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RESEARCH ARTICLE

Free commutative dimonoids

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on the occasion of his 60-th birthday

Abstract. We construct a free commutative dimonoid and characterize the least idempotent congruence on this dimonoid.

1. Introduction

Jean-Louis Loday introduced the notion of a dimonoid [1]. Dimonoids are a tool to study Leibniz algebras. A dimonoid is a set equipped with two binary associative operations satisfying some axioms (see below). If the operations of a dimonoid coincide, then the dimonoid becomes a semigroup. The first result about dimonoids is the description of the free dimonoid generated by a given set [1]. T. Pirashvili [2] introduced the notion of a duplex which generalizes the notion of a dimonoid and constructed a free duplex. In [3] it is proved that every commutative dimonoid is a semilattice of archimedean subdimonoids.

In this paper we construct a free commutative dimonoid (Theorem 3), characterize the least idempotent congruence on this dimonoid and the classes of this congruence (Theorem 4). Also we describe the free commutative dimonoids of the small ranks (Propositions 3 and 4). In section 4 we give some properties of commutative dimonoids.

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2. Preliminaries

A set D equipped with two binary associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$, is called a dimonoid.

A map f from a dimonoid D_1 to a dimonoid D_2 is a homomorphism, if $(x \prec y)f = xf \prec yf$, $(x \succ y)f = xf \succ yf$ for all $x, y \in D_1$. If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then corresponding congruence on D_1 we denote by Δ_f.

A subset T of a dimonoid $(D, \prec, \succ)$ is called a subdimonoid, if for any $a, b \in D$, a, $b \in T$ implies $a \prec b$, $a \succ b \in T$.

Now we give the necessary information about varieties of dimonoids.

A class H of algebraic systems is called a variety, if there exists such family $\wp$ of identities of a signature Ω that H consists from that and only that systems of the signature Ω in which all the formulas from $\wp$ are true.

Let H be some class of algebraic systems. We call an arbitrary algebraic system H' a H-system, if $H' \in H$.

Theorem 1. *(Birkhoff [4]) A nonempty class H of algebraic systems is a variety if and only if the following conditions hold:*

a) the Cartesian product of an arbitrary sequence of H-systems is a H-system,

b) any subsystem of an arbitrary H-system is a H-system,

c) any homomorphic image of an arbitrary H-system is a H-system.

Observe that the class Dim of all dimonoids satisfies the conditions of Birkhoff's theorem and therefore it is a variety. Indeed, if suppose

$$\wp = \{(x \prec y) \prec z = x \prec (y \succ z), (x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z = x \succ (y \succ z)\}, \quad \Omega = \{\prec, \succ\},$$

then $H = Dim$ is a variety.

Let U be a dimonoid and let R be some class of dimonoids. A nonempty set X of some elements from U is called independent in U with respect to the class R, if an arbitrary map from X into any R-dimonoid M can be extended to a homomorphism from $\bar{X}$ into M, where $\bar{X}$ is a subdimonoid generated by the elements of X in U.

A dimonoid U is called free concerning the class R, if in U there exists a set X of elements which is independent with respect to R and which generates the dimonoid U. The set X satisfying these properties is called a R-free basis of the dimonoid U. The dimonoid U is called a free dimonoid of rank m in the class R, if $U \in R$ and in U there exists a R-free basis of cardinality m.

The next assertion follows from Malchev's book [5].

Proposition 1. *If in the class R there exist free dimonoids of rank m, then all they are isomorphic and any R-dimonoid having a generating set of cardinality m is a homomorphic image of the free dimonoid of rank m in R. In particular, if in R there exist free dimonoids of an arbitrary rank, then every R-dimonoid U is a homomorphic image of the free dimonoid of rank $|U|$ in R.*

The free dimonoids in the class Dim of all dimonoids are called absolutely free. Note that the absolutely free dimonoid was constructed by Loday [1]. It is clear that the variety Dim completely is defined by the absolutely free dimonoids.

A variety R is called minimal, if R contains a dimonoid with more than 1 element, and there are not others subvarieties in R, except R and the trivial variety (containing 1-element dimonoids only). For any class R let $\hat{R}$ be a minimal variety which contains the class R.

The next assertion follows from Malchev's book [5].

Proposition 2. *A dimonoid U is free in some class if and only if it has an independent generating set of elements. In this case the dimonoid U is free in the variety $\hat{U}$.*

A dimonoid $(D, \prec, \succ)$ will be called a commutative (idempotent) dimonoid, if both semigroups $(D, \prec)$ and $(D, \succ)$ are commutative (idempotent).

Observe that the class of commutative dimonoids is a subvariety of the variety Dim. A dimonoid which is free in the variety of commutative dimonoids will be called a free commutative dimonoid.

We finish this section with the formulations of some results from [3].

Lemma 1. *([3], Lemma 2) In a commutative dimonoid $(D, \prec, \succ)$ the equalities*

$$(x \prec y) \prec z = x \prec (y \succ z) =$$
$$= (x \succ y) \prec z = x \succ (y \prec z) =$$
$$= (x \prec y) \succ z = x \succ (y \succ z)$$

hold for all $x, y, z \in D$.

From Lemma 1 it follows that the operations $\prec$ and $\succ$ of a commutative dimonoid $(D, \prec, \succ)$ are indistinguishable for three and more multipliers and the product of these elements doesn't depend on the parenthesizing.

A commutative idempotent semigroup will be called a semilattice. If ρ is a congruence on the dimonoid $(D, \prec, \succ)$ such that $(D, \prec, \succ)/_{\rho}$ is an idempotent dimonoid, then we say that ρ is an idempotent congruence.

Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\prec$, $a, b \in D$. We say that a $\prec$-divide b and write $a_{\prec}|b$, if there exists such element x from $(D, \prec)$ with an identity that $a \prec x = b$.

As usual N denotes the set of positive integers.

Let $(D, \prec, \succ)$ be a dimonoid, $a \in D$, $n \in N$. Denote by a^n the degree n of an element a concerning the operation $\prec$. Define a relation η on the dimonoid $(D, \prec, \succ)$ with a commutative operation $\prec$ by

$$\begin{gathered} a\eta b \text{ if and only if there exist positive integers} \\ m,\ n,\ m \neq 1, n \neq 1 \text{ such that } a_{\prec}|b^m,\ b_{\prec}|a^n. \end{gathered}$$

Theorem 2. *([3], Theorem 1) The relation η on the dimonoid $(D, \prec, \succ)$ with a commutative operation $\prec$ is the least idempotent congruence, and $(D, \prec, \succ)/_{\eta}$ is a commutative idempotent dimonoid which is a semilattice.*

3. Constructions

In this section we construct a free commutative dimonoid, describe the least idempotent congruence η on this dimonoid and characterize the corresponding classes of the congruence. We also consider separately the free commutative dimonoids of the rank 1 and 2.

Let A be an alphabet, $F[A]$ be a free commutative semigroup over A, G be a set of non-ordered pairs (p, q), $p, q \in A$. Define the operations $\prec$ and $\succ$ on the set $F[A] \bigcup G$ by

$$a_1...a_m \prec b_1...b_n = a_1...a_m b_1...b_n,$$

$$a_1...a_m \succ b_1...b_n = \begin{cases} a_1...a_m b_1...b_n,\ mn > 1, \\ (a_1, b_1),\ m = n = 1, \end{cases}$$

$$a_1...a_m \prec (p, q) = a_1...a_m \succ (p, q) = a_1...a_m pq,$$

$$(p, q) \prec a_1...a_m = (p, q) \succ a_1...a_m = pqa_1...a_m,$$

$$(p, q) \prec (r, s) = (p, q) \succ (r, s) = pqrs$$

for all $a_1...a_m,\ b_1...b_n \in F[A]$, $(p, q), (r, s) \in G$. An immediate verification shows that axioms of a dimonoid hold concerning the operations $\prec, \succ$

and thus, $(F[A]\bigcup G, \prec, \succ)$ is a dimonoid. It is clear that the operations $\prec, \succ$ are commutative.

Theorem 3. *$(F[A]\bigcup G, \prec, \succ)$ is a free commutative dimonoid.*

Proof. Show that $(F[A]\bigcup G, \prec, \succ)$ is free.

Let $(T, \prec', \succ')$ be an arbitrary commutative dimonoid, $\alpha : A \to T$ an arbitrary map. Define a map

$$\theta : (F[A]\bigcup G, \prec, \succ) \to (T, \prec', \succ') : w \mapsto w\theta,$$

assuming

$$w\theta = \begin{cases} a_1\alpha \prec' \ldots \prec' a_m\alpha, \ w = a_1 \ldots a_m, \\ p\alpha \succ' q\alpha, \ w = (p, q) \end{cases}$$

for all $w \in F[A]\bigcup G$.

We show that θ is a homomorphism. For arbitrary elements $a_1 \ldots a_m, \ b_1 \ldots b_n \in F[A]$, $(p, q), (r, s) \in G$ we obtain

$$(a_1 \ldots a_m \prec b_1 \ldots b_n)\theta = (a_1 \ldots a_m b_1 \ldots b_n)\theta =$$

$$= a_1\alpha \prec' \ldots \prec' a_m\alpha \prec' b_1\alpha \prec' \ldots \prec' b_n\alpha =$$

$$= (a_1 \ldots a_m)\theta \prec' (b_1 \ldots b_n)\theta,$$

$$(a_1 \ldots a_m \prec (p, q))\theta = (a_1 \ldots a_m pq)\theta =$$

$$= a_1\alpha \prec' \ldots \prec' a_m\alpha \prec' p\alpha \prec' q\alpha =$$

$$= (a_1 \ldots a_m)\theta \prec' p\alpha \prec' q\alpha =$$

$$= (a_1 \ldots a_m)\theta \prec' (p\alpha \succ' q\alpha) =$$

$$= (a_1 \ldots a_m)\theta \prec' (p, q)\theta,$$

$$((p, q) \prec (r, s))\theta = (pqrs)\theta =$$

$$= p\alpha \prec' q\alpha \prec' r\alpha \prec' s\alpha =$$

$$= (p\alpha \succ' q\alpha) \prec' (r\alpha \succ' s\alpha) =$$

$$= (p, q)\theta \prec' (r, s)\theta$$

by Lemma 1. So, $(w_1 \prec w_2)\theta = w_1\theta \prec' w_2\theta$ for all $w_1, w_2 \in F[A]\bigcup G$. If $mn > 1$, then

$$(a_1 \ldots a_m \succ b_1 \ldots b_n)\theta = (a_1 \ldots a_m b_1 \ldots b_n)\theta =$$

$$= a_1\alpha \prec' \ldots \prec' a_m\alpha \prec' b_1\alpha \prec' \ldots \prec' b_n\alpha =$$

$$= (a_1...a_m)\theta \succ' (b_1...b_n)\theta$$

by Lemma 1. In the case $m = n = 1$,

$$(a_1 \succ b_1)\theta = (a_1, b_1)\theta =$$

$$= a_1\alpha \succ' b_1\alpha = a_1\theta \succ' b_1\theta.$$

Moreover,

$$(a_1...a_m \succ (p, q))\theta = (a_1...a_m pq)\theta =$$

$$= a_1\alpha \prec' ... \prec' a_m\alpha \prec' p\alpha \prec' q\alpha =$$

$$= (a_1...a_m)\theta \prec' p\alpha \prec' q\alpha =$$

$$= (a_1...a_m)\theta \succ' (p\alpha \succ' q\alpha) =$$

$$= (a_1...a_m)\theta \succ' (p, q)\theta,$$

$$((p, q) \succ (r, s))\theta = (pqrs)\theta =$$

$$= p\alpha \prec' q\alpha \prec' r\alpha \prec' s\alpha =$$

$$= p\alpha \succ' q\alpha \succ' r\alpha \succ' s\alpha =$$

$$= (p, q)\theta \succ' (r, s)\theta$$

by Lemma 1.

So, $(w_1 \succ w_2)\theta = w_1\theta \succ' w_2\theta$ for all $w_1, w_2 \in F[A] \bigcup G$. □

Now we describe the least idempotent congruence η (see section 2) on the free commutative dimonoid and characterize the corresponding classes of this congruence.

Recall that N denotes the set of positive integers. Define the operations $\prec$ and $\succ$ on the set $N \bigcup \{\tilde{2}\}$ by

$$m \prec n = m + n,$$

$$m \prec \tilde{2} = \tilde{2} \prec m = m \succ \tilde{2} = \tilde{2} \succ m = m + 2,$$

$$m \succ n = \begin{cases} \tilde{2}, \ m = n = 1, \\ m + n \quad \text{otherwise}, \end{cases}$$

$$\tilde{2} \prec \tilde{2} = \tilde{2} \succ \tilde{2} = 4$$

for all $m, \ n \in N$. The set $N \bigcup \{\tilde{2}\}$ with the operations $\prec$ and $\succ$ is a dimonoid. We denote the dimonoid obtained by $N_{(\tilde{2})}$.

Define the operation $\prec$ on the set $N^2 \bigcup \{1\}$ by

$$(m, n) \prec (p, l) = (m + p, n + l),$$

$$(m,n) \prec 1 = 1 \prec (m,n) =$$
$$= (m+1,\, n+1), 1 \prec 1 = (2,\, 2)$$

for all $(m,n),\ (p,l) \in N^2$. The set $N^2 \bigcup \{1\}$ concerning this operation is a semigroup. We denote by $N^2_{(1)}$ this semigroup.

Denote by N^k the Cartesian product of k copies of the additive semigroup of positive integers.

For every $w \in F[A]$ the set of all elements $x \in A$ occurring in w will be denoted by $c(w)$ and assume

$$d(u) = \begin{cases} \{p,q\},\ u = (p,q) \in G, \\ c(u),\ u \in F[A] \end{cases}$$

for all $u \in F[A] \bigcup G$. The equivalence

$$w_1 \eta w_2 \Leftrightarrow d(w_1) = d(w_2)$$

for all $w_1, w_2 \in (F[A] \bigcup G, \prec, \succ)$ follows immediately from Theorem 2. Denote by F the dimonoid $(F[A] \bigcup G, \prec, \succ)/_{\eta}$.

Theorem 4. *The dimonoid F is a semilattice isomorphic to the semilattice $\Omega(A)$ of nonempty finite subsets of the set A with respect to the operation of a union. Let F_w be a class of the congruence η on the dimonoid $(F[A] \bigcup G, \prec, \succ)$ with a representative $w \in F_w$. Then*

1) if $|d(w)| = 1$, then $F_w \cong N_{(\tilde{2})}$,

2) if $|d(w)| = 2$, then $F_w \cong N^2_{(1)}$,

3) if $|d(w)| = k \geq 3$, then $F_w \cong N^k$.

Proof. It is immediate to cheek that the map

$$d : F \to \Omega(A) : F_w \mapsto d(w)$$

is an isomrphism.

Let $|d(w)| = 1$ and let $d(w) = \{x\}$. It is easy to cheek that the map

$$\alpha_1 : F_w \to N_{(\tilde{2})} :\ \ u \mapsto u\alpha_1 = \begin{cases} s,\ u = x^s, \\ \tilde{2},\ u = (x,x) \end{cases}$$

is an isomorphism.

If $|d(w)| = 2$, $d(w) = \{x,y\}$ and $u = x^{s_1}y^{s_2}$ is the canonical form of a word $u \in F_w \backslash \{(x,y)\}$, $s_1, s_2 \in N$, then we can show that the map

$$\alpha_2 : F_w \to N^2_{(1)} :\ \ v \mapsto v\alpha_2 = \begin{cases} (s,t),\ \ v = x^s y^t, \\ 1,\ \ v = (x,y) \end{cases}$$

is an isomorphism.

Finally, let $|d(w)| = k \geq 3$ and let $u = x_1^{p_1}x_2^{p_2}...x_k^{p_k}$ be the canonical form of a word $u \in F_w$, $x_i \in A, p_i \in N, 1 \leq i \leq k$. It is not difficult to show that the map

$$\alpha_3 : F_w \to N^k : \; u = x_1^{p_1}x_2^{p_2}...x_k^{p_k} \mapsto u\alpha_3 = (p_1, p_2, ..., p_k)$$

is an isomorphism. □

Now we consider the free commutative dimonoids of the small ranks.

Proposition 3. *If* $|A| = 1$, *then*

$$(F[A]\bigcup G, \prec, \succ) \cong N_{(\tilde{2}).}$$

Proof. Let $A = \{a\}$. Define a map

$$\mu : (F[A]\bigcup G, \prec, \succ) \to N_{(\tilde{2})} : w \mapsto w\mu,$$

where

$$w\mu = \begin{cases} k, \; w = a^k \in F[A], \\ \tilde{2}, \; w = (a, a). \end{cases}$$

An immediate verification shows that μ is an isomorphism. □

Let $\tilde{N} = (N^0 \times N^0)\backslash\{(0,0)\}$, where N^0 is the additive semigroup of positive integers with a zero, and let $\{1, \tilde{2}, \bar{2}\}$ be an arbitrary three-element set. Define the operations $\prec$ and $\succ$ on the set $\tilde{N}\bigcup\{1, \tilde{2}, \bar{2}\}$ by

$$(m, n) * 1 = 1 * (m, n) = (m + 1, \, n + 1),$$
$$(m, n) * \tilde{2} = \tilde{2} * (m, n) = (m + 2, \, n),$$
$$(m, n) * \bar{2} = \bar{2} * (m, n) = (m, \, n + 2),$$
$$1 * 1 = (2, \, 2), \quad \tilde{2} * \tilde{2} = (4, 0), \quad \bar{2} * \bar{2} = (0, 4),$$
$$1 * \tilde{2} = \tilde{2} * 1 = (3, 1), \quad \tilde{2} * \bar{2} = \bar{2} * \tilde{2} = (2, 2),$$
$$1 * \bar{2} = \bar{2} * 1 = (1, 3),$$

where $* = \prec$ or $\succ$, and

$$(m, n) \prec (p, l) = (m + p, n + l),$$

$$(m, n) \succ (p, l) = \begin{cases} 1, \text{ if } (m, n) = (1, 0), \; (p, l) = (0, 1), \\ 1, \text{ if } (m, n) = (0, 1), \; (p, l) = (1, 0), \\ \tilde{2}, \text{ if } (m, n) = (p, l) = (1, 0), \\ \bar{2}, \text{ if } (m, n) = (p, l) = (0, 1), \\ (m + p, n + l) \quad \text{otherwise} \end{cases}$$

for all $(m, n), (p, l) \in \tilde{N}$. A long verification shows that $(\tilde{N}\bigcup\{1, \tilde{2}, \bar{2}\}, \prec, \succ)$ is a dimonoid. We denote this dimonoid by $N_{(1, \tilde{2}, \bar{2})}$.

Proposition 4. *If* $|A| = 2$, *then*

$$(F[A] \bigcup G, \prec, \succ) \cong N_{(1,\tilde{2},\bar{2})}.$$

Proof. Let $A = \{a, b\}$ and let $u = a^m b^n$ be the canonical form of a word $u \in F[A], m, n \in N^0$ (m and n are not equal to a zero concurrently). Define a map

$$\tau : (F[A] \bigcup G, \prec, \succ) \to N_{(1,\tilde{2},\bar{2})} : w \mapsto w\tau,$$

where

$$w\tau = \begin{cases} (m, n), \ w = a^m b^n \in F[A], \\ 1, \ w = (a, b) \in G, \\ \tilde{2}, \ w = (a, a) \in G, \\ \bar{2}, \ w = (b, b) \in G. \end{cases}$$

An immediate verification shows that τ is an isomorphism. □

4. Some properties

In this section we describe some properties of commutative dimonoids.

Recall the definitions of Green's relations on a semigroup S. Green's relations on S are called the binary relations:

$$\mathfrak{L} = \{(x; y) \in S \times S \,|S^1 x = S^1 y\},$$

$$\mathfrak{R} = \{(x; y) \in S \times S \,|xS^1 = yS^1\},$$

$$\mathfrak{J} = \{(x; y) \in S \times S \,|S^1 x S^1 = S^1 y S^1\},$$

$$\mathfrak{H} = \mathfrak{L} \bigcap \mathfrak{R}, \ \mathfrak{D} = \mathfrak{L} \circ \mathfrak{R},$$

where S^1 is a semigroup with an identity, $\mathfrak{L} \circ \mathfrak{R}$ is the composition of binary relations.

Let $(D, \prec, \succ)$ be a dimonoid and let K be one of Green's relations on $(D, \prec)$. Then we will call K a Green's relation on the dimonoid $(D, \prec, \succ)$.

Lemma 2. *In a commutative dimonoid* $(D, \prec, \succ)$ *all Green's relations coincide and are congruences.*

Proof. An equality of Green's relations on $(D, \prec, \succ)$ follows from the equality of Green's relations on the commutative semigroup $(D, \prec)$ (see [6]). It is well-known also that the relation $\mathfrak{L}$ is a congruence on the semigroup $(D, \prec)$ (see [6]). We show that $\mathfrak{L}$ is compatible with the operation $\succ$.

Let $x\mathfrak{L}y$, $x, y, c \in D$. Then $y = t_1 \prec x$, $x = t_2 \prec y$ for some $t_1, t_2 \in D$. Hence,

$$c \succ y = c \succ (t_1 \prec x) = (c \succ t_1) \prec x =$$

$$= (t_1 \succ c) \prec x = t_1 \prec (c \succ x),$$

$$c \succ x = c \succ (t_2 \prec y) = (c \succ t_2) \prec y =$$

$$= (t_2 \succ c) \prec y = t_2 \prec (c \succ y)$$

according to Lemma 1. So, $c \succ x \,\mathfrak{L}\, c \succ y$. From the commutativity of the operation $\succ$ it follows that $x \succ c \,\mathfrak{L}\, y \succ c$. Thus, $\mathfrak{L}$ is a congruence on $(D, \prec, \succ)$. □

Corollary 1. *Green's relations on the free commutative dimonoid* $(F[A] \bigcup G, \prec, \succ)$ *are equal to the diagonal of* $F[A] \bigcup G$.

Let $(D, \prec, \succ)$ be a commutative dimonoid, $n \in N$, $n > 1$. Recall that we denote by a^n the degree n of an element $a \in D$ concerning the operation $\prec$.

Lemma 3. *The map* $\beta : x \mapsto x^n$ *is an endomorphism of the commutative dimonoid* $(D, \prec, \succ)$ *and the operations of the dimonoid* ${}^{(D,\prec,\succ)}/_{\Delta_\beta}$ *coincide.*

Proof. If $a, b \in D$, then

$$(a \prec b)\beta = (a \prec b)^n =$$

$$= a^n \prec b^n = a\beta \prec b\beta,$$

$$(a \succ b)\beta = (a \succ b)^n = a^n \prec b^n =$$

$$= a^n \succ b^n = a\beta \succ b\beta$$

according to Lemma 1. As, in a commutative dimonoid, $(a \prec b)^n = (a \succ b)^n$ for all $a, b \in D$, $n \in N$, $n > 1$, then the operations of ${}^{(D,\prec,\succ)}/_{\Delta_\beta}$ coincide. □

Corollary 2. *The map* β *is a homomorphism from the commutative dimonoid* $(D, \prec, \succ)$ *to the semigroup* $(D, \prec)$.

For all $h = (x, y) \in G$ let $[h]$ be an element $xy \in F[A]$.

Corollary 3. *Let* $(F[A]\bigcup G,\prec,\succ)$ *be a free commutative dimonoid. For all* $w,u\in(F[A]\bigcup G,\prec,\succ)$ *we have* $w\Delta_\beta u$ *if and only if one of the following statements holds:*

(i) if $w,u\in F[A]$*, then* $w=u$*,*

(ii) if $w\in G$*, then* $u=[w]$ *or* $u=w$*,*

(iii) if $u\in G$*, then* $w=[u]$ *or* $w=u$*.*

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