

## Integrability by quadratures for systems of involutive vector fields

### Інтегрування в квадратурах інволютивних систем векторних полів

Starting from results and ideas of S. Lie and E. Cartan, we give a systematic and geometric treatment of integrability by quadratures of involutive systems of vector fields, showing how a generalization of the usual multiplier can be constructed with the aid of closed differential forms and enough symmetry vector fields. This leads us to explicit formulas for the independent integrals. These results allow us to identify symmetries with integral invariants in the sense of Poincaré and Cartan. A further (new) result gives the equivalence of integrability by quadratures and the existence of solvable structures, these latter being generalizations of solvable algebras.

Ісходя из результатов и идей С. Ли и Е. Картана приводим систематизацию и геометрическое трактование интегрирования в квадратурах инволютивных систем векторных полей, показав как обобщение обычного интегрирующего множителя может быть построено с помощью замкнутых дифференциальных форм и достаточно симметричных векторных полей. Это приводит к явным формулам независимых интегралов. Полученные результаты позволили отождествить симметрии с интегральными инвариантами в смысле Пуанкаре — Картана. Последующий (новый) результат дает эквивалентность интегрируемости в квадратурах и существование разрешимых структур, которые являются обобщением разрешимых алгебр.

Виходячи із результатів та ідей С. Лі та Е. Картана наводимо систематизацію та геометричне трактування інтегрування в квадратурах інволютивних систем векторних полів, показавши як узагальнення звичайного інтегрувального множника може бути побудоване за допомогою замкнених диференціальних форм та достатньо симетричних векторних полів. Це приводить до явних формул незалежних інтегралів. Одержані результати дозволили отожнити симетрії з інтегральними інваріантами в розумінні Пуанкаре — Картана. Наступний (новий) результат дає еквівалентність інтегровності в квадратурах та існування розв'язуваних структур, які є узагальненням розв'язуваних алгебр.

**1. Introduction.** This note has its root in an article by S. Lie concerning the group-theoretical foundations of Jacobi's last multiplier, and a generalization of this multiplier to involutive systems of vector fields [1]. Subsequently, Cartan discussed the integration of a Pfaffian equation in which the multiplier appeared as an integrating factor—that is, it made certain exterior form closed [2, p. 93]. However, the sources [1, 2] are somewhat difficult to follow, and, indeed, a workable definition of the multiplier is not given in [1] (however, in Chapter 15, section 5 of [3] such a definition is given for the case of one linear partial differential equation of the first order). Moreover, a geometrical treatment of Lie's generalization has not been given previously. This is one reason for this note. A further motivation is that recent work on the integrability by quadratures of linear first-order partial differential equations [4] leads one to consider systematic methods which can give explicit formulas for the first integrals of involutive systems of vector fields. These are also of interest in control theory, where exact linearization of non-linear systems gives rise to a system of partial differential equations whose solution is sought through Frobenius' theorem [5]. Furthermore, one is led to consider symmetries of systems of vector fields in the theory of conditional symmetries [6]. Here we give a systematic account of the construction of integrals of motion for such systems.

We use the notation of modern differential geometry, for which we refer the reader to [7]. In particular,  $\iota_X \omega$  denotes the interior product of a differential form  $\omega$  by a vector field  $X$ . This product has the following properties:

$$\begin{aligned} \iota_X^2 \omega &= 0, \\ \iota_X(\omega_1 \wedge \omega_2) &= \iota_X \omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \iota_X \omega_2, \\ \iota_{X \wedge Y} \omega &= \iota_Y \iota_X \omega \end{aligned} \quad (1)$$

for all forms  $\omega$ ,  $\omega_2$  and  $p$ -forms  $\omega_1$  and all vector fields  $X, Y$ . Furthermore

$$\iota_X f = 0,$$

$$\iota_X df = Xf, \quad (2)$$

$$\iota_{X_1 \wedge \dots \wedge X_p} \omega = \omega(X_1, \dots, X_p)$$

for all functions  $f$ ,  $p$ -forms  $\omega$  and vector fields  $X_1, \dots, X_p$ . The Lie derivative  $L_X$  with respect to the vector field  $X$  can be defined as [7]

$$L_X = d\iota_X + \iota_X d.$$

We use the summation convention.

2. Closed forms and Jacobi's last multiplier. Suppose that we are given a system  $A = \{A_1, \dots, A_{n-p}\}$  of vector fields on  $\mathbb{R}^n$ , which are in involution and independent over the ring of  $C^\infty$  functions. Then Frobenius theorem [7] tells us that in the neighbourhood of a regular point (that is, one where the vector fields are independent) there exists a coordinate system  $u^1, \dots, u^n$  so that

$$\text{span}\{A_1, \dots, A_{(n-p)}\} = \text{span}\left\{\frac{\partial}{\partial u^{p+1}}, \dots, \frac{\partial}{\partial u^n}\right\}$$

and hence that  $u^1, \dots, u^p$  are  $p$  common integrals of  $A$ , namely

$$A_i u^j = 0 \quad i = 1, \dots, n-p; \quad j = 1, \dots, p. \quad (3)$$

However, Frobenius' theorem does not tell us how to construct such a system of coordinates. If we do know this change of coordinates, then we have solved, at least locally, the problem of integrability by quadratures. The existence of enough symmetries is helpful in this problem, as was noted by Lie [8].

Definition 1. A system  $X = \{X_1, \dots, X_p\}$  of independent vector fields is said to be a system of symmetries of the involutive system  $A = \{A_1, \dots,$

$\dots, A_{(n-p)}\}$  if

$$1) \{A_1, \dots, A_{(n-p)}, X_1, \dots, X_p\}$$

are independent

$$2) [X_i, A_j] = c_{ij}^k A_k \quad i = 1, \dots, p; \quad j, k = 1, \dots, n-p.$$

Now, suppose that  $f^1, \dots, f^p$  are  $p$  independent, locally defined common integrals of  $A$ . From Definition 1 (2), it follows that

$$X_i f^j = F_{ij}(f^1, \dots, f^p), \quad i, j = 1, \dots, p, \quad (4)$$

where  $(F_{ij})$  is a  $p \times p$  matrix of  $C^\infty$  functions, as any integral of  $A$  is a function of a given set of  $p$  common, independent integrals.

Lemma 1. The matrix  $F = (F_{ij})$  defined by equation (4) is invertible if and only if the  $\{X_i : i = 1, \dots, p\}$  is a system of symmetries of  $A$ .

Proof. If the  $\{X_i\}$  are a system of symmetries of  $A$ , then  $F$  can not have rank less than  $p$ . If its rank were less than  $p$ , then (at least) one row, say  $(X_1 f^1, X_1 f^2, \dots, X_1 f^p)$  would be a linear combination of the other rows. This would imply that a linear combination of the vector fields  $X_1, \dots, X_p$  would have each of the  $f^1, \dots, f^p$  as a common integral. It follows then that the system  $\{A, X\}$  could not be linearly independent, because a linear combination of the  $X_i$  would belong to  $A$ . However, the system  $\{A, X\}$  is a linearly independent system of  $n$  vector fields on  $\mathbb{R}^n$ , and therefore the rows of  $F$  are independent. Hence,  $F$  is invertible. On the other hand, the matrix elements  $F_{ij}$  depend only on the integrals  $f_1, \dots, f_p$  if and only if the  $\{X_i\}$  are symmetries of  $A$ . If the matrix  $(F_{ij})$  is invertible, then the system  $\{A, X\}$  must be linearly independent, from the previous argument. Therefore the  $\{X_i\}$  must be a system of symmetries of the system  $A$ .

Definition 2. If  $\{x^1, \dots, x^n\}$  is a local coordinate system on a neighbourhood in  $\mathbb{R}^n$  on which none of the  $A_i, X_j$  vanish, define

- 1)  $\Omega = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ ;
- 2)  $\iota_A \Omega = \iota_{A_1} \wedge \dots \wedge \iota_{A_{(n-p)}} \Omega = \iota_{A_{(n-p)}} \iota_{A_{(n-p-1)}} \dots \iota_{A_1} \Omega$ ;
- 3)  $\iota_{A \wedge X} \Omega = \iota_{A_1} \wedge \dots \wedge \iota_{A_{(n-p)}} \wedge \iota_{X_1} \wedge \dots \wedge \iota_{X_p} \Omega$ .

Lemma 2. The  $p$ -form

$$\omega = \frac{\iota_A \Omega}{\iota_{A \wedge X} \Omega}$$

is closed.

Proof. It follows from equations (1) and Definition 2 (2) that  $\iota_{A_i} \iota_A \Omega = 0$ ,  $i = 1, \dots, n-p$ . Furthermore, if a vector field  $Y$  satisfies  $\iota_Y \iota_A \Omega = 0$  then  $Y$  is a linear combination of the  $A_i$ : if not, then  $A \wedge Y \neq 0$  and so  $\iota_{A \wedge Y} \Omega \neq 0$ . These statements are also true of the  $p$ -form  $df^1 \wedge \dots \wedge df^p$ . It then follows that  $\iota_A \Omega$  and  $df^1 \wedge \dots \wedge df^p$  are proportional, namely there exists a function  $Q$  so that

$$\iota_A \Omega = Q df^1 \wedge \dots \wedge df^p.$$

Then we have:

$$\begin{aligned} \iota_{A \wedge X} \Omega &= \iota_X \iota_A \Omega = Q \iota_{X_1} \wedge \dots \wedge \iota_{X_p} (df^1 \wedge \dots \wedge df^p) = \\ &= Q (df^1 \wedge \dots \wedge df^p) (X_1, \dots, X_p) = Q \det (X_i^{f^j}) = Q \det F \neq 0 \end{aligned}$$

by Lemma 1. Therefore,

$$\omega = \frac{1}{\det F} df^1 \wedge \dots \wedge df^p.$$

The right-handside of this last equation is a closed  $p$ -form, since  $F$ , and hence  $\det F$ , depend only on the functions  $f^1, \dots, f^p$ .

This means that, when we know enough symmetries, we are able to construct a closed  $p$ -form, rather than merely know of its existence. The above construction is important in calculating the integrals. Let us remark that the formula for  $\omega$  gives us a way of constructing a quasi-invariant measure on the quotient group of a Lie group by its normal subgroup, using left- or right-invariant vector fields.

Lemma 3. The  $p$ -form  $\omega$  is invariant under the system  $A$  and quasi-invariant under the system  $X$ .

Proof. For each  $A_i$ , apply the Lie derivative:

$$L_{A_i} \omega = d \iota_{A_i} \omega + \iota_{A_i} d\omega = 0$$

since  $\omega$  is closed and  $\iota_{A_i} \omega = 0$ , as noted above. To prove quasi-invariance with respect to the  $X_i$ , we apply the Lie derivative  $L_{X_i}$  to  $\omega$ . As  $\omega$  is closed, then

$$L_{X_i} \omega = d \iota_{X_i} \omega = d \iota_{X_i} \left( \frac{1}{\det F} df^1 \wedge \dots \wedge df^p \right) = \rho_i \omega$$

where  $\rho_i$  is a function of the integrals  $f^1, \dots, f^p$ , the last line following from the fact that the matrix  $F$  depends only on the  $f^i$  and that  $X_i^{f^j}$  is a function of these integrals.

Remark. The connection with the usual theory of Jacobi's last multiplier can be seen from the following:

$$\begin{aligned} \iota_{A \wedge X} \Omega &= \iota_{A_1} \wedge \dots \wedge \iota_{A_{(n-p)}} \wedge \iota_{X_1} \wedge \dots \wedge \iota_{X_p} (dx^1 \wedge \dots \wedge dx^n) = \\ &= (dx^1 \wedge \dots \wedge dx^n) (A_1, \dots, A_{(n-p)}, X_1, \dots, X_p) = \end{aligned}$$

$$= \det \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \dots & \vdots \\ A_{(n-p)1} & \dots & A_{(n-p)n} \\ X_{11} & \dots & X_{1n} \\ \vdots & \dots & \vdots \\ X_{p1} & \dots & X_{pn} \end{pmatrix} = \frac{1}{M}.$$

The expression  $M$  appears in Lie's article [1], and in Chapter 15, section 5 of [3] for the case of one vector field. In the case of one vector field  $A$ , the function  $M$  is just Jacobi's last multiplier. One easily calculates then that

$$\iota_A \Omega = \sum_{i=1}^n (-1)^{(i-1)} A_i \cdot dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

where  $A_i$  is the  $i$ -th component of the vector field  $A$ , and  $\widehat{dx^i}$  denotes omission of  $dx^i$ . Furthermore,  $M \cdot \iota_A \Omega$  is closed, by Lemma 2. Hence we have:

$$\begin{aligned} 0 = d(M \iota_A \Omega) &= \sum_{i=1}^n (-1)^{(i-1)} \frac{\partial(A_i M)}{\partial x^r} dx^r \wedge \dots \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n = \\ &= \left( \frac{\partial(A_1 M)}{\partial x^1} + \dots + \frac{\partial(A_n M)}{\partial x^n} \right) \Omega \end{aligned}$$

Since  $\Omega \neq 0$ , it follows that

$$\frac{\partial(MA_1)}{\partial x^1} + \dots + \frac{\partial(MA_n)}{\partial x^n} = 0$$

and this shows that  $M$  is Jacobi's multiplier. Thus in the case of one vector field  $A$ , the knowledge of  $n - 1$  independent symmetries allows us to give an explicit formula for Jacobi's last multiplier. Let us note that in Section 8 of [4] another explicit formula for it is obtained in terms of  $n - 1$  functionally independent first integrals of the vector field  $A$ . Moreover, we see that

$M = \frac{1}{\iota_{A \wedge X \Omega}}$  is a generalization to systems of vector fields of Jacobi's multiplier (and, of course, of the integrating factor in the case of first order ordinary differential equations).

3. Solvable structures and integrability by quadratures. Definition 3. Put

$$\beta_i = \frac{\iota_{A_1 \wedge \dots \wedge A_{(n-p)} \wedge X_1 \wedge \dots \wedge \widehat{X_i} \wedge \dots \wedge X_p} \Omega}{\iota_{A \wedge X} \Omega}$$

where  $\widehat{X_i}$  denotes omission of  $X_i$ ,  $i = 1, \dots, p$ .

Proposition 1.

- 1) The  $\beta_i$ ,  $i = 1, \dots, p$ , are independent;
- 2) If the  $\beta_i$  are closed, then the functions  $\varphi_i = \int \beta_i$  are  $p$  independent, local common integrals of the system  $A$ ;
- 3) The  $\beta_i$  are either absolute or relative integral invariants (locally) of the system  $A$ .

Proof. Suppose that there are non-zero functions  $\lambda_1, \dots, \lambda_p$  such that  $\lambda_1 \beta_1 + \dots + \lambda_p \beta_p = 0$  then apply  $\iota_{X_j}$  to the left-handside of the above exp-

ression to 0 in  $\lambda_j = 0$  since  $\iota_{X_j} \int_i = (-1)^{(i-1)} \delta_{ij}$ , on using equations (1).

This establishes independence (at least locally).

If the  $\beta_i$  are closed, the by Poincaré's lemma [4], there exist (locally) functions  $\varphi_i$  such that  $\beta_i = d\varphi_i$ . The  $\varphi_i$  are functionally independent, since the  $\beta_i$  are independent. Moreover, they are integrals of the system **A** because

$$A_j \varphi_i = \iota_{A_j} (d\varphi_i) = \iota_{A_j} \beta_i = 0$$

for every  $j$  and each  $i$ .

To establish the last statement, we remark that either  $\beta_i$  is closed or  $d\beta_i \neq 0$ . In the first case,  $\beta_i$  is an absolute integral invariant of **A**. In the second case we have for each  $k = 1, \dots, n-p$  and every vector field  $Z \in \{A_1, \dots, A_{(n-p)}, X_1, \dots, X_p\}$ :

$$\iota_{A_k} b\beta_i(Z) = L_{A_k} \beta_i(Z) = L_{A_k} (\beta_i(Z)) - \beta_i([A_k, Z]) = 0$$

since  $\beta_i(Z)$  is either 0 or 1 by construction, and the commutator  $[A_k, Z]$  is always in the system **A**, as  $Z$  is a symmetry of the system. Now the vector fields  $\{A_h, X_i\}$  are locally independent, so they form a local basis for vector fields. This implies that  $\iota_{A_k} d\beta_i = 0$  locally, which is the condition that  $\beta_i$  be a (local) relative integral invariant [7, p. 166—172].

**Proposition 2.** *The form  $\beta_i$  is closed if and only if the system  $\{A, X_1, \dots, \hat{X}_i, \dots, X_p\}$  is closed and  $X_i$  is a symmetry of it.*

**Proof.** If  $X_i$  is a symmetry of the closed system  $\{A, X_1, \dots, \hat{X}_i, \dots, X_p\}$ , then  $\beta_i$  is closed by Lemma 2. Conversely, if  $\beta_i$  is closed, we have for all vector fields  $Z_1, Z_2 \in \{A, X_1, \dots, X_p\}$ :

$$0 = \iota_{Z_1} d\beta_i(Z_2) = L_{Z_1} \beta_i(Z_2) = L_{Z_1} (\beta_i(Z_2)) - \beta_i([Z_1, Z_2]) = -\beta_i([Z_1, Z_2]).$$

Since  $\beta_i(Z_2) = 0$  or 1 by construction. Now,  $\beta_i([Z_1, Z_2]) = 0$  if and only if  $[Z_1, Z_2] \in \{A, X_1, \dots, \hat{X}_i, \dots, X_p\}$ . This proves the contention.

**Corollary.** *All the  $\beta_i$  are closed if only if the symmetries  $\{X_i\}$  are such that  $[X_i, X_j] \in A$  for  $i, j = 1, \dots, p$ . In particular, if the  $\{X_i\}$  are in involution, then the  $\beta_i$  are all closed if and only if the  $X_i$  commute.*

The above results imply that symmetries are in one to one correspondence with integral invariants. Moreover, if the symmetries are in involution, the kernel in  $X$  of those  $\beta_i$  which are closed is an ideal. This follows from an argument using the calculation in Proposition 2.

**Definition 4.** *We say that the system  $\{A, X\}$  is a solvable structure with respect to **A** if*

- 1) **A** is in involution;
- 2)  $S_j = \{A, X_1, \dots, X_j\}$   $j = 1, \dots, p$  is in involution;
- 3)  $X_1$  is a symmetry of **A** and  $X_{j+1}$  is a symmetry of  $S_j$  for  $j = 1, \dots, p-1$ .

In other words,  $S_j$  is an ideal, of codimension 1, in  $S_{j+1}$  for each  $j = 1, \dots, p-1$ . This is the weakest generalization of a solvable algebra.

**Proposition 3.** *If the system  $\{A, X\}$  is a solvable structure with respect to **A**, then one can find, at least locally, the integrals of **A** by quadratures alone.*

**Proof.** By definition,  $X_p$  is a symmetry of  $S_{(p-1)}$ , so by Lemma 2, the 1-form

$$\beta_p = \frac{\iota_{S_{(p-1)}} \Omega}{\iota_{S_{(p-1)} \wedge X_p} \Omega}$$

is closed. Proposition 1 (2) then gives us that

$$\varphi_p = \int \beta_p$$

is one integral. Now define the change of coordinates  $(x^1, \dots, x^n) \rightarrow (x^1, \dots, x^{(n-1)}, \varphi_p)$ . Now, a vector field

$$V = V_1 \frac{\partial}{\partial x^1} + \dots + V_n \frac{\partial}{\partial x^n}$$

becomes, in the new system,

$$V' = V(x^1) \frac{\partial}{\partial x^1} + \dots + V(x^{(n-1)}) \frac{\partial}{\partial x^{(n-1)}} + V(\varphi_p) \frac{\partial}{\partial \varphi_p} = \\ = V'_1 \frac{\partial}{\partial x^1} + \dots + V'_{(n-1)} \frac{\partial}{\partial x^{(n-1)}}$$

whenever  $V\varphi_p = 0$ , and where  $V'_i$  is just  $V_i$  expressed in the new coordinates. Thus the vector fields  $A_1, \dots, A_{(n-p)}, X_1, \dots, X_{(p-1)}$  in  $S_{(p-1)}$  are reduced to vector fields on  $R^{(n-1)}$ , on suppressing the dependence on  $\varphi_p$ . Keeping  $\varphi_p$  constant, we find that the (new) vector field  $X_{(p-1)}$  is a symmetry of the (new) system  $S_{(p-2)}$  in  $R^{(n-1)}$ . Then we construct the 1-form

$$\beta_{(p-1)} = \frac{v_{S_{(p-2)}} \Omega}{v_{S_{(p-1)}} \Omega}$$

which is closed, by Lemma 2. Again, by Proposition 1 (2), it follows that

$$\varphi_{(p-1)} = \int \beta_{(p-1)}$$

is another integral. Then change coordinates again:

$$(x^1, \dots, x^{(n-1)}, \varphi_p) \rightarrow (x^1, \dots, x^{(n-2)}, \varphi_{(p-1)}, \varphi_p)$$

and proceed as before. At each stage we have the same situation, until we arrive at  $S_1$ , and this is the last step to perform, as above. In this way we construct the full number of independent integrals of  $A$ . The above method of proof goes back to Lie [3]. One can also find the same procedure in [9], in the reduction of order of differential equations using groups and differential invariants.

As a converse to this result, one has the following:

**Proposition 4.** *If the  $p$  independent integrals of  $A$  are known, then one can construct a local coordinate system in which there exist  $p$  independent, commuting symmetries of the system  $A$ .*

**Proof.** If  $f^1, \dots, f^p$  are these integrals, then change coordinates:

$$(x^1, \dots, x^n) \rightarrow (x^1, \dots, x^{(n-p)}, f^1, \dots, f^p).$$

From the considerations in the proof of Proposition 3, the vector fields  $A_1, \dots, A_{(n-p)}$  are expressed solely in terms of the  $\partial/\partial x^i$ . We can then take  $X_j = \partial/\partial f^j$  as the independent commuting symmetries, using Lemma 1.

We can apply the above to  $n - th$  order ordinary differential equations. Each such equation

$$y^{(n)} = g(x, y, y^{(1)}, \dots, y^{(n-1)})$$

where  $g$  is a  $C^\infty$  function and  $y^{(j)} = d^j y/dx^j$ , for  $j = 1, \dots, n$ , is associated with the vector field

$$A = \frac{\partial}{\partial x} + y^{(1)} \frac{\partial}{\partial y} + y^{(2)} \frac{\partial}{\partial y^{(1)}} + \dots + g(x, y, y^{(1)}, \dots, y^{(n-1)}) \frac{\partial}{\partial y^{(n-1)}}$$

defined on some open subset of  $R^{n+1}$ . Then a symmetry of the equation is a vector field

$$X = X_0 \frac{\partial}{\partial x} + X_1 \frac{\partial}{\partial y} + \dots + X_{(n-1)} \frac{\partial}{\partial y^{(n-1)}}$$

defined on some open subset of  $R^{n+1}$  such that  $[X, A] = \rho A$  for some function  $\rho$ . Our results then give us:

**Proposition 5.** *The  $n - th$  order ordinary differential equation, above, is integrable by quadratures, at least locally, if there exist  $n$  independent sym-*

metries  $X_1, \dots, X_n$  of the associated vector field  $A$  such that  $\{A, X_1, \dots, X_n\}$  forms a solvable structure.

If the equation is integrable by quadratures, then one can construct, by quadratures alone, a local coordinate system in which there are  $n$  independent, commuting symmetries of the equation.

The proof of this is an elementary combination of Propositions 3 and 4.

The above reasoning and results lead us to the following:

**Proposition 6.** *The involutive system  $A$  is locally integrable by quadratures if and only if there exists a solvable structure with respect to  $A$ .*

The proof is a combination of Propositions 3 and 4. It is not difficult to see that a special case of this situation arises when a system of dimension  $p$  in  $\mathbb{R}^n$  has a solvable symmetry group of dimension  $n - p$ . The above results give a one to one correspondence between integrability by quadratures and solvable structures. This generalizes the usual result concerning solvable groups and integrability by quadratures (Theorem 2.64 in [9]).

**4. An example.** We now apply the above theory to the integration of the vector field

$$X = \left( xz - \frac{xy}{AB} \right) \frac{\partial}{\partial x} + (Ayz + yx) \frac{\partial}{\partial y} + (Bxz + zy) \frac{\partial}{\partial z}$$

where  $A, B \neq 0$ .  $X$  corresponds to the Lotka — Volterra system with the constants  $A, B, C$  satisfying  $ABC + 1 = 0$ . See [4].  $X$  is in involution with the vector fields

$$Y_1 = z \frac{\partial}{\partial x} + Ay \frac{\partial}{\partial y} + (y + Bz) \frac{\partial}{\partial z},$$

$$Y_2 = x \frac{\partial}{\partial x} - ABz \frac{\partial}{\partial z}$$

as can be verified (after some calculation). Furthermore, the vector field  $Z$  given by

$$Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$$

is a symmetry of  $X$  and commutes with  $Y_1$  and  $Y_2$ . Thus  $Z$  is a symmetry of the two involutive systems  $\{X, Y_1\}$  and  $\{X, Y_2\}$ , and therefore we can calculate the two corresponding integrals  $\varphi_1$  and  $\varphi_2$  of the vector field  $X$ . Indeed, we find that  $\omega_i/M_i$  is a closed 1-form, for  $i = 1, 2$ , where  $\omega_i = \iota_{X \wedge Y_i} \Omega$  and  $M_i = \iota_Z \omega_i$  and  $\Omega$  is a 3-form. Then we put

$$\varphi_i = \int \frac{\omega_i}{M_i}.$$

For  $i = 1$ , we take  $\Omega = dx \wedge dy \wedge dz$  and a routine calculation gives:

$$\omega_1 = \frac{\Delta}{AB} (ABdx + dy - Adz)$$

where

$$\Delta = (ABz - y)y - Byz(Az + x).$$

Also,

$$M_1 = \frac{\Delta}{AB} (ABx + y - Az).$$

Thus we obtain the closed form

$$\frac{\omega_1}{M_1} = \frac{(ABdx + dy - Adz)}{(ABx + y - Az)} = d \ln |ABx + y - Az|.$$

From this we obtain the integral  $\varphi_1 = ABx + y - Az$ . The case  $i = 2$  gives:

$$\omega_2 = AByz(Az + x) dx - Bxz(Az + x) dy + xy(Az + x) dz$$

and

$$M_2 = xyz(Az + x)(AB - B + 1).$$

From this we obtain

$$\begin{aligned} \frac{\omega_2}{M_2} &= \frac{AB}{(AB - B + 1)} \frac{dx}{x} - \frac{B}{(AB - B + 1)} \frac{dy}{y} + \frac{dz}{(AB - B + 1)z} = \\ &= \frac{1}{(AB - B + 1)} d \left( \ln \frac{|x|^{AB} |z|}{|y|^{AB}} \right) \end{aligned}$$

and this then gives us the integral  $\varphi_2 = |x|^{AB} |y|^{-B} |z|$ . It is easy to verify that the integrals  $\varphi_1, \varphi_2$  are independent: indeed,  $\varphi_1$  is not an integral of  $Y_2$ , since  $X \wedge Y_1 \wedge Y_2 \neq 0$  as can be verified. We have therefore been able to calculate two independent integrals of the Lotka — Volterra equations, in the case where  $ABC + 1 = 0$ . For a thorough study of integrals of these equations, see [4]. We merely remark that our method is equivalent to the use of Jacobi's last multiplier in [4].

**Conclusion.** We have given a geometrical description of Lie's generalization of Jacobi's multiplier, and shown how it generalizes the usual integrating factor. This was previously thought not to exist [9, p. 140]. However, the method of differential invariants as expounded in [9] is equivalent to the use of the integrating factor. The advantage of the method presented here is that one can see the connection between symmetries and conserved quantities, which are either absolute or relative integral invariants. Moreover, we have given explicit formulas for the integrals of involutive systems of vector fields admitting symmetries. We have also shown by an example, how the theory is used to obtain integrals of motion. This example shows also how symmetries and compatible (in the sense of [4]) vector fields can be combined to obtain solutions. Our example also shows how conditional symmetries in the sense of [6] help us with integration by quadratures.

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