УДК 517.9

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ON THE DEFINITION OF SINGULAR BILINEAR FORMS AND SINGULAR LINEAR OPERATORS

ПРО ОЗНАЧЕННЯ СИНГУЛЯРНИХ БІЛІНІЙНИХ ФОРМ ТА СИНГУЛЯРНИХ ЛІНІЙНИХ ОПЕРАТОРІВ

We revise various definitions of the notions of singular operator and singular form and propose the most suitable ones. We also present the simplest properties of these objects.

Аналізуються різні означення понять сингулярного оператора та сингулярної білінійної форми і пропонуються найбільш вдалі. Розглядаються також найпростіші властивості таких об'єктів.

The singular bilinear forms and singular linear operators in a Hilbert space are remarkable objects. Roughly speaking, they vanish on a dense subset but are still capable to produce a nontrivial perturbation effect.

This note is devoted to the analysis of various definitions of notions of singular operator and singular form. We also sketch the simplest properties of singular objects.

In the physical literature the singular objects are often associated with the potentials supported by null sets. In mathematics the term "singular" (for a form or an operator) originates from the canonical Lebesgue decompositions of one measure with respect to another into absolutely continuous and singular parts [1]. Later, it became clear that the concept of singularity for the forms and operators is of intrinsic nature and admits appropriate definition [2–4].

1. Regular and singular bilinear forms. In our approach, the properties of regularity and singularity are not alternative [1–4].

Let γ be a symmetric bilinear form with the domain $Q(\gamma)$ dense in a Hilbert space $\mathcal{H}$. We shall use the notation $\gamma[\varphi] \equiv \gamma(\varphi, \varphi)$, $\gamma \in Q(\gamma)$.

***Definition* 1.** *A vector* $\varphi \in \mathcal{H}$ *is called regular for* γ *if* $\varphi_n \in Q(\gamma)$, $\varphi_n \xrightarrow{\mathcal{H}} \varphi$, $\gamma[\varphi_n - \varphi_m] \to 0 \Rightarrow \gamma[\varphi_n] \to a \in \mathbb{R}$ *and the number* a *is independent of the choice of the sequence* φ_n.

The set of all vectors regular for γ will be denoted by $R(\gamma)$.

Following facts from the theory of bilinear forms [5] are well-known.

Proposition 1.

$$Q(\gamma) = R(\gamma) \Leftrightarrow \gamma = \bar{\gamma}, \tag{1}$$

where $\bar{\gamma}$ *stands for the closure of* γ.

Proposition 2.

$$0 \in R(\gamma) \Leftrightarrow \gamma \subseteq \bar{\gamma}. \tag{2}$$

If $Q(\gamma) \subseteq R(\gamma)$, we say that γ is *regular*. Thus, γ is regular iff it is closable. In other words if γ is regular , then all vectors $\varphi \in Q(\gamma)$ are regular for γ. A sufficient condition for this to hold is $0 \in R(\gamma)$ [5]. The set of all densely defined symmetric bilinear regular forms is denoted by $\mathcal{T}_r$.

If γ is not closable in $\mathcal{H}$, then $0 \notin R(\gamma)$. What can we say about other vectors $\varphi \in Q(\gamma)$?

***Definition* 2.** $\varphi \in \mathcal{H}$ *is said to be of singular type for* γ *if there exist at least* *two different sequences* $\varphi_n, \varphi'_n \in \mathcal{Q}(\gamma)$ *such that*

$$\varphi_n \xrightarrow{\mathcal{H}} \varphi, \quad \gamma[\varphi_n - \varphi_m] \to 0, \quad \gamma[\varphi_n] \to a, \tag{3}$$

$$\varphi'_n \xrightarrow{\mathcal{H}} \varphi, \quad \gamma[\varphi'_n - \varphi'_m] \to 0, \quad \gamma[\varphi'_n] \to a', \tag{4}$$

and $a \neq a'$.

The set of all vectors of singular type for γ is denoted by $\mathcal{S}(\gamma)$.

It is clear that if γ is not closable, then $\mathcal{S}(\gamma)$ is not empty. Indeed, then at least $0 \in \mathcal{S}(\gamma)$. It may even happen that $\mathcal{H} = \mathcal{S}(\gamma)$. This is the case, for example, if we set $\mathcal{H} = L_2$, $\gamma[\cdot] = \|\cdot\|_{L_2} + \gamma_\delta[\cdot]$, where $\gamma_\delta[\varphi] = |\varphi(0)|^2$.

***Definition* 3.** *A singular type vector* φ ($\varphi \in \mathcal{S}(\gamma)$) *is called singular for* γ *if one of the numbers* a, a' *of* (3), (4) *is zero. The set of all vectors singular for* γ *is denoted by* $S(\gamma)$. *A form* γ *is called singular if* $S(\gamma)$ *is dense in* $\mathcal{H}$. *In this case, we write* $\gamma \in \mathcal{T}_s$.

Proposition 3. *If* $\gamma \neq 0$, $\gamma \geq 0$ *and the set*

$$\Phi_0 := \{\varphi \in \mathcal{Q}(\gamma) \mid \exists \varphi_n \in \mathcal{Q}(\gamma), \ \varphi_n \xrightarrow{\mathcal{H}} \varphi, \ \gamma[\varphi_n] \to 0\} \tag{5}$$

is dense in $\mathcal{H}$, *then* $\gamma \in \mathcal{T}_s$.

Proof. The set

$$\operatorname{Ker} \gamma := \{\varphi \in \mathcal{Q}(\gamma) \mid \gamma[\varphi] = 0\} \tag{6}$$

is evidently contained in Φ_0. Moreover, if Ker γ is dense in $\mathcal{H}$, then $\gamma \in \mathcal{T}_s$. Indeed, let $\psi \in \mathcal{Q}(\gamma)$, $\gamma[\psi] = a' \neq 0$ and $\psi_n \to \psi$, $\psi_n \in \operatorname{Ker}\gamma$. Then for each $\varphi \in \operatorname{Ker} \gamma$, we consider two sequences $\varphi_n = \varphi$ and $\varphi'_n = \varphi + \psi - \psi_n$ which satisfy (3), (4). Thus, $\operatorname{Ker} \gamma \subset S(\gamma)$ and, according to definition (3), $\gamma \in \mathcal{T}_s$.

If $\operatorname{Ker} \gamma$ is not dense in $\mathcal{H}$, then the reasoning is different. Denote

$$\Phi^{\neq 0} := \{\varphi \in \mathcal{Q}(\gamma) \mid \gamma[\varphi] \neq 0\}. \tag{7}$$

In this case,

$$\Phi^{\neq 0} \cap \Phi_0 \neq \emptyset \tag{8}$$

because $\operatorname{Ker} \gamma$ is not dense in $\mathcal{H}$. Moreover, any φ that belongs to both $\Phi^{\neq 0}$ and Φ_0 is singular, i. e.,

$$\Phi^{\neq 0} \cap \Phi_0 \subset S(\gamma). \tag{9}$$

Indeed, let us take, for the sequences in (3), the one given by (5) and $\varphi'_n = \varphi$ with $a' = \gamma[\varphi] \neq a = 0$. We now have to show that each $\varphi_0 \in \Phi_0$ is singular. For this, we construct two sequences: $\varphi_{n,0}$ from (5) and $\varphi'_{n,0} = \varphi_0 - \varphi - \varphi_n$, where $\varphi \in \Phi^{\neq 0}$ is fixed and $\varphi_n \to \varphi$ as in (5). Then $\varphi_{n,0} \to \varphi_0$, $\varphi'_{n,0} \to \varphi_0$, and $a = 0$, $a' = \gamma[\varphi] \neq 0$.

By virtue of Proposition 3, we can reformulate the definition of a singular form as follows.

***Definition* 4.** *A positive bilinear form* $\gamma \neq 0$ *is said to be singular in* $\mathcal{H}$ ($\gamma \in \mathcal{T}_s$) *if for any*

$$\varphi \in Q(\gamma),\ \exists\ \varphi_n \in Q(\gamma),\ \varphi_n \xrightarrow{\mathcal{H}} \varphi,\ \gamma[\varphi_n] \to 0. \tag{10}$$

***Remark* 1.** If $\Phi_0 \subseteq Q(\gamma) \cap S(\gamma)$ is dense in $\mathcal{H}$, then $Q(\gamma) \subset S(\gamma)$.

The above formulations of the concepts of regularity and singularity for bilinear forms has the following important consequence.

Theorem 1 [1, 3]. *Every symmetric semibounded bilinear form* γ *densely defined in the Hilbert space* $\mathcal{H}$ *admits the unique decomposition into regular and singular components*

$$\gamma = \gamma_r + \gamma_s \tag{11}$$

where $\gamma_r \in \mathcal{T}_r$, $\gamma_r \geq -M$, $\gamma_s \in \mathcal{T}_s$, $\gamma_s \geq 0$.

In the case $\gamma \geq 0$, the decomposition (11) was first obtained in [1], where γ_s was defined as $\gamma - \gamma_r$ and γ_r was the largest regular form obeying $\gamma_r \leq \gamma$.

2. Relative regularity and singularity for a pair of forms. In this section, we consider a pair of bilinear forms defined on a linear complex vector space and discuss the concepts of their relative regularity (respectively, singularity).

Let Φ be a linear complex vector space and let t be a positive bilinear form on Φ. We shall use following notation [6, 7]:

$$\Sigma_t := \left\{ \{\varphi_n\}_{n=1}^{\infty} \mid \varphi_n \in \Phi,\ t[\varphi_n - \varphi_m] \to 0 \right\},$$
$$\theta_t := \left\{ \{\varphi_n\}_{n=1}^{\infty} \mid \varphi_n \in \Phi,\ t[\varphi_n] \to 0 \right\}. \tag{12}$$

In other words, Σ_t is the set of all t-Cauchy sequences and $\theta_t \subset \Sigma_t$ is the set of all zero-t-Cauchy sequences. It is clear that the factor space Σ_t/θ_t is a Hilbert space with the inner quasiproduct

$$(\varphi, \psi)_t := t(\varphi, \psi), \quad \varphi, \psi \in \Phi. \tag{13}$$

We denote this space by $\mathcal{H}_t$.

Consider now two positive bilinear forms γ and χ defined on Φ. By using (12), we introduce the sets

$$\Sigma_t := \Sigma_\gamma \cap \Sigma_\chi, \quad \theta := \theta_\gamma \cap \theta_\chi,$$
$$\Pi_{\gamma,\chi} := \Sigma_\gamma \cap \Sigma_\chi, \quad \Pi_{\chi,\gamma} := \Sigma_\chi \cap \theta_\gamma. \tag{14}$$

***Definition* 5.** *A positive bilinear form* γ *on* Φ *is called regular with respect to another positive bilinear form* χ on Φ *(we write* $\gamma \parallel \chi$*) if*

$$\Pi_{\gamma,\chi} = \theta. \tag{15}$$

Relation (15) *implies that* γ *is closable* (*regular*) *in the Hilbert space* $\mathcal{H}_\chi$. *The forms* γ *and* χ *are mutually regular* ($\gamma \parallel \chi$, $\chi \parallel \gamma$) *if*

$$\Pi_{\chi,\gamma} = \Pi_{\gamma,\chi} = \theta. \tag{16}$$

Proposition 4 [7]. *For* $\gamma, \chi \geq 0$ *on* Φ,

$$\gamma \parallel \chi \Rightarrow \gamma \parallel \chi \tag{17}$$

iff Ker $\overline{\gamma} = 0$ *in* $\mathcal{H}_\chi$.

For $t \geq 0$ on Φ and $\varphi \in \Phi$, we denote

$$E_t^{\varphi} := \left\{ \{\varphi_n\}_{n=1}^{\infty} \mid t\chi[\varphi_n - \varphi] \to 0, \text{ i. e. } \varphi_n \xrightarrow{t} \varphi \right\}.$$

Definition 6. *A form* $\gamma \geq 0$ *on* Φ *is is called singular with respect to the form* $\chi \geq 0$ *on* Φ $(\gamma \perp \chi)$ *if for any* $\varphi \in \Phi$,

$$\Sigma_{\chi}^{\varphi} \cap \theta_{\gamma} \neq \varnothing. \tag{18}$$

Proposition 5.

$$\gamma \perp \chi \Leftrightarrow \chi \perp \gamma. \tag{19}$$

Proof. Let $\gamma \perp \chi$ and, for $\varphi \in \Phi$, we have

$$\varphi_n \xrightarrow{\chi} \varphi, \quad \varphi_n \xrightarrow{\gamma} 0. \tag{20}$$

Then, clearly, $\psi_n = \varphi - \varphi_n \xrightarrow{\gamma} \varphi, \quad \psi_n \xrightarrow{\chi} 0.$

Proposition 6 [3].

$$\gamma \perp \chi \Leftrightarrow \mathcal{H}_{\gamma+\chi} = \mathcal{H}_{\gamma} \oplus \mathcal{H}_{\chi}. \tag{21}$$

Theorem 2 [3]. *Let* χ, γ *be symmetric bilinear forms on* Φ. *Assume that* $\chi \geq$ ≥ 0, $\operatorname{Ker} \chi \subseteq \operatorname{Ker} \gamma$, *and* γ *is bounded below in* $\mathcal{H}_{\chi}$, *i. e.,* $\gamma[\varphi] \geq -M \|\varphi\|_{\chi}^2$ *for some* $M \geq 0$. *Then* γ *admits the unique decomposition*

$$\gamma = \gamma_r + \gamma_s, \tag{22}$$

where $\gamma_r \geq -M$, γ_r *is closable in* $\mathcal{H}_{\chi}$, $\gamma_s \perp \chi$, $\gamma_s \geq 0$ *in* $\mathcal{H}_{\chi}$. *If* $M = 0$, *then* $\gamma_r \| \chi$ *and* $\gamma_s \perp \gamma_r$.

3. Singular forms in the scale of Hilbert space. Let Φ and γ be fixed and let χ_{α}, $\alpha \geq 0$, denote a family of positive bilinear forms on Φ; also let

$$\mathcal{H}_{-\alpha} \supset \mathcal{H}_0 \supset \mathcal{H}_{\alpha} \supseteq \Phi, \quad \alpha \geq 0; \tag{23}$$

we define $\chi_{\alpha}(\cdot,\cdot) \equiv (\cdot,\cdot)_{\mathcal{H}_{\alpha}}$, $\alpha \geq 0$.

If $\gamma \| \chi_0$ (i. e., γ is closable in $\mathcal{H}_0$), then, evidently, $\gamma \| \chi_{\alpha}$ for all $\alpha > 0$. But if $\gamma \perp \chi_0$ (or $\gamma \perp \chi_{\alpha_0}$, $\alpha_0 > 0$), then it is possible that $\gamma \| \chi_{\alpha}$ for some $\alpha > \alpha_0$. Conversely, let $\gamma \| \chi_{\alpha}$ for some $\alpha > 0$. Under what condition $\gamma \perp \chi_{\beta}$, $0 \leq \beta < \alpha$?

Theorem 3 . *Let* $\gamma \geq 0$ *on* Φ *and* $\gamma \| \chi_{\alpha}$, $\alpha > 0$. *Assume that*

$$\operatorname{Ker} \bar{\gamma}^{(\alpha)} \subset \mathcal{H}_{\alpha} \tag{24}$$

is dense in $\mathcal{H}_{\beta}$, $0 \leq \beta < \alpha$. *Then* $\gamma \perp \chi_{\beta}$, *where* $\bar{\gamma}^{(\alpha)}$ *denotes the closure of* γ *in* $\mathcal{H}_{\alpha}$.

Proof. (24) $\Rightarrow$ (18) with $\Sigma_{\chi_{\alpha}}^{\varphi}$ for all $\varphi \in \Phi$.

Further, let us study a geometrical question for the scale (23), namely, when a subspace $F_0 \subset \mathcal{H}_{\alpha}$, $\alpha > 0$, is dense in $\mathcal{H}_{\beta}$, $0 \leq \beta < \alpha$?

For $0 \leq \beta < \alpha$, we introduce the rigged Hilbert space

$$\mathcal{H}_- \supset \mathcal{H}_0 \supset \mathcal{H}_+, \tag{25}$$

where $\mathcal{H}_- \equiv \mathcal{H}_\beta$, $\mathcal{H}_0 \equiv \mathcal{H}_\alpha$, $\mathcal{H}_+ \equiv \mathcal{H}_{\alpha+d}$, $d = \alpha - \beta$.

Theorem 4 [3, 8]. *The linear space F_0 closed in $\mathcal{H}_0$ is dense in $\mathcal{H}_-$ iff:*

i) $(\mathcal{H}_0 \ominus F_0) \cap \mathcal{H}_+ = \{0\}$; (26)

or

ii) $(\mathbf{1}_{-,0}(\mathcal{H}_0 \ominus F_0)) \cap \mathcal{H}_0 = \{0\}$; (27)

where $\mathbf{1}_{-,0}$ is canonical isomorphism from $\mathcal{H}_0$ to $\mathcal{H}_-$ [9].

Thus, every positive closed operator V in $\mathcal{H}_\alpha$ such that $\operatorname{Ker} V \cap \mathcal{H}_{\alpha+d} = \{0\}$, $0 < d < \alpha$, generates the bilinear form $\gamma_V(\varphi, \psi) = (V\varphi, \psi)_{\mathcal{H}_\alpha}$ which is singular in $\mathcal{H}_{\alpha-d} \equiv \mathcal{H}_\beta$ $(\gamma_V \perp \chi_\beta)$.

4. Singular operators. According to [4], an operator A densely defined in $\mathcal{H}$ is called *singular* if for any $\psi \in \mathcal{R}(A)$ there exists a sequence $\psi_n \in \mathcal{D}(A)$ such that $\psi_n \to 0$ and $A\psi_n \to \psi$.

Equivalently, A is singular if for any $\varphi \in \mathcal{D}(A)$; there exists a sequence $\varphi_n \in \mathcal{D}(A)$ such that $\varphi_n \to \varphi$ and $A\varphi_n \to 0$ (we can put $\psi = A\varphi$, $\psi_n = \varphi_n - \psi$).

This shows that the formulation of the concept of singularity in terms of forms has some advantage over that in terms of operators. Namely, if A is positive and singular, then the form $\gamma_A(\varphi, \psi) = (A\varphi, \psi)$ is singular in the sense of definition 4. On the other hand, a positive form γ defined on $\Phi \subset \mathcal{H}$ does not necessarily admits an operator representation in this space. However, according to the discussion in Sec. 3, it might be possible to find a positive form $\chi \parallel \gamma$ and then γ will possess an operator representation in $\mathcal{H}_\chi$.

As a conclusion, we present a theorem on connection between singular forms and operators under some additional assumptions.

Assume that a rigged Hilbert space of the form (25) is given. We write $\gamma \in \mathcal{T}_{s_0, \bar{r}_+}$ if the following conditions hold:

i) $Q(\gamma) \subset \mathcal{H}_+$ and is dense in $\mathcal{H}_+$;

ii) $\gamma = \gamma^* = \bar{\gamma}^{(+)} > -\infty$, i. e., γ is symmetric, closed, and bounded below in $\mathcal{H}_+$;

iii) $\operatorname{Ker} \gamma$ is dense in $\mathcal{H}_0$. (28)

It is clear that each $\gamma \in \mathcal{T}_{s_0, \bar{r}_+}$ has the operator representation in $\mathcal{H}_+$, $\gamma(\varphi, \psi) = (V_\gamma \varphi, \psi)_+$, $\varphi, \psi \in \mathcal{D}(V_\gamma) \subset Q(\gamma)$, where $V_\gamma = V_\gamma^* > -\infty$ is the self-adjoint bounded below operator in $\mathcal{H}_+$ such that the set $\operatorname{Ker} V_\gamma = \operatorname{Ker} \gamma$ is dense in $\mathcal{H}_0$. Consider $T_\gamma := \mathbf{1}_{0,+} V_\gamma$ as an operator in $\mathcal{H}_0$, where $\mathbf{1}_{0,+}$ is the canonical isomorphism from $\mathcal{H}_+$ to $\mathcal{H}_0$. Evidently, T_γ is singular in $\mathcal{H}_0$ because the set $\operatorname{Ker} T_\gamma = \operatorname{Ker} V_\gamma$ is dense in $\mathcal{H}_0$. Thus, we can call T_γ the associated singular operator corresponding to the singular bilinear form γ given in the rigged Hilbert space (25).

Conversely, let T be a singular operator in $\mathcal{H}_0$ from (25) with the following properties:

i) the domain $\mathcal{D}(T) \subset \mathcal{H}_+$ and is dense in $\mathcal{H}_+$;

ii) $\mathbf{1}_{-,0}T$ is self-adjoint and bounded below as an operator from $\mathcal{H}_+$ to $\mathcal{H}_0$;

iii) the closure of the range $\mathcal{R}(T)$ in $\mathcal{H}_0$ has zero intersection with $\mathcal{H}_+$, i. e.,

$$\overline{\mathcal{R}(T)}^{(0)} \cap \mathcal{H}_+ = \{0\}. \tag{29}$$

We denote by $\mathcal{E}_{s_0,\bar{r}_+}$ the set of all operators of this sort.

Theorem 5. *For the fixed rigged Hilbert space* (25), *there exists the one-to-one correspondence between the sets* $\mathcal{T}_{s_0,\bar{r}_+}$ *and* $\mathcal{E}_{s_0,\bar{r}_+}$.

Proof. For each $\gamma \in \mathcal{T}_{s_0,\bar{r}_+}$, the singular associated operator T_γ belongs to the class $\mathcal{E}_{s_0,\bar{r}_+}$. Indeed, $\mathbf{1}_{-,0}T_\gamma$ is self-adjoint as $V_\gamma = V_\gamma^*$ and $\mathbf{1}_{-,+} = \mathbf{1}_{-,0}\mathbf{1}_{0,+}$ is a unitary operator. Further, condition (28) implies (29) by theorem 4. Conversely, each $T \in \mathcal{E}_{s_0,\bar{r}_+}$ defines the form $\dot{\gamma}(\varphi, \psi) = \langle \mathbf{1}_{-,0}T\varphi, \psi \rangle$ in $\mathcal{H}_+$ ($\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{H}_-$ and $\mathcal{H}_+$) which is symmetric and bounded below. In $\mathcal{H}_+$, the following representation is true: $\dot{\gamma}(\varphi, \psi) = (V\varphi, \psi)_+ = (\varphi, V\psi)_+$, where $V := \mathbf{1}_{+,-}\mathbf{1}_{-,0}T$ is a self-adjoint operator. Let γ be the closure of $\dot{\gamma}$ in $\mathcal{H}_+$. Finally, the set $\operatorname{Ker}\gamma$ is dense in $\mathcal{H}_0$ by virtue of (29). In fact, the equivalence (29) and

$$\overline{\mathcal{R}(\mathbf{1}_{-,0}T)}^{(-)} \cap \mathcal{H}_0 = \{0\} \tag{30}$$

give

$$0 = (\psi, \operatorname{Ker}\gamma) = \langle \psi, \operatorname{Ker}\gamma \rangle = (\mathbf{1}_{+,-}\psi, \operatorname{Ker}\gamma)_+.$$

for each $\psi \in \mathcal{H}_0$ such that $\psi \perp \operatorname{Ker}\gamma$. This means that $\psi \in \overline{\mathcal{R}(\mathbf{1}_{-,0}T)}^{(-)}$, and therefore, due to (30), $\psi \equiv 0$. Thus, $\gamma \in \mathcal{T}_{s_0,\bar{r}_+}$. It is clear that the singular operator T_γ associated with γ coincides with T.

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Received 09.04.92