +(-1)^{x(y+z)}\psi(Y)\psi(ZX),\quad \forall_{X\in M_x,Y\in M_y,Z\in M_z}.\end{aligned}$$

- *(ii) Consider two Malcev representations $\psi: M\longrightarrow End(V)$ and $\phi: M\longrightarrow End(W)$ of M, where V and W are $\mathbb{Z}_2$-graded vector spaces. We say that ψ and ϕ are equivalent if there exists a bijective even linear map $\delta: V\longrightarrow W$ such that $\delta\circ\psi(X)=\phi(X)\circ\delta$, $\forall_{X\in M}$.*

Example 1.7. Let $M=M_{\bar 0}\oplus M_{\bar 1}$ be a Malcev superalgebra. It is easy to see that the map $ad: M\longrightarrow \mathrm{End}(M)$ defined by $ad(X)(Y)=XY$, $\forall_{X,Y\in M}$, is a Malcev representation of M in itself. It is called the *adjoint representation* of M. Besides, this representation induces a Malcev representation $ad: M_{\bar 0}\longrightarrow \mathrm{End}(M_{\bar 1})$ of the Malcev algebra $M_{\bar 0}$ in the vector space $M_{\bar 1}$ defined by $ad(X)(Y)=XY$, $\forall_{X\in M_{\bar 0},Y\in M_{\bar 1}}$. It is called the *adjoint representation* of $M_{\bar 0}$ in $M_{\bar 1}$.

Example 1.8. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra and M^* its dual vector space. We easily show that the map $ad^* : M \longrightarrow \mathrm{End}(M^*)$ defined by

$$ad^*(X)(f)(Y) = -(-1)^{x\alpha} f(XY), \qquad \forall_{X\in M_x, Y\in M_y, f\in (M^*)_\alpha},$$

is a Malcev representation of M in M^*. It is called the *co-adjoint representation* of M. Moreover, it is also clear that $ad^* : M_{\bar{0}} \longrightarrow \mathrm{End}(M_{\bar{1}}^*)$ defined by $ad^*(X)(f)(Y) = -f(XY)$, $\forall_{X\in M_{\bar{0}}, Y\in M_{\bar{1}}, f\in (M_{\bar{1}}^*)_\alpha}$, is a Malcev representation of $M_{\bar{0}}$ in $M_{\bar{1}}^*$. It is called the *co-adjoint representation* of $M_{\bar{0}}$ in $M_{\bar{1}}^*$.

Odd-quadratic Lie superalgebras were characterized in [3]. Here, we obtain an analogue characterization of odd-quadratic Malcev superalgebras.

Proposition 1.9. *Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra. Consider the Malcev representation $ad : M_{\bar{0}} \longrightarrow End(M_{\bar{0}})$ of $M_{\bar{0}}$ in itself defined by $ad(X)(Y) = XY$, $\forall_{X,Y\in M_{\bar{0}}}$ and the Malcev representation $ad^* : M_{\bar{0}} \longrightarrow End(M_{\bar{1}}^*)$ of $M_{\bar{0}}$ in $M_{\bar{1}}^*$ defined by $ad^*(X)(f)(Y) = -f(XY)$, $\forall_{X\in M_{\bar{0}}, Y\in M_{\bar{1}}, f\in (M_{\bar{1}}^*)_\alpha}$. Then the Malcev superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is odd-quadratic if and only if there exists an equivalence $\varphi : M_{\bar{0}} \longrightarrow M_{\bar{1}}^*$ of ad and ad^* such that*

$$\varphi(XY)(Z) = \varphi(YZ)(X), \qquad \forall_{X,Y,Z\in M_{\bar{1}}}.$$

In this case, $\dim M_{\bar{0}} = \dim M_{\bar{1}}$ *and dimension of M is even.*

Proof. It is similar to the proof of Proposition 1.10 in [3]. □

We use the following result to reduce the study of odd-quadratic Malcev superalgebra (M, B) to the B-irreducible case.

Proposition 1.10. *Let (M, B) be an odd-quadratic Malcev superalgebra. Then $M = \bigoplus_{k=1}^m M_k$, where each M_k is a B-irreducible graded ideal of M such that $B(M_k, M_{k'}) = \{0\}$, when $k, k' \in \{1, \ldots, m\}$ and $k \neq k'$.*

2. Odd-quadratic Malcev superalgebras of dimension ≤ 4

Motivated by the goal of gaining a better understanding of the theory, we present a classification of the odd-quadratic Malcev non-Lie superalgebras of dimension ≤ 4. First, notice that a Malcev superalgebra M of $\dim M \leq 2$ is always a Lie superalgebra [1]. Now, we are going to determine the odd-quadratic Malcev superalgebras in the list of Malcev non-Lie superalgebras of dim ≤ 4 presented in [8].

Example 2.1. The solvable Malcev non-Lie superalgebra $M^2(2,2) = M_{\bar{0}} \oplus M_{\bar{1}}$ of dimension 4, where $\{a, b\}$ is a basis of $M_{\bar{0}}$ and $\{v, u\}$ is a basis of $M_{\bar{1}}$, with the multiplication defined by $ab = b$, $av = -v$, $bv = u$, $v^2 = b$, is odd-quadratic. Indeed, we easily show that the symmetric bilinear form $B : M \times M \longrightarrow \mathbb{K}$ defined by $B(a, u) = B(b, v) = 1$ is an odd scalar product defined on M.

Lemma 2.2. *Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra with* $\dim M = 4$. *Suppose that* $\dim M_{\bar{0}} = 2$ *and $M_{\bar{0}}$ is abelian. If there exist bases $\{a, b\}$ of $M_{\bar{0}}$ and $\{v, u\}$ of $M_{\bar{1}}$ such that $u = bv$ then M is not odd-quadratic.*

Proof. Let B be an invariant odd bilinear from on M. Then

$$B(a, u) = B(a, bv) = B(ab, v) = 0$$
$$B(b, u) = B(b, bv) = B(bb, v) = 0.$$

Consequently $B(M, u) = \{0\}$, so B is degenerate. □

Corollary 2.3. *The Malcev non-Lie superalgebras $M^1(2, 2)$, $M(\mu, 2, 2)$, and $M(2, 2, 1)$ presented in [1, 8] are not odd-quadratic.*

We recall that we are considering Malcev superalgebras over a field of characteristic zero.

Lemma 2.4. *Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra with* $\dim M = 4$. *Suppose that* $\dim M_{\bar{0}} = 2$. *If there exist bases $\{a, b\}$ of $M_{\bar{0}}$ and $\{v, u\}$ of $M_{\bar{1}}$ such that $av = \alpha v$ and $au = \beta u$, with $\alpha, \beta \in \mathbb{K} \setminus \{0\}$, then M is not odd-quadratic.*

Proof. Let B be an invariant odd bilinear from on M. Then

$$B(a, v) = B(a, \alpha^{-1} av) = \alpha^{-1} B(aa, v) = 0$$
$$B(a, u) = B(a, \beta^{-1} au) = \beta^{-1} B(aa, u) = 0.$$

Then $B(a, M) = \{0\}$ and we conclude that B is degenerate. □

Corollary 2.5. *The Malcev non-Lie superalgebras $M^0(2, 2)$, $M(2, 2, \mu)$, $M^1(2, 2, \gamma)$, $M(2, 2, 2, 1)$, $M(2, 2, 2, 0)$, $M(2, 2, \frac{1}{2}, 0)$, and $M(2, 2, \frac{1}{2}, \delta)$ given in [1, 8] are not odd-quadratic.*

Lemma 2.6. *Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra such that $M_{\bar{1}} = M_{\bar{1}} M_{\bar{0}}$ and $aM_{\bar{1}} = \{0\}$, for some $a \in M_{\bar{0}}$. Then M is not odd-quadratic.*

Proof. Consider B an invariant odd bilinear from on M. As

$$B(a, M_{\bar{1}}) = B(a, M_{\bar{1}}M_{\bar{0}}) = B(aM_{\bar{1}}, M_{\bar{0}}) = 0,$$

then $B(a, M) = \{0\}$, which implies that B is degenerate. □

Corollary 2.7. *The Malcev non-Lie superalgebras $M^3(2,2)$ and $M^4(2,2)$ presented in [1, 8] are not odd-quadratic.*

The statements above lead us to the following interesting characterization.

Theorem 2.8. *Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev non-Lie superalgebra such that $\dim M \leq 4$. Then M is odd-quadratic if and only if M is isomorphic to $M^2(2,2)$.*

3. Odd double extension of odd-quadratic Malcev superalgebras

Because of the odd-quadratic structure, occasionally we have to change the gradation of the Malcev superalgebras as follows. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra. Denote by $P(M) = V_{\bar{0}} \oplus V_{\bar{1}}$ the $\mathbb{Z}_2$-graded vector space obtained from M with gradation defined by

$$V_{\bar{0}} = M_{\bar{1}} \quad \text{and} \quad V_{\bar{1}} = M_{\bar{0}}.$$

Note that the dual spaces $P(M^*)$ and M^* are equal as $\mathbb{Z}_2$-graded vector spaces, however

$$V_{\bar{0}}^* = M_{\bar{1}}^* \quad \text{and} \quad V_{\bar{1}}^* = M_{\bar{0}}^*.$$

Denote by $\pi_M : M \longrightarrow End(P(M^*))$ the linear map defined for homogeneous elements as follows

$$\pi_M(Z)(f)(Y) = -(-1)^{z\delta} f(ZY), \qquad \forall_{f \in (P(M^*))_\delta, Z \in M_z, Y \in M}.$$

It is quite easy to show that π_M is a representation of M in $P(M^*)$, but it is not the co-adjoint representation of M.

Definition 3.1. *Let M be a Malcev superalgebra and V a $\mathbb{Z}_2$-graded vector space. Let $\omega : M \times M \longrightarrow V$ be a homogeneous bilinear map. If the following assertions are satisfied: $\forall_{X \in M_x, Y \in M_y, Z \in M_z, T \in M_t}$,*

(i) $\omega(X, Y) = -(-1)^{xy}\omega(Y, X)$,

(ii) $(-1)^{yz}\omega(XZ, YT) = \omega((XY)Z, T) + (-1)^{x(y+z+t)}\omega((YZ)T, X) + (-1)^{(x+y)(z+t)}\omega((ZT)X, Y) + (-1)^{t(x+y+z)}\omega((TX)Y, Z)$,

we say that ω is a Malcev 2-cocycle on M with values in V.

Definition 3.2. *Let M be a Malcev superalgebra and $\phi : M \longrightarrow M$ an endomorphism of M. We say that ϕ is a Malcev operator of M if*

$$\begin{aligned}\phi((XY)Z) \;&= (\phi(X)Y)Z - (-1)^{xy}\phi(Y)(XZ) - (-1)^{z(x+y)}(\phi(Z)X)Y \\ &\quad -(-1)^{x(y+z)}\phi(YZ)X, \qquad \forall_{X\in M_x, Y\in M_y, Z\in M_z}.\end{aligned}$$

We denote by $\Big(Op(M)\Big)_\alpha$ the vector subspace of $End(M)$ formed by the Malcev operators of M of degree α ($\alpha \in \mathbb{Z}_2$). Then $Op(M) = \Big(Op(M)\Big)_{\bar{0}} \oplus \Big(Op(M)\Big)_{\bar{1}}$.

Definition 3.3. *Let (M, B) be an odd-quadratic Malcev superalgebra. A homogeneous map $\phi \in End(M)$ of degree α (with $\alpha \in \mathbb{Z}_2$) is called skew-supersymmetric if*

$$B(\phi(X), Y) = -(-1)^{\alpha x} B(X, \phi(Y)), \qquad \forall_{X\in M_x, Y\in M}.$$

We denote by $\Big(Op_a(M)\Big)_\alpha$ the vector subspace of the skew-supersymmetric elements of $\Big(Op(M)\Big)_\alpha$. We write $Op_a(M) = \Big(Op_a(M)\Big)_{\bar{0}} \oplus \Big(Op_a(M)\Big)_{\bar{1}}$ which is a super-vector subspace of $Op(M)$.

Lemma 3.4. *Let (M, B) be an odd-quadratic Malcev superalgebra and $\omega : M \times M \longrightarrow P(\mathbb{K})$ a bilinear form of degree $\alpha \in \mathbb{Z}_2$.*

(i) There exists a homogeneous map $\phi \in End(M)$ of degree α such that

$$\omega(X, Y) = B(\phi(X), Y), \qquad \forall_{X,Y\in M},$$

(ii) ω is a Malcev 2-cocycle on M if and only if ϕ is a skew-supersymmetric Malcev operator of M.

In the sequel we shall need the notion of central extension of Malcev superalgebras, so we explicitly state Proposition 4.1 of [5].

Proposition 3.5. *Let M be a Malcev superalgebra, V a graded vector space, and $\omega : M \times M \longrightarrow V$ be an even bilinear map. We consider in the space $L = M \oplus V$ the multiplication defined by*

$$(X + Z)(Y + T) \;=\; XY + \omega(X, Y), \qquad \forall_{X,Y\in M, Z,T\in V}.$$

Then L with this multiplication is a Malcev superalgebra if and only if ω is a Malcev 2-cocycle on M with values in V. In this case, $L = M \oplus V$ is called the central extension of V by M (by means of ω).

We also need the concept of semi-direct product of Malcev superalgebras.

Definition 3.6. *Let M and V be two Malcev superalgebras and ψ a Malcev representation of M in V. We say that ψ is an admissible Malcev representation of M in V if $L = M \oplus V$ with the multiplication*

$$(X+Z)(Y+T) = (XY)_M + \psi(X)(T) - (-1)^{xy}\psi(Y)(Z) + (ZT)_V,$$

$\forall_{(X+Z)\in L_x,(Y+T)\in L_y}$, is a Malcev superalgebra. In this conditions, the Malcev superalgebra $L = M \oplus V$ is called the semi-direct product of V by M (by means of ψ).

Proposition 3.7. *Let M and V be Malcev superalgebras and $\Omega : M \longrightarrow End(V)$ a Malcev representation of M in V. Then Ω is an admissible Malcev representation of M in V if and only if the following assertions are verified:*

(i) $\forall_{X\in M_x,Y\in M_y,h\in V_z,i\in V_t}$,

$$\begin{aligned}\big(\Omega(XY)(h)\big)i = \Omega(X)\big(\Omega(Y)(h)i\big) + (-1)^{yz}\big(\Omega(X)(h)\big)\big(\Omega(Y)(i)\big)\\ -(-1)^{xy}\Omega(Y)\big(\Omega(X)(hi)\big) - (-1)^{tz+xy}\Omega(Y)\big(\Omega(X)(i)\big)h;\end{aligned}$$

(ii) $\forall_{X\in M_x,Z\in M_z,g\in V_y,i\in V_t}$,

$$\begin{aligned}\Omega(XZ)(gi) = -(-1)^{zx}\Omega(Z)\big(\Omega(X)(g)\big)i + \Omega(X)\big(\Omega(Z)(g)i\big)\\ -(-1)^{yt}\Omega(X)\big(\Omega(Z)(i)\big)g + (-1)^{ty+xz}\Omega(Z)\big(\Omega(X)(i)g\big);\end{aligned}$$

(iii) $\forall_{X\in M_x}$, *$\Omega(X)$ is a Malcev operator of V.*

Now we are prepared to present the notion of odd double extension of odd-quadratic Malcev superalgebras.

Proposition 3.8. *Let (M,B) be an odd-quadratic Malcev superalgebra and N a Malcev superalgebra. Suppose that there is an admissible Malcev representation $\psi : N \longrightarrow End(M)$ such that $\psi(X) \in Op_a(M)$, $\forall_{X\in N}$. Define a bilinear map $\varphi : M \times M \longrightarrow P(N^*)$ by*

$$\varphi(X,Y)(Z) = (-1)^{z(x+y)}B(\psi(Z)(X),Y), \qquad \forall_{X\in M_x,Y\in M_y,Z\in N_z}.$$

Then the graded vector space $L = M \oplus P(N^*)$ endowed with the multiplication defined by

$$(X+f)(Y+h) = (XY)_M + \varphi(X,Y), \qquad \forall_{(X+f),(Y+h)\in(M\oplus P(N^*))}, \tag{3.1}$$

is the central extension of $P(N^*)$ by M (by means of φ).

Proof. Since $\psi(Z)$, $\forall_{Z\in N}$, is a skew-supersymmetric Malcev operator of degree z then the bilinear map φ is an even Malcev 2-cocycle of M with values in $P(N^*)$. Consequently, $M\oplus P(N^*)$ with the multiplication (3.1) is a Malcev superalgebra called the central extension of $P(N^*)$ by M (by means of φ) as required. □

Theorem 3.9. *Let (M,B) be an odd-quadratic Malcev superalgebra and N a Malcev superalgebra. Let $\psi : N \longrightarrow End(M)$ be an admissible Malcev representation of N in M such that $\psi(X)\in Op_a(M)$, $\forall_{X\in N}$. Define the linear map $\widetilde{\psi} : N \longrightarrow End(M\oplus P(N^*))$ by*

$$\widetilde{\psi}(Z)(X+f) \quad = \quad \psi(Z)(X)+ad_N^*(Z)(f), \qquad \forall_{(X+f)\in(M\oplus P(N^*)),Z\in N},$$

where $ad_N^* : N \longrightarrow End(P(N^*))$ is the linear map defined as follows

$$ad_N^*(Z)(f)(Y) \quad = \quad -(-1)^{z\delta}f(ZY), \qquad \forall_{f\in(P(N^*))_\delta,Z\in N_z,Y\in N}.$$

Then $\widetilde{\psi}(Z)$, $\forall_{Z\in N_z}$, is an admissible Malcev representation of N in $M\oplus P(N^*)$ (the central extension of $P(N^*)$ by M (by means of φ) introduced in Proposition 3.8). Moreover, $L=N\oplus M\oplus P(N^*)$ with the multiplication defined by

$$\begin{aligned}(Z+X+f)(W+Y+h) =& \\ =(ZW)_N+(XY)_M+\psi(Z)(Y)-(-1)^{xy}\psi(W)(X)+& \\ +ad_N^*(Z)(h)-(-1)^{xy}ad_N^*(W)(f)+\varphi(X,Y),& \qquad (3.2)\end{aligned}$$

$\forall_{(Z+X+f)\in L_x,(W+Y+h)\in L_y}$, is the semi-direct product of the central extension $M\oplus P(N^*)$ by N (by means of $\widetilde{\psi}$). Furthermore, let γ be an odd supersymmetric invariant bilinear form on N (not necessarily non-degenerate). Then the bilinear form $\widetilde{B} : L\times L \longrightarrow \mathbb{K}$ defined by

$$\widetilde{B}(Z+X+f,W+Y+h)=B(X,Y)+\gamma(Z,W)+f(W)+(-1)^{xy}h(Z),$$

$\forall_{(Z+X+f)\in L_x,(W+Y+h)\in L_y}$, is an odd invariant scalar product on L. We say that the odd-quadratic Malcev superalgebra $(L,\widetilde{B})$ is an odd double extension of (M,B) by N (by means of ψ).

Proof. First we have to show that $\widetilde{\psi}$ is an admissible Malcev representation of N in $M\oplus P(N^*)$. As $\widetilde{\psi}$ is the direct sum of two Malcev representations of N in $M\oplus P(N^*)$, it is also a Malcev representation. Let us prove that $\widetilde{\psi}$ satisfies the conditions of Proposition 3.7. First, we have to see that, $\forall_{X\in N_x,Y\in N_y,V+f\in(M\oplus P(N^*))_v,W+h\in(M\oplus P(N^*))_w}$

$$\big(\widetilde{\psi}(XY)(V+f)\big)(W+h) \quad = \quad \widetilde{\psi}(X)\big(\widetilde{\psi}(Y)(V+f)(W+h)\big)$$

$$+(-1)^{yv}\big(\widetilde{\psi}(X)(V+f)\big)\big(\widetilde{\psi}(Y)(W+h)\big)$$
$$-(-1)^{xy}\widetilde{\psi}(Y)\widetilde{\psi}(X)\big((V+f)(W+h)\big)$$
$$-(-1)^{xy+vw}\big(\widetilde{\psi}(Y)\widetilde{\psi}(X)(W+h)\big)(V+f).$$

Analyzing the terms separately, we get

$$\big(\widetilde{\psi}(XY)(V+f)\big)(W+h) = \big(\psi(XY)(V)\big)W + \varphi\big(\psi(XY)(V),W\big),$$
$$\widetilde{\psi}(X)\big(\widetilde{\psi}(Y)(V+f)(W+h)\big) = \psi(X)\big(\psi(Y)(V)W\big) + ad_N^*(X)\big(\varphi(\psi(Y)(V),W)\big)$$
$$\big(\widetilde{\psi}(X)(V+f)\big)\big(\widetilde{\psi}(Y)(W+h)\big) = \psi(X)(V)\psi(Y)(W) + \varphi\big(\psi(X)(V),\psi(Y)(W)\big),$$
$$\widetilde{\psi}(Y)\widetilde{\psi}(X)\big((V+f)(W+h)\big) = \psi(Y)\big(\psi(X)(VW)\big) + ad_N^*(Y)\big(ad_N^*(X)(\varphi(V,W))\big)$$
$$\big(\widetilde{\psi}(Y)\widetilde{\psi}(X)(W+h)\big)(V+f) = \big(\psi(Y)\psi(X)(W)\big)V + \varphi\big(\psi(Y)\psi(X)(W),V\big)$$

As ψ is an admissible Malcev representation of N in M, it remains to prove that

$$\varphi\big(\psi(XY)(V),W\big) = ad_N^*(X)\big(\varphi(\psi(Y)(V),W)\big)+$$
$$+(-1)^{yv}\varphi\big(\psi(X)(V),\psi(Y)(W)\big) - (-1)^{xy}ad_N^*(Y)\big(ad_N^*(X)(\varphi(V,W))\big)-$$
$$-(-1)^{xy+vw}\varphi\big(\psi(Y)\psi(X)(W),V\big).$$

But, $\forall_{T\in N_t}$,

$$\varphi\big(\psi(XY)(V),W\big)(T) = (-1)^{t(x+y+v+w)}B\big(\psi(T)\psi(XY)(V),W\big)$$
$$ad_N^*(X)\big(\varphi(\psi(Y)(V),W)\big)(T) = -(-1)^{t(y+v+w)}B\big(\psi(XT)(\psi(Y)(V)),W\big)$$
$$(-1)^{yv}\varphi\big(\psi(X)(V),\psi(Y)(W)\big)(T) = -(-1)^{xy+t(x+v+w)}B\big(\psi(Y)\psi(T)\psi(X)(V),W\big)$$
$$-(-1)^{xy}ad_N^*(Y)\big(ad_N^*(X)(\varphi(V,W))\big)(T) = -(-1)^{t(v+w)}B\big(\psi(X(YT))(V),W\big)$$
$$-(-1)^{xy+vw}\varphi\big(\psi(Y)\psi(X)(W),V\big)(T) = (-1)^{t(v+w)}B\big(\psi(X)\psi(Y)\psi(T)(V),W\big).$$

As B is non-degenerate it follows that

$$(-1)^{t(x+y)}\psi(T)\psi(XY) = -(-1)^{ty}\psi(XT)\psi(Y)-$$
$$-(-1)^{x(y+t)}\psi(Y)\psi(T)\psi(X) - \psi(X(YT)) + \psi(X)\psi(Y)\psi(T).$$

A simple computation shows that the last equality is verified because ψ is an admissible Malcev representation of N in M.

Now we shall prove condition (ii) of Proposition 3.7, that is, $\forall_{X\in N_x, Z\in N_z, Y+f\in(M\oplus P(N^*))_y, T+g\in(M\oplus P(N^*))_t}$,

$$\widetilde{\psi}(XZ)\big((Y+f)(T+g)\big) = -(-1)^{zx}\widetilde{\psi}(Z)\big(\widetilde{\psi}(X)(Y+f)\big)(T+g)$$
$$+\widetilde{\psi}(X)\big(\widetilde{\psi}(Z)(Y+f)(T+g)\big)$$
$$-(-1)^{yt}\widetilde{\psi}(X)\big(\widetilde{\psi}(Z)(T+g)\big)(Y+f)$$

$$+(-1)^{ty+xz}\widetilde{\psi}(Z)\big(\widetilde{\psi}(X)(T+g)(Y+f)\big).$$

Doing some calculations we get

$$\begin{aligned}
&\widetilde{\psi}(XZ)\big((Y+f)(T+g)\big) = \psi(XZ)(YT) + ad^*_N(XZ)\big(\varphi(Y,T)\big)\\
&\widetilde{\psi}(Z)\big(\widetilde{\psi}(X)(Y+f)\big)(T+g) = \psi(Z)\big(\psi(X)(Y)\big)T + \varphi\big(\psi(Z)(\psi(X)(Y)),T\big)\\
&\widetilde{\psi}(X)\big(\widetilde{\psi}(Z)(Y+f)(T+g)\big) = \psi(X)\big(\psi(Z)(Y)T\big) + ad^*_N(X)\big(\varphi(\psi(Z)(Y),T)\big)\\
&\widetilde{\psi}(X)\big(\widetilde{\psi}(Z)(T+g)\big)(Y+f) = \psi(X)\big(\psi(Z)(T)\big)Y + \varphi\big(\psi(X)(\psi(Z)(T)),Y\big)\\
&\widetilde{\psi}(Z)\big(\widetilde{\psi}(X)(T+g)(Y+f)\big) = \psi(Z)\big(\psi(X)(T)Y\big) + ad^*_N(Z)\big(\varphi(\psi(X)(T),Y)\big).
\end{aligned}$$

As ψ is an admissible Malcev representation, $\widetilde{\psi}$ satisfies condition (ii) of Proposition 3.7 if and only if

$$\begin{aligned}
ad^*_N(XZ)\big(\varphi(Y,T)\big) = &-(-1)^{zx}\varphi\big(\psi(Z)(\psi(X)(Y)),T\big)+\\
&+ ad^*_N(X)\big(\varphi(\psi(Z)(Y),T)\big) - (-1)^{yt}\varphi\big(\psi(X)(\psi(Z)(T)),Y\big)+\\
&+ (-1)^{ty+xz}ad^*_N(Z)\big(\varphi(\psi(X)(T),Y)\big).
\end{aligned}$$

Analyzing each term, $\forall_{W\in N_w}$,

$$\begin{aligned}
&ad^*_N(XZ)\big(\varphi(Y,T)\big)(W) = -(-1)^{w(y+t)}B\big(\psi((XZ)W)(Y),T\big)\\
&\varphi\big(\psi(Z)(\psi(X)(Y)),T\big)(W) =\\
&\qquad = (-1)^{w(x+y+z+t)}B\big(\psi(W)(\psi(Z)(\psi(X)(Y))),T\big)\\
&ad^*_N(X)\big(\varphi(\psi(Z)(Y),T)\big)(W) = -(-1)^{w(y+z+t)}B\big(\psi(XW)(\psi(Z)(Y)),T\big)\\
&\varphi\big(\psi(X)(\psi(Z)(T)),Y\big)(W) =\\
&\qquad = -(-1)^{xz+yt+w(y+t)}B\big(\psi(Z)(\psi(X)(\psi(W)(Y))),T\big)\\
&ad^*_N(Z)\big(\varphi(\psi(X)(T),Y)\big)(W) =\\
&\qquad = -(-1)^{xz+yt+w(y+t)}B\big(\psi(X)(\psi(ZW)(Y)),T\big).
\end{aligned}$$

Since B is non-degenerate, it follows that

$$\begin{aligned}
\psi((XZ)W)(Y) = &(-1)^{zx+w(x+z)}\psi(W)\big(\psi(Z)(\psi(X)(Y))\big)+\\
&+ (-1)^{wz}\psi(XW)(\psi(Z)(Y)) - (-1)^{xz}\psi(Z)\big(\psi(X)(\psi(W)(Y))\big)+\\
&+ \psi(X)(\psi(ZW)(Y)).
\end{aligned}$$

Since ψ is a Malcev representation of N in M, it implies that the last condition is verified. Finally, let us prove that, $\forall_{X\in N_x}$, $\widetilde{\psi}(X)$ is a Malcev operator, that is,
$\forall_{X\in N_x, Y+f\in(M\oplus P(N^*))_y, Z+g\in(M\oplus P(N^*))_z, T+h\in(M\oplus P(N^*))_t}$,

$$\widetilde{\psi}(X)\big(((Y+f)(Z+g))(T+h)\big) =$$

$$
\begin{aligned}
&= \big((\widetilde{\psi}(X)(Y+f))(Z+g)\big)(T+h)- \\
&- (-1)^{yz}\big(\widetilde{\psi}(X)(Z+g)\big)\big((Y+f)(T+h)\big)- \\
&- (-1)^{t(y+z)}\big((\widetilde{\psi}(X)(T+h))(Y+f)\big)(Z+g)- \\
&- (-1)^{y(z+t)}\widetilde{\psi}(X)\big((Z+g)(T+h)\big)(Y+f).
\end{aligned}
$$

But

$$
\begin{aligned}
\widetilde{\psi}(X)\big(((Y+f)(Z+g))(T+h)\big) &= \psi(X)\big((YZ)T\big) + ad_N^*(X)\big(\varphi(YZ,T)\big) \\
\big((\widetilde{\psi}(X)(Y+f))(Z+g)\big)(T+h) &= \big((\psi(X)(Y))Z\big)T + \varphi\big((\psi(X)(Y))Z,T\big) \\
\big(\widetilde{\psi}(X)(Z+g)\big)\big((Y+f)(T+h)\big) &= \big(\psi(X)(Z)\big)(YT) + \varphi\big(\psi(X)(Z),YT\big) \\
\big((\widetilde{\psi}(X)(T+h))(Y+f)\big)(Z+g) &= \big((\psi(X)(T))Y\big)Z + \varphi\big((\psi(X)(T))Y,Z\big) \\
\widetilde{\psi}(X)\big((Z+g)(T+h)\big)(Y+f) &= \big(\psi(X)(ZT)\big)Y + \varphi\big(\psi(X)(ZT),Y\big).
\end{aligned}
$$

Since $\psi(X)$ is a Malcev operator then $\widetilde{\psi}(X)$ is a Malcev operator if and only if

$$
\begin{aligned}
ad_N^*(X)\big(\varphi(YZ,T)\big) &= \varphi\big((\psi(X)(Y))Z,T\big) - (-1)^{yz}\varphi\big(\psi(X)(Z),YT\big)- \\
&- (-1)^{t(y+z)}\varphi\big((\psi(X)(T))Y,Z\big) - (-1)^{y(z+t)}\varphi\big(\psi(X)(ZT),Y\big).
\end{aligned}
$$

Let us show that the last assertion is true. $\forall_{S\in N_s}$,

$$
\begin{aligned}
ad_N^*(X)\big(\varphi(YZ,T)\big)(S) &= -(-1)^{s(y+z+t)}B\big(\psi(XS)(YZ),T\big) \\
\varphi\big((\psi(X)(Y))Z,T\big)(S) &= (-1)^{s(x+y+z+t)}B\big(\psi(S)(\psi(X)(Y)Z),T\big) \\
\varphi\big(\psi(X)(Z),YT\big)(S) &= (-1)^{s(x+y+z+t)}B\big((\psi(S)(\psi(X)(Z)))Y,T\big) \\
\varphi\big((\psi(X)(T))Y,Z\big)(S) &= (-1)^{sz+t(y+z+s)}B\big(\psi(X)(Y(\psi(S)(Z))),T\big) \\
\varphi\big(\psi(X)(ZT),Y\big)(S) &= -(-1)^{ys+xz+t(y+s)}B\big(Z(\psi(X)(\psi(S)(Y))),T\big).
\end{aligned}
$$

As B is non-degenerate then

$$
\begin{aligned}
\psi(SX)(YZ) \quad &= \quad \psi(S)\big((\psi(X)(Y))Z\big) - (-1)^{yz}\big(\psi(S)(\psi(X)(Z))\big)Y \\
&\quad (-1)^{sx+yz}\psi(X)\big((\psi(S)(Z))Y\big) - (-1)^{xs}\big(\psi(X)(\psi(S)(Y))\big)Z.
\end{aligned}
$$

Since ψ is an admissible Malcev representation, then last statement is true by Proposition 3.7. We have that $\widetilde{\psi}$ is an admissible Malcev representation of N in $M\oplus P(N^*)$ and consequently $L = N\oplus M\oplus P(N^*)$ with multiplication (3.2) is a Malcev superalgebra. Let γ be an odd supersymmetric invariant bilinear form on N (not necessarily non-degenerate) and define the bilinear form $\widetilde{B} : L\times L \longrightarrow \mathbb{K}$ by

$$
\widetilde{B}(Z+X+f, W+Y+h) = B(X,Y) + \gamma(Z,W) + f(W) + (-1)^{xy}h(Z),
$$

$\forall_{(Z+X+f)\in L_x,(W+Y+h)\in L_y}$. It is immediate that $\widetilde{B}$ is odd and supersymmetric. Now we shall prove that $\widetilde{B}$ is non-degenerate. Let $Y+X+f\in L$ such that $\widetilde{B}(Y+X+f,L)=\{0\}$. As $\widetilde{B}(Y+X+f,g)=\{0\}$, $\forall_{g\in P(N^*)}$ then $Y=0$. From $\widetilde{B}(X+f,Z)=\{0\}$, $\forall_{Z\in M}$, and B is non-degenerate it follows that $X=0$. Since $\widetilde{B}(f,W)=\{0\}$, $\forall_{W\in N}$, we conclude that $f=0$. Finally, let us care about the $\widetilde{B}$-invariance. It comes straightforward that $\forall_{(W+X+f)\in L_x,(T+Y+g)\in L_y,(S+Z+h)\in L_z}$

$$\begin{aligned}
&\widetilde{B}\big((W+X+f)(T+Y+g),S+Z+h\big)=\\
&\quad=\gamma(WT,S)+B(XY+\psi(W)(Y)-(-1)^{xy}\psi(T)(X),Z)+\\
&\quad+\varphi(XY)(S)+ad^*_N(W)(g)(S)-(-1)^{xy}ad^*_N(T)(f)(S)+\\
&\quad+(-1)^{z(x+y)}h(WT)=\gamma(WT,S)+B(XY,Z)+\\
&+B(\psi(W)(Y),Z)-(-1)^{xy}B(\psi(T)(X),Z)+(-1)^{z(x+y)}B(\psi(S)(X),Y)-\\
&\quad-(-1)^{xy}g(WS)+f(TS)+(-1)^{z(x+y)}h(WT).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\widetilde{B}\big(W+X+f,(T+Y+g)(S+Z+h)\big)=\\
&\quad=\gamma(W,TS)+B(X,YZ)-(-1)^{xy}B(\psi(T)(X),Z)+\\
&\quad+(-1)^{z(x+y)}B(\psi(S)(X),Y)+f(TS)+B(\psi(W)(Y),Z)+\\
&\quad+(-1)^{z(x+y)}h(WT)-(-1)^{xy}g(WS).
\end{aligned}$$

By invariance of B and γ, we conclude the invariance of $\widetilde{B}$, completing the proof. □

A simple calculation involving the invariant scalar product reveals that the two following conditions are equivalent.

Lemma 3.10. *Let $(M=M_{\bar 0}\oplus M_{\bar 1},B)$ be an odd quadratic Malcev superalgebra and $D\in(Op_a(M))_{\bar 0}$. The following assertions are equivalent:* $\forall_{X\in M_x,Y\in M_y}$,

(i) $D^2(XY)=D(X)D(Y)+D(D(X)Y)+XD^2(Y)$;

(ii) $D^2(X)Y-D(D(X)Y)=-(-1)^{xy}\big\{D^2(Y)X-D(D(Y)X)\big\}$.

Proof. Let us assume that condition (i) is true (for condition (ii) it is proved in a similar way). Doing elementary calculations, from $\forall_{X\in M_x,Y\in M_y,Z\in M_z}$,

$$B\big(D^2(XY),Z\big)=B\big(D(D(X)Y),Z\big)+B\big(D(X)D(Y),Z\big)+B\big(XD^2(Y),Z\big)$$

we obtain

$$B\big(X, YD^2(Z)\big) = B\big(X, D(YD(Z))\big) - B\big(X, D(D(Y)Z)\big) + B\big(X, D^2(Y)Z\big).$$

Since B is non-degenerate it follows (ii) as required. $\square$

Consider $\mathbb{K}e$ the one-dimensional Lie algebra and $(\mathbb{K}e)^* = \mathbb{K}e^*$ its dual vector space.

Corollary 3.11. *Let $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ be an odd quadratic Malcev superalgebra and $D \in (Op_a(M))_{\bar{0}}$ such that*

$$D^2(X)Y - D(D(X)Y) = -(-1)^{xy}\big\{D^2(Y)X - D(D(Y)X)\big\}, \forall_{X \in M_x, Y \in M_y}.$$

Then $L = \mathbb{K}e \oplus M \oplus \mathbb{K}e^*$ with the multiplication defined by, $\forall_{a,b,\alpha,\beta \in \mathbb{K}}, \forall_{X,Y \in M}$,

$$(ae + X + \alpha e^*)(be + Y + \beta e^*) = aD(Y) - bD(X) + XY + B(D(X), Y)e^*,$$

is a Malcev superalgebra. Moreover, the bilinear form $\widetilde{B} : L \times L \longrightarrow \mathbb{K}$ defined by

$$\widetilde{B}(ae + X + \alpha e^*, be + Y + \beta e^*) = B(X, Y) + a\beta + b\alpha, \forall_{a,b,\alpha,\beta \in \mathbb{K}}, \forall_{X,Y \in M},$$

is an odd-invariant scalar product on L.

Proof. Since $D \in (Op_a(M))_{\bar{0}}$ and satisfies the property presented above it follows that $\psi : \mathbb{K}e \longrightarrow End(M)$ defined by $\psi(ae) = aD$, $\forall_{a \in \mathbb{K}}$, is an admissible Malcev representation of the one-dimensional Lie algebra $\mathbb{K}e$ in M. By last theorem we have the result. $\square$

Now we shall prove the converse of the Theorem 3.9.

Theorem 3.12. *Let $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ be a B-irreducible odd-quadratic Malcev superalgebra such that $\dim M > 1$. If $\mathfrak{z}(M) \cap M_{\bar{1}} \neq \{0\}$ then (M, B) is an odd double extension of an odd-quadratic Malcev superalgebra $(N, \widetilde{B})$ such that $\dim N = \dim M - 2$, by the one-dimensional Lie algebra.*

We shall prove the result by using a procedure similar to the one employed in case of odd-quadratic Lie superalgebras [3]. First, we will determine the odd-quadratic Malcev superalgebra $(N, \widetilde{B})$; then we will show that the odd-quadratic Malcev superalgebra (M, B) is the odd double extension of $(N, \widetilde{B})$ by the one-dimensional Lie algebra.

Proof. Let us assume that (M,B) is a B-irreducible odd-quadratic Malcev superalgebra such that $\dim M > 1$ and $\mathfrak{z}(M) \cap M_{\bar{1}} \neq \{0\}$. We set e^* a non-zero element of $\mathfrak{z}(M) \cap M_{\bar{1}}$ and denote $I = \mathbb{K}e^*$. As B is odd we have that $M_{\bar{1}} \subseteq J$, where J is the orthogonal of I with respect to B. Since B is non-degenerate and odd then there exists $e \in M_{\bar{0}}$ such that $B(e^*, e) \neq 0$. We may assume that $B(e^*, e) = 1$. As $e \notin J$ and $\dim J = \dim M - 1$ we infer that $M = J \oplus \mathbb{K}e$. Consider the two-dimensional vector subspace $A = \mathbb{K}e^* \oplus \mathbb{K}e$ of M. Since $B\mid_{A\times A}$ is non-degenerate we have $M = A \oplus N$, where N is the orthogonal of A with respect to B. It comes that $\widetilde{B} = B\mid_{N\times N}$ is non-degenerate. As B is odd we have $\mathbb{K}e^* \subseteq J$, and so $\mathbb{K}e^* \oplus N \subseteq J$. From $\dim\,(\mathbb{K}e^* \oplus N) = \dim M - 1 = \dim J$ it comes that $J = \mathbb{K}e^* \oplus N$. So N is a graded vector subspace of M contained in the graded ideal $J = N \oplus \mathbb{K}e^*$ of M. Then we have

$$XY \;=\; \alpha(X,Y) + \varphi(X,Y)e^*, \qquad \forall_{X,Y\in N},$$

where $\alpha(X,Y) \in N$ and $\varphi(X,Y) \in \mathbb{K}$. Further,

$$eX \;=\; D(X) + \psi(X)e^*, \qquad \forall_{X\in N},$$

where $D(X) \in N$ and $\psi(X) \in \mathbb{K}$. Acting as in the proof of Theorem 2.7 in [4], observing that $\forall_{X\in N_x, Y\in N_y, Z\in N_z, T\in N_t}$,

$$(XZ)(YT) = \alpha(\alpha(X,Z),\alpha(Y,T)) + \varphi(\alpha(X,z),\alpha(Y,T))e^*,$$

$$((XY)Z)T = \alpha(\alpha(\alpha(X,Y)Z)T) + \varphi(\alpha(\alpha(X,Y),Z),T)e^*,$$

and using the axioms of the definition of Malcev superalgebra we conclude that N endowed with the multiplication α is a Malcev superalgebra. It is immediate that $\widetilde{B}$ is an odd-invariant scalar product on N.

Claim. Then D is an even skew-supersymmetric Malcev operator of $(N,\widetilde{B})$ such that $\forall_{X\in M_x, Y\in M_y}$,

$$D^2(X)Y - D(D(X)Y) = -(-1)^{xy}\big\{D^2(Y)X - D(D(Y)X)\big\}. \tag{3.3}$$

Moreover, (M,B) is the odd double extension of $(N,\widetilde{B})$ by the one-dimensional Lie algebra $\mathbb{K}e$ (by means of D).

Proof of the Claim: We start by showing that D is an even skew-supersymmetric Malcev operator of $(N,\widetilde{B})$. Clearly D is a homogeneous map of degree $\bar{0}$. From the second property of definition of Malcev superalgebras $\forall_{Y\in N_y, Z\in N_z, T\in N_t}$,

$$\begin{aligned}(-1)^{yz}(eZ)(YT) = ((eY)Z)T + ((YZ)T)e+ \\ + (-1)^{y(z+t)}((ZT)e)Y + (-1)^{t(y+z)}((Te)Y)Z,\end{aligned}$$

follows that D is a Malcev operator of N. On the other hand, using the invariance of B, $B(eX,Y) = -B(X,eY)$, $\forall_{X\in N_x, Y\in N}$, we obtain

$$\widetilde{B}\big(D(X),Y\big) \quad = \quad -\widetilde{B}\big(X,D(Y)\big), \qquad \forall_{X\in N_x, Y\in N},$$

which means that $D \in \Big(Op_a(N,\widetilde{B})\Big)_{\bar{0}}$. Using the second property of definition of Malcev superalgebras $\forall_{X\in M_x, Y\in M_y}$,

$$(ee)(XY) = ((eX)e)Y + ((Xe)Y)e + (-1)^{xy}((eY)e)X + (-1)^{yx}((Ye)X)e,$$

we obtain (3.3). Notice that from $\forall_{X\in M_x, Y\in M_y}$,

$$(eX)(eY) = ((ee)X)Y + ((eX)Y)e + ((XY)e)e + (-1)^{yx}((Ye)e)X,$$

we infer condition (i) of Lemma 3.10 which is equivalent to (3.3). From $B(eX,Y) = B(e,XY)$, $\forall_{X,Y\in N}$, we get that

$$\varphi(X,Y) \quad = \quad B(D(X),Y), \qquad \forall_{X,Y\in N}.$$

Due to $B(eX,e) = B(e,Xe)$, $\forall_{X\in N_x}$, we conclude that $\psi(X) = 0$, $\forall_{X\in N_x}$. This states the claim and consequently the theorem. $\square$

Definition 3.13. *The trivial odd double extension of a Malcev superalgebra M is the odd double extension $(\mathfrak{k} = M \oplus P(M^*), \widetilde{B})$ of $\{0\}$ by M.*

Example 3.14. Consider the well known simple Malcev non-Lie algebra C with $\dim C = 7$. Let $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be a basis of C, with multiplication

$$\begin{array}{lll}
[e_1,e_2] = 2e_2 & [e_2,e_3] = 2e_7 & [e_4,e_7] = e_1 \\
[e_1,e_3] = 2e_3 & [e_2,e_4] = -2e_6 & [e_5,e_6] = -2e_4 \\
[e_1,e_4] = 2e_4 & [e_2,e_5] = e_1 & [e_5,e_7] = 2e_3 \\
[e_1,e_5] = -2e_5 & [e_3,e_4] = 2e_5 & [e_6,e_7] = -2e_2 \\
[e_1,e_6] = -2e_6 & [e_3,e_6] = e_1 & \\
[e_1,e_7] = -2e_7 & &
\end{array}$$

and the others are zero. Now, let $\{e_1^*, e_2^*, e_3^*, e_4^*, e_5^*, e_6^*, e_7^*\}$ be the correspondent basis of the dual C^*. On the trivial odd double extension $(C \oplus P(C^*), B)$ of the simple Malcev non-Lie algebra C, the non-zero values of the graded skew-symmetric multiplication for the elements of the basis are

$$\begin{array}{lll}
[e_1,e_2]=2e_2 & [e_1,e_2^*]=-2e_2^* & [e_4,e_4^*]=2e_1^* \\
[e_1,e_3]=2e_3 & [e_1,e_3^*]=-2e_3^* & [e_4,e_5^*]=2e_3^* \\
[e_1,e_4]=2e_4 & [e_1,e_4^*]=-2e_4^* & [e_4,e_6^*]=-2e_2^* \\
[e_1,e_5]=-2e_5 & [e_1,e_5^*]=2e_5^* & [e_5,e_1^*]=e_2^* \\
[e_1,e_6]=-2e_6 & [e_1,e_6^*]=2e_6^* & [e_5,e_3^*]=-2e_7^* \\
[e_1,e_7]=-2e_7 & [e_1,e_7^*]=2e_7^* & [e_5,e_4^*]=2e_6^* \\
[e_2,e_3]=2e_7 & [e_2,e_1^*]=e_5^* & [e_5,e_5^*]=2e_1^* \\
[e_2,e_4]=-2e_6 & [e_2,e_2^*]=2e_1^* & [e_6,e_1^*]=e_3^* \\
[e_2,e_5]=e_1 & [e_2,e_6^*]=2e_4^* & [e_6,e_2^*]=2e_7^* \\
[e_3,e_4]=2e_5 & [e_2,e_7^*]=-2e_3^* & [e_6,e_4^*]=-2e_5^* \\
[e_3,e_6]=e_1 & [e_3,e_1^*]=-e_6^* & [e_6,e_6^*]=-2e_1^* \\
[e_4,e_7]=e_1 & [e_3,e_3^*]=e_1^* & [e_7,e_1^*]=e_4^* \\
[e_5,e_6]=-2e_4 & [e_3,e_5^*]=-2e_4^* & [e_7,e_2^*]=-2e_6^* \\
[e_5,e_7]=2e_3 & [e_3,e_7^*]=2e_2^* & [e_7,e_3^*]=2e_5^* \\
[e_6,e_7]=-2e_2 & [e_4,e_1^*]=-e_7^* & [e_7,e_7^*]=-2e_1^*
\end{array}$$

Moreover, the non-zero values of the supersymmetric bilinear form B are

$$B(e_i,e_i^*)=1, \qquad \forall_{i\in\{1,2,3,4,5,6,7\}}.$$

Since the superjacobian $\mathrm{SJ}(e_1,e_2,e_3)=12e_7\neq 0$ (see [5] for definition) then the Malcev superalgebra $C\oplus P(C^*)$ is not a Lie superalgebra.

4. Generalized odd double extension of odd-quadratic Malcev superalgebras

In the following we shall need the notion of generalized semi-direct product of two Malcev superalgebras introduced in [4].

Proposition 4.1. *Consider two Malcev superalgebras M and V, an even linear map $\Omega : M \longrightarrow End(V)$ (not necessarily a Malcev representation), and an even skew-supersymmetric bilinear map $\zeta : M\times M \longrightarrow V$ such that*

(i) $\forall_{X\in M_x, Y\in M_y, h\in V_z, i\in V_t}$,

$$\begin{aligned}
\big(\Omega(XY)(h)\big)i - \Omega(X)\big(\Omega(Y)(h)i\big) - (-1)^{yz}\big(\Omega(X)(h)\big)\big(\Omega(Y)(i)\big)\\
+(-1)^{xy}\Omega(Y)\big(\Omega(X)(hi)\big) + (-1)^{tz+xy}\Omega(Y)\big(\Omega(X)(i)\big)h\\
+\big(\zeta(X,Y)h\big)i = 0;
\end{aligned}$$

(ii) $\forall_{X\in M_x, Z\in M_z, g\in V_y, i\in V_t}$,

$$(-1)^{yz}\big\{\zeta(X,Z)(gi)+\Omega(XZ)(gi)\big\} = -(-1)^{z(x+y)}\Omega(Z)\big(\Omega(X)(g)\big)i$$

$$+(-1)^{yz}\Omega(X)\big(\Omega(Z)(g)i\big) - (-1)^{y(z+t)}\Omega(X)\big(\Omega(Z)(i)\big)g$$
$$+(-1)^{ty+(x+y)z}\Omega(Z)\big(\Omega(X)(i)g\big);$$

(iii) $\forall_{X\in M_x, Y\in M_y, Z\in M_z, T\in M_t}$,

$$\begin{aligned}
&(-1)^{yz}\big(\zeta(X,Z)\zeta(Y,T)\big)_V + (-1)^{yz}\Omega(XZ)\big(\zeta(Y,T)\big)\\
&-(-1)^{zt+x(y+t)}\Omega(YT)\big(\zeta(X,Z)\big) + (-1)^{yz}\zeta(XZ,YT) =\\
&\quad = \zeta((XY)Z,T) - (-1)^{t(x+y+z)}\Omega(T)\big(\zeta(XY,Z)\big)\\
&\quad\ \ +(-1)^{z(x+y)+t(x+y+z)}\Omega(T)\big(\Omega(Z)(\zeta(X,Y))\big)\\
&\quad\ \ +(-1)^{x(y+z+t)}\zeta((YZ)T,X) - \Omega(X)\big(\zeta(YZ,T)\big)\\
&\quad\ \ +(-1)^{t(y+z)}\Omega(X)\big(\Omega(T)(\zeta(Y,Z))\big)\\
&\quad\ \ +(-1)^{(x+y)(z+t)}\zeta((ZT)X,Y) - (-1)^{x(y+z+t)}\Omega(Y)\big(\zeta(ZT,X)\big)\\
&\quad\ \ +(-1)^{xy}\Omega(Y)\big(\Omega(X)(\zeta(Z,T))\big)\\
&\quad\ \ +(-1)^{t(x+y+z)}\zeta((TX)Y,Z) - (-1)^{(x+y)(t+z)}\Omega(Z)\big(\zeta(TX,Y)\big)\\
&\quad\ \ +(-1)^{yz+x(y+z+t)}\Omega(Z)\big(\Omega(Y)(\zeta(T,X))\big);
\end{aligned}$$

(iv) $\forall_{X\in M_x, Y\in M_y, Z\in M_z}$,

$$\begin{aligned}
&\Omega\big((XY)Z\big) - \Omega(X)\Omega(Y)\Omega(Z) + (-1)^{x(y+z)}\Omega(YZ)\Omega(X)+\\
&\quad + (-1)^{z(x+y)}\Omega(Z)\Omega(X)\Omega(Y) - (-1)^{x(y+z)}\Omega(Y)\Omega(ZX) =\\
&\qquad = -(-1)^{x(y+z)}ad_V\big(\zeta(Y,Z)\big)\Omega(X)+\\
&\qquad + (-1)^{z(x+y)}ad_V\big(\Omega(Z)(\zeta(X,Y))\big)-\\
&\qquad - (-1)^{xy}\Omega(Y)ad_V\big(\zeta(X,Z)\big) - ad_V\big(\zeta(XY,Z)\big);
\end{aligned}$$

(v) $\forall_{X\in M_x}$, *$\Omega(X)$ is a Malcev operator of V.*

Then the $\mathbb{Z}_2$-graded vector space $M\oplus V$ endowed with the multiplication

$$\begin{aligned}
&(X+f)(Y+h) =\\
&\qquad = (XY)_M + (fh)_V + \Omega(X)(h) - (-1)^{xy}\Omega(Y)(f) + \zeta(X,Y),
\end{aligned}$$

where $(X+f)\in(M\oplus V)_x$ and $(Y+h)\in(M\oplus V)_y$, is a Malcev superalgebra. The Malcev superalgebra $M\oplus V$ is called the generalized semi-direct product of V by M (by means of Ω and ζ).

We consider $\mathbb{K}e = (\mathbb{K}e)_{\bar{1}}$ the one-dimensional abelian Lie superalgebra with even part zero and $\mathbb{K}e^*$ its dual vector space. Using a procedure familiar from Lie superalgebras we start by making a central extension.

Proposition 4.2. *Let (M, B) be an odd-quadratic Malcev superalgebra and D an odd skew-supersymmetric Malcev operator of (M, B). Let us consider the bilinear map $\varphi : M \times M \longrightarrow P(\mathbb{K}e^*)$ defined by*

$$\varphi(X, Y) \quad = \quad B(D(X), Y)e^*, \qquad \forall_{X,Y \in M}.$$

Then φ is a Malcev 2-cocycle on M with values in $P(\mathbb{K}e^*)$. Moreover, the $\mathbb{Z}_2$-graded vector space $M \oplus P(\mathbb{K}e^*)$ with the multiplication defined by

$$(X + \alpha e^*)(Y + \beta e^*) = XY + \varphi(X, Y), \qquad \forall_{(X+\alpha e^*),(Y+\beta e^*)\in(M\oplus P(\mathbb{K}e^*))},$$

is the central extension of $P(\mathbb{K}e^*)$ by M (by means of φ).

Proof. Since D is odd, we have that, $\forall_{X\in M_x, Y\in M_y}$,

$$\varphi(X, Y) = B(D(X), Y)e^* \in (\mathbb{K}e^*)_{x+y+\bar{1}} = (P(\mathbb{K}e^*))_{x+y},$$

meaning that φ is even. As D is an odd skew-supersymmetric Malcev operator of (M, B), by Lemma 3.4 it comes straightforward that φ is an even Malcev 2-cocycle on $P(\mathbb{K}e^*)$. Applying Proposition 3.5 we conclude that $M \oplus P(\mathbb{K}e^*)$ is the central extension of $P(\mathbb{K}e^*)$ by M (by means of φ) as desired. □

Now, we are in position to present the notion of generalized double extension. We proceed in two steps, a central extension following by a generalized semi-direct product.

Theorem 4.3. *Let $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ be an odd-quadratic Malcev superalgebra. Suppose that $D \in (Op_a(M))_{\bar{1}}$, $A_0 \in M_{\bar{0}}$, and $\lambda_0 \in \mathbb{K}$ such that*

$$D(A_0X) = A_0D(X) - D(A_0)X, \tag{4.1}$$

$$A_0(XY) = D(D(X)Y) + D^2(X)Y - \\ - (-1)^{xy}\{D^2(Y)X + D(D(Y)X)\}, \tag{4.2}$$

where $X \in M_x, Y \in M_y$ are arbitrary. Define a map $\Omega : (\mathbb{K}e)_{\bar{1}} \longrightarrow Op(M \oplus P(\mathbb{K}e^*))$ by $\Omega(e) = \widetilde{D}$, where $\widetilde{D} : M \oplus P(\mathbb{K}e^*) \longrightarrow M \oplus P(\mathbb{K}e^*)$ satisfies $\widetilde{D}(e^*) = 0$ and

$$\widetilde{D}(X) \quad = \quad D(X) - (-1)^x B(X, A_0)e^*, \qquad \forall_{X\in M_x}.$$

Consider the map $\zeta : \mathbb{K}e \times \mathbb{K}e \longrightarrow M \oplus P(\mathbb{K}e^*)$ defined by $\zeta(e, e) = A_0 + \lambda_0 e^*$. Then $N = \mathbb{K}e \oplus M \oplus P(\mathbb{K}e^*)$ equipped with the even skew-symmetric bilinear map on N defined by

$$ee \quad = \quad A_0 + \lambda_0 e^*,$$

$$\begin{aligned} eX &= D(X) - (-1)^x B(X, A_0)e^*, && \forall_{X \in M_x}, \\ XY &= (XY)_M + B(D(X), Y)e^*, && \forall_{X,Y \in M}, \\ e^*N &= \{0\}, \end{aligned} \tag{4.3}$$

is the generalized semi-direct product of $M \oplus P(\mathbb{K}e^*)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of Ω and ζ). Moreover, the supersymmetric bilinear form $\widetilde{B} : N \times N \longrightarrow \mathbb{K}$ defined by

$$\begin{aligned} &\widetilde{B}\mid_{M \times M} = B, \\ &\widetilde{B}(e, e^*) = 1, \\ &\widetilde{B}(M, e) = \widetilde{B}(M, e^*) = \{0\}, \\ &\widetilde{B}(e, e) = \widetilde{B}(e^*, e^*) = 0, \end{aligned}$$

is an odd invariant scalar product on N. The odd-quadratic Malcev superalgebra $(N, \widetilde{B})$ is called the generalized odd double extension of (M, B) by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of D, A_0, and λ_0).

Proof. We have to care about the several conditions of Proposition 4.1. First, we show that $\widetilde{D}$ is a Malcev operator of $M \oplus P(\mathbb{K}e^*)$. For all $(X + \alpha e^*) \in (M \oplus P(\mathbb{K}e^*))_x$, $(Y + \beta e^*) \in (M \oplus P(\mathbb{K}e^*))_y$, $(Z + \gamma e^*) \in (M \oplus P(\mathbb{K}e^*))_z$ we have to ensure that

$$\begin{aligned} \widetilde{D}\big(((X + \alpha e^*)(Y + \beta e^*))(Z + \gamma e^*)\big) = &\big(\widetilde{D}(X + \alpha e^*)(Y + \beta e^*)\big)(Z + \gamma e^*) \\ &-(-1)^{xy}\widetilde{D}(Y + \beta e^*)\big((X + \alpha e^*)(Z + \gamma e^*)\big) \\ &-(-1)^{z(x+y)}\big(\widetilde{D}(Z + \gamma e^*)(X + \alpha e^*)\big)(Y + \beta e^*) \\ &-(-1)^{x(y+z)}\widetilde{D}\big((Y + \beta e^*)(Z + \gamma e^*)\big)(X + \alpha e^*), \end{aligned}$$

which is equivalent to

$$\begin{aligned} D((XY)Z) - (-1)^{x+y+z}&B((XY)Z, A_0)e^* = \\ &= (D(X)Y)Z + B(D(D(X)Y), Z)e^* \\ &\quad -(-1)^{xy}D(Y)(XZ) - (-1)^{xy}B(D^2(Y), XZ)e^* \\ &\quad -(-1)^{z(x+y)}(D(Z)X)Y - (-1)^{z(x+y)}B(D(D(Z)X), Y)e^* \\ &\quad -(-1)^{x(y+z)}D(YZ)X - (-1)^{x(y+z)}B(D^2(YZ), X)e^*. \end{aligned}$$

Since D is a Malcev operator, it remains to see that

$$\begin{aligned} -(-1)^{x+y+z}B((XY)Z, A_0) = &B(D(D(X)Y), Z) - (-1)^{xy}B(D^2(Y), XZ) \\ &-(-1)^{z(x+y)}B(D(D(Z)X), Y) - (-1)^{x(y+z)}B(D^2(YZ), X). \end{aligned}$$

As D is skew-supersymmetric, since B is odd and non-degenerate, the last condition is assertion (4.2). Condition (iii) is trivial and (iv) comes from (4.1). Relative to condition (i), we have to verify that for all $(Z+\gamma e^*) \in (M \oplus \mathbb{K}e^*)_z$, $(T+\delta e^*) \in (M \oplus \mathbb{K}e^*)_t$,

$$\begin{aligned}
&\big(\Omega(ee)(Z+\gamma e^*)\big)(T+\delta e^*) - \Omega(e)\big(\Omega(e)(Z+\gamma e^*)(T+\delta e^*)\big) \\
&-(-1)^z\big(\Omega(e)(Z+\gamma e^*)\big)\big(\Omega(e)(T+\delta e^*)\big) - \Omega(e)\big(\Omega(e)((Z+\gamma e^*)(T+\delta e^*))\big) \\
&-(-1)^{tz}\Omega(e)\big(\Omega(e)(T+\delta e^*)\big)(Z+\gamma e^*) + \big(\zeta(e,e)(Z+\gamma e^*)\big)(T+\delta e^*) = 0,
\end{aligned}$$

that is

$$\begin{aligned}
&-D(D(Z)T) - (-1)^z D(Z)D(T) - D^2(ZT) - (-1)^{zt}D^2(T)Z + (A_0Z)T \\
&-(-1)^{z+t}B(D(Z)T, A_0)e^* - (-1)^{z+t}B(D(ZT), A_0)e^* + B(D(A_0Z), T)e^* \\
&\qquad - (-1)^z B(D^2(Z), D(T))e^* - (-1)^{zt}B(D^3(T), Z)e^* = 0.
\end{aligned}$$

From (4.1) and (4.2), since B is odd and non-degenerate, and D is an odd skew-supersymmetric map, this expression comes straightforward. Finally, for condition (ii) we have to prove that for all $(Y+\beta e^*) \in (M \oplus P(\mathbb{K}e^*))_y$, $(T+\delta e^*) \in (M \oplus P(\mathbb{K}e^*))_t$,

$$\begin{aligned}
(-1)^y\big\{\zeta(e,e)((Y+\beta e^*)(T+\delta e^*)) + \Omega(ee)((Y+\beta e^*)(T+\delta e^*))\big\} \\
= (-1)^y\Omega(e)\big(\Omega(e)(Y+\beta e^*)\big)(T+\delta e^*) \\
+(-1)^y\Omega(e)\big(\Omega(e)(Y+\beta e^*)(T+\delta e^*)\big) \\
-(-1)^{y+yt}\Omega(e)\big(\Omega(e)(T+\delta e^*)\big)(Y+\beta e^*) \\
-(-1)^{y+yt}\Omega(e)\big(\Omega(e)(T+\delta e^*)(Y+\beta e^*)\big),
\end{aligned}$$

which is

$$\begin{aligned}
A_0(YT) + B(D(A_0), YT)e^* = {} & D^2(Y)T + B(D^3(Y), T)e^* \\
& +D(D(Y)T) + (-1)^{y+t}B(D(Y)T, A_0)e^* \\
& -(-1)^{yt}\big\{D^2(T)Y + B(D^3(T), Y)e^*\big\} \\
& -(-1)^{yt}\big\{D(D(T)Y) + (-1)^{y+t}B(D(T)Y, A_0)e^*\big\}.
\end{aligned}$$

From (4.1) and (4.2), since B is odd and non-degenerate, and D is an odd skew-supersymmetric map, we obtain assertion (ii) of Proposition 4.1. It is easy to see that $\mathbb{K}e \oplus M \oplus P(\mathbb{K}e^*)$ equipped with multiplication defined above is the generalized semi-direct product of $M \oplus P(\mathbb{K}e^*)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of Ω and ζ). $\square$

Next, we will show the inverse of the Theorem 4.3.

Theorem 4.4. *Let $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ be a B-irreducible odd-quadratic Malcev superalgebra such that $\dim M > 1$. If $\mathfrak{z}(M) \cap M_{\bar{0}} \neq \{0\}$ then (M, B) is a generalized odd double extension of an odd-quadratic Malcev superalgebra $(N, \widetilde{B})$ such that $\dim N = \dim M - 2$ by the one-dimensional Lie superalgebra with even part zero.*

Following the usual procedure, first we will determine the odd-quadratic Malcev superalgebra $(N, \widetilde{B})$, after we will show that the odd-quadratic Malcev superalgebra (M, B) is the generalized odd double extension of $(N, \widetilde{B})$ by the one-dimensional Lie superalgebra with even part zero.

Proof. Let us assume that (M, B) is a B-irreducible odd-quadratic Malcev superalgebra such that $\dim M > 1$ and $\mathfrak{z}(M) \cap M_{\bar{0}} \neq \{0\}$. We consider e^* a non-zero element of $\mathfrak{z}(M) \cap M_{\bar{0}}$ and denote $I = \mathbb{K}e^*$. As B is odd we have $M_{\bar{0}} \subseteq J$, where J is the orthogonal of I with respect to B. Since B is non-degenerate and odd then there exists $e \in M_{\bar{1}}$ such that $B(e^*, e) \neq 0$. We may assume that $B(e^*, e) = 1$. As $e \notin J$ and $\dim J = \dim M - 1$ we infer that $M = J \oplus \mathbb{K}e$. Consider the two-dimensional vector subspace $A = \mathbb{K}e^* \oplus \mathbb{K}e$ of M. Since $B\mid_{A \times A}$ is non-degenerate we have $M = A \oplus N$, where N is the orthogonal of A with respect to B. It comes that $\widetilde{B} = B\mid_{N \times N}$ is non-degenerate. As B is odd we have $\mathbb{K}e^* \subseteq J$, and so $\mathbb{K}e^* \oplus N \subseteq J$. From $\dim(\mathbb{K}e^* \oplus N) = \dim M - 1 = \dim J$ it comes that $J = \mathbb{K}e^* \oplus N$. So N is a graded vector subspace of M contained in the graded ideal $J = N \oplus \mathbb{K}e^*$ of M. Then we have

$$XY = \alpha(X, Y) + \varphi(X, Y)e^*, \qquad \forall_{X,Y \in N},$$

where $\alpha(X, Y) \in N$ and $\varphi(X, Y) \in \mathbb{K}$. Further,

$$eX = D(X) + \psi(X)e^*, \qquad \forall_{X \in N},$$

where $D(X) \in N$ and $\psi(X) \in \mathbb{K}$. As usually, we conclude that N provided with the multiplication α is a Malcev superalgebra and $\widetilde{B}$ is an odd-invariant scalar product on N. Since $e \in M_{\bar{1}}$ then ee is not necessarily zero, and we may write

$$ee = X_0 + \lambda_0 e^*,$$

where $X_0 \in N_{\bar{0}}$ and $\lambda_0 \in \mathbb{K}$.

Claim. Then D is an odd skew-supersymmetric Malcev operator of $(N, \widetilde{B})$ such that $\forall_{X \in N_x, Y \in N_y}$,

$$D(A_0 X) = A_0 D(X) - D(A_0)X, \tag{4.4}$$

$$A_0(XY) = D(D(X)Y) + D^2(X)Y - (-1)^{xy}\big\{D^2(Y)X + D(D(Y)X)\big\}. \tag{4.5}$$

Furthermore, (M, B) is the generalized odd double extension of $(N, \widetilde{B})$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of D and A_0).
Proof of Claim: We start by showing that D is an odd skew-supersymmetric Malcev operator of $(N, \widetilde{B})$. Clearly, D is a homogeneous map of degree $\bar{1}$. From the axiomatic of Malcev superalgebras $\forall_{Y \in N_y, Z \in N_z, T \in N_t}$,

$$\begin{aligned}(-1)^{yz}(eZ)(YT) &= ((eY)Z)T + (-1)^{y+z+t}((YZ)T)e \\ &\quad +(-1)^{(\bar{1}+y)(z+t)}((ZT)e)Y + (-1)^{t(\bar{1}+y+z)}((Te)Y)Z,\end{aligned}$$

follows that D is a Malcev operator of N. The invariance of B in M, $\forall_{X \in N_x, Y \in N}$, $B(eX, Y) = -(-1)^x B(X, eY)$ enables us to say that the odd Malcev operator D of $(N, \widetilde{B})$ is skew-supersymmetric. The condition (4.5) follows from the second property of definition of Malcev superalgebras $\forall_{X \in M_x, Y \in M_y}$,

$$\begin{aligned}(-1)^x(ee)(XY) &= ((eX)e)Y - (-1)^{x+y}((Xe)Y)e \\ &\quad -(-1)^{x+y+xy}((eY)e)X + (-1)^{yx}((Ye)X)e.\end{aligned}$$

Moreover, using $\forall_{X \in M_x}$,

$$-(ee)(eX) = ((ee)e)X + (-1)^x((ee)X)e + ((eX)e)e + (-1)^x((Xe)e)e,$$

we conclude (4.4). The fact that $B(eX, Y) = B(e, XY)$, $\forall_{X,Y \in N}$, implies that

$$\varphi(X, Y) = B(D(X), Y), \qquad \forall_{X,Y \in N}.$$

Using $B(eX, e) = -(-1)^x B(X, ee)$, $\forall_{X \in N_x}$, we obtain that $\psi(X) = -(-1)^x B(X, A_0)$, $\forall_{X \in N_x}$. This permits to conclude the claim and consequently the theorem. $\square$

5. Odd-quadratic Malcev superalgebras with reductive even part

The results concerning odd-quadratic Malcev superalgebras with reductive even part has now precisely the same formulation as in Lie superalgebras context. We shall omit the proofs, since a careful analysis of the proofs of the results in Lie case [3] reveals that they also work in Malcev case.

Definition 5.1. *A Malcev algebra M is reductive if $M = \mathfrak{z}(M) \oplus s$, where s is the greatest semisimple ideal of M.*

Proposition 5.2. *Let* $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ *be an odd-quadratic Lie superalgebra. Then* $\mathfrak{z}(M_{\bar{0}}) = \mathfrak{z}(M) \cap M_{\bar{0}}$*. Furthermore, if* $M_{\bar{0}}$ *is a reductive Malcev algebra, then* $\mathfrak{z}(M_{\bar{0}}) = \{0\}$ *if and only if* $\mathfrak{z}(M) = \{0\}$*.*

Corollary 5.3. *Let* $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ *be a* B*-irreducible odd-quadratic Malcev superalgebra with reductive Malcev algebra* $M_{\bar{0}}$*. Suppose that* M *is neither simple with* $\dim M \neq 2$ *nor abelian of dimension* 2*. Then* $\mathfrak{z}(M) = \mathfrak{z}(M_{\bar{0}}) = \mathfrak{z}(M) \cap M_{\bar{0}}$*.*

For odd-quadratic Malcev superalgebras with reductive even part, we still have the following special property, the action of the even part in the odd part is completely reducible.

Lemma 5.4. *Consider* $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ *an odd-quadratic Malcev superalgebra such that* $M_{\bar{0}}$ *is a reductive Malcev algebra. Then* $M_{\bar{1}}$ *is a* $M_{\bar{0}}$*-module completely reducible.*

We can characterize the minimal graded ideals of the B-irreducible odd-quadratic Malcev superalgebras with reductive even part as follows.

Proposition 5.5. *Consider* $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ *a non-simple* B*-irreducible odd-quadratic Malcev superalgebra such that* $M_{\bar{0}}$ *is a reductive Malcev algebra. Then a graded ideal* I *of* M *is minimal if and only if* $I \subseteq \mathfrak{z}(M)$ *and* $\dim I = 1$ *or* I *is a non trivial irreducible* s*-submodule of* $M_{\bar{1}}$ *such that* $M_{\bar{1}} I = \{0\}$*, where* s *is the greatest semisimple ideal of* $M_{\bar{0}}$*.*

The next auxiliary lemma will help us to prove the following proposition.

Lemma 5.6. *Let* $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ *be a non-simple* B*-irreducible odd-quadratic Malcev superalgebra such that* $M_{\bar{0}}$ *is a semisimple Malcev algebra. Suppose that* I *is a non trivial irreducible* $M_{\bar{0}}$*-submodule of* $M_{\bar{1}}$ *such that* $M_{\bar{1}} I = \{0\}$*. If we denote* $s' = M_{\bar{0}} \cap I^{\perp}$ *then* $M_{\bar{0}} = s' \oplus s$*, where* s *is a semisimple ideal of* $M_{\bar{0}}$*. Moreover,* $sI = I$ *and* $\dim I = \dim s = \dim\ sM_{\bar{1}}$*.*

The next result says that the non trivial superalgebras with center zero, are precisely the trivial odd double extensions of a simple Malcev algebra.

Proposition 5.7. *Consider* $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ *a* B*-irreducible odd-quadratic Malcev superalgebra such that* $M_{\bar{0}}$ *is a reductive Malcev algebra, that it is neither zero nor simple. Then the following assertions are equivalent:*

(i) $\mathfrak{z}(M) = \{0\}$*;*

(ii) $\mathfrak{z}(M_{\bar{0}}) = \{0\}$;

(iii) $M = s \oplus P(s^*)$, *where s is a simple Malcev algebra.*

6. Inductive description of odd-quadratic Malcev superalgebras with reductive even part

It is known that a simple Malcev superalgebra is either a simple Lie superalgebra or is a simple Malcev algebra [20], and that a simple Malcev algebra does not admit an odd-invariant scalar product. In following example we recall the family of classical simple Lie superalgebras admitting an odd-invariant scalar product [19].

Example 6.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded vector space with $\dim\ V_{\bar{0}} = \dim\ V_{\bar{1}} = n$ and an odd linear map $c : V \longrightarrow V$ such that $c^2 = -I$ (where, as usually, I represents the identity of V). Denote by $L(c)$ the $\mathbb{Z}_2$-graded subalgebra of $pl(V)$ formed by the elements which leave c invariant. It corresponds to the subalgebra of spl (n, n) defined by

$$L(n) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \in \text{ spl } (n,n) \ : \ A, B \in gl(n) \right\}.$$

The commutator algebra of $L(n)$ is determined by

$$d(n) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \in \text{ spl } (n,n) \ : \ A \in gl(n), B \in sl(n) \right\}$$

and $\mathbb{K}I_{2n}$ is a graded ideal of $d(n)$. The Lie superalgebra $d(n)/\mathbb{K}I_{2n}$ is isomorphic to the algebra (f,d) of Gell-Mann, Michel, and Radicati. Recall that (f,d) is the Lie superalgebra such that the even subspace is equal to $sl(n) \times \{0\}$ and the odd subspace is equal to $\{0\} \times sl(n)$. The multiplication on (f,d) is defined by

$$[(X,Y),(Z,W)] \quad = \quad \Big([X,Z] + \{Y,W\} - \frac{2}{n}\text{ tr }(YW)I_n, [X,W] + [Y,Z]\Big),$$

for every $X, Y, Z, W \in sl(n)$, where $[,]$ (respectively, $\{,\}$) denotes the commutator (respectively, anticommutator) of two matrices. For $n \geq 3$, the Lie superalgebra (f,d) is simple and the bilinear form $B : \Big(sl(n) \times sl(n)\Big) \times \Big(sl(n) \times sl(n)\Big) \longrightarrow \mathbb{K}$ defined by

$$B\Big((X,Y),(Z,W)\Big) \quad = \quad \text{tr }(XW + YZ), \qquad \forall_{X,Y,Z,W \in sl(n)},$$

is an odd-invariant scalar product on (f,d), so $((f,d),B)$ is an odd-quadratic Lie superalgebra.

Now we state our must important theorem concerning odd-quadratic Malcev superalgebras. Let $\mathfrak{U}$ be the set formed by $\{0\}$ and the simple Lie superalgebras $d(n)/\mathbb{K}I_{2n}$ $(n \geq 3)$.

Theorem 6.2. *Let $(M = M_{\bar{0}} \oplus M_{\bar{1}}, B)$ be an odd-quadratic Malcev superalgebra such that $M_{\bar{0}}$ is a reductive Malcev algebra. Then M is either an element of $\mathfrak{U}$ or is obtained from a finite number of elements of $\mathfrak{U}$ by a finite sequence of generalized odd double extensions by the one-dimensional Lie superalgebra, and/or by trivial odd double extensions of either a simple Lie algebra or the simple Malcev non-Lie algebra C $(\dim C = 7)$, and/or by orthogonal direct sums.*

Proof. Applying the procedure familiar from Lie superalgebras, we proceed by induction on the even dimension of M. If $\dim M = 0$ then $M = \{0\} \in \mathfrak{U}$. If $\dim M = 2$ then M is a Lie superalgebra [1]. Consequently M is either the abelian two-dimensional odd-quadratic Malcev superalgebra which is a trivial odd double extension of $\{0\}$ by the one-dimensional Lie algebra or the two-dimensional odd-quadratic Malcev superalgebra which is a generalized odd double extension of $\{0\}$ by the one-dimensional Lie superalgebra with even part zero. Suppose that the theorem is true for $\dim M < n$, with $n \geq 4$. We consider $\dim M = n$. We have to analyze two cases.

First case: Suppose that M is B-irreducible. If $\mathfrak{z}(M) = \{0\}$ we apply Proposition 5.7 to infer that M is a trivial odd double extension of simple Malcev algebras. It is known that a simple Malcev algebra is either a simple Lie algebra or is the simple Malcev non-Lie algebra C $(\dim C = 7)$ [11, 12, 16, 17]. If $\mathfrak{z}(M) \neq \{0\}$, by Corollary 5.3 we infer that $\mathfrak{z}(M) \cap M_{\bar{0}} \neq \{0\}$. Applying Theorem 4.4 we get that M is a generalized odd double extension of an odd-quadratic Malcev superalgebra N by the one-dimensional Lie superalgebra. In this case, $\dim N = \dim M - 2 < n$ and the even part of N is a reductive Malcev algebra, so applying the induction hypothesis to N we infer the theorem for M.

Second case: Now we assume that M is not B-irreducible. In view of Proposition 1.10, $M = \bigoplus_{k=1}^{m} M_k$, where $\{M_k | 1 \leq k \leq m\}$ is a set of B-irreducible graded ideals of M such that $B(M_k, M_{k'}) = \{0\}$, for all $k, k' \in \{1, \ldots, m\}$ and $k \neq k'$, and $(M_k)_{\bar{0}}$ is a reductive Malcev algebra, whenever $k \in \{1, \ldots, m\}$. We apply the result to M_k, for all $k \in \{1, \ldots, m\}$. □

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