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STRUCTURE OF BANACH ALGEBRAS OF BOUNDED CONTINUOUS FUNCTIONS IN THE OPEN DISK, WHICH CONTAIN H^∞, HOFFMAN ALGEBRA, AND NONTANGENTIAL LIMITS

СТРУКТУРА БАНАХОВИХ АЛГЕБР ОБМЕЖЕНИХ НЕПЕРЕРВНИХ ФУНКЦІЙ У ВІДКРИТОМУ КРУЗІ, ЩО МІСТЯТЬ H^∞, АЛГЕБРА ГОФМАНА ТА НЕДОТИЧНІ ГРАНИЦІ

Let $\mathcal{H}_s^G$ be an algebra of bounded continuous functions in an open disk $\mathbb{D}$, of the form $\mathcal{H}_B \cap G$, where $G \overset{\text{def}}{=} C(M(H^\infty)) = \mathrm{alg}(H^\infty, \overline{H^\infty})$ and $\mathcal{H}_B$ is a closed subalgebra in $C(D)$ which consists of all the functions which have nontangential limits a. e. on $\mathbf{T}$ belonging to the Douglas algebra B. The goal of this paper is to describe the maximal ideal space $M(\mathcal{H}_s^G)$ of the algebra $\mathcal{H}_s^G$. We prove that $M(\mathcal{H}_s^G) = M(B) \cup M(\mathcal{H}_0^G)$, where $\mathcal{H}_0^G$ is a closed ideal in G which consists of all the functions having nontangential limits a. e. on $\mathbf{T}$ and these limits are equal to zero. We prove that $H^\infty[\bar{z}] \neq \mathcal{H}^G_{H^\infty + C}$ in the disk. We generalize Chang–Marshall theorem on Banach algebras $\mathcal{H}_s^G$ and prove that $\mathcal{H}_s^G = \mathrm{alg}(\mathcal{H}^G_{H^\infty}, \overline{IB})$ for any Douglas algebra B, where $IB = \{u_\alpha\}_B$ is a set of inner functions such that $\bar{u}_\alpha \in B$ on $\mathbf{T}$.

Нехай $\mathcal{H}_s^G$ – алгебра обмежених неперервних функцій у відкритому крузі $\mathbb{D}$, зображена у вигляді $\mathcal{H}_B \cap G$, де $G \overset{\text{def}}{=} C(M(H^\infty)) = \mathrm{alg}(H^\infty, \overline{H^\infty})$ і $\mathcal{H}_B$ – замкнена підалгебра у $C(D)$, що складається з функцій, які мають недотичні границі, зокрема, на $\mathbf{T}$, що належать алгебрі Дугласа B. У статті наведено опис простору максимальних ідеалів $M(\mathcal{H}_s^G)$ алгебри $\mathcal{H}_B^G$. Доводиться, що $M(\mathcal{H}_s^G) = M(B) \cup M(\mathcal{H}_0^G)$, де $\mathcal{H}_0^G$ – замкнений ідеал в G, який складається з функцій, що мають недотичні границі, зокрема, на $\mathbf{T}$, і ці границі рівні нулю. Крім того, доведено, що в крузі $H^\infty[\bar{z}] \neq \mathcal{H}^G_{H^\infty + C}$. Узагальнюється теорема Чанга-Маршалла про банахові алгебри $\mathcal{H}_B^G$ і доводиться, що $\mathcal{H}_B^G = \mathrm{alg}(\mathcal{H}^G_{H^\infty}, \overline{IB})$ для будь-якої алгебри Дугласа B, де $IB = \{u_\alpha\}_B$ – внутрішні функції, такі, що $\bar{u}_\alpha \in B$ на $\mathbf{T}$.

1. Introduction. The closed uniform Banach subalgebras of $L^\infty(\mathbb{T})$ containing $H^\infty(\mathbb{T})$ are called the *Douglas* algebras. These algebras play an important role in the theory of Hankel and Toeplitz operators. The Chang–Marshall theorem is one of the most important results on subalgebras $L^\infty(\mathbb{T})$. It states that any closed subalgebra B of $L^\infty(\mathbb{T})$ containing $H^\infty(\mathbb{T})$ is generated by $H^\infty(\mathbb{T})$ and the complex conjugates of the interpolating Blaschke products invertible in B [1, Ch. IX]. Unfortunately, this theorem has no generalization to the case of an open disk. The main reason is a very complicated structure of these algebras. Our theorem 4 and remark 1 show that there exist many algebras which are not generated by H^∞ and an arbitrary collection of bounded harmonic functions in the open disk. The algebra $G = \mathrm{alg}(H^\infty, \overline{H^\infty})$ is an analog of the algebra $L^\infty(\mathbb{T})$ in the disk. The algebra G is called a *Hoffman* algebra. The goal of this paper is to study algebras of bounded continuous function $\mathcal{H}, H^\infty \subset \mathcal{H} \subset G$, with the usual uniform norm in the disk.

We will need the properties of the maximal ideal space of $H^\infty(\mathbb{D})$, denoted by

$M(H^\infty)$, where $\mathbb{D} = \{z : |z| < 1\}$ is the unit open disk. The maximal ideal space is the space of nonzero multiplicative linear functional on $H^\infty(\mathbb{D})$. We give $M(H^\infty)$ the weak-star topology and with this topology it is a compact Hausdorff space. It is easy to see that the open unit disk may be regarded as a subset of maximal ideal space. The Corona theorem of Carleson tells us that $M(H^\infty)$ is a compactification of the open disk [1]. We identify a function in H^∞ with its Gelfand transform. Doing this allows us to regard H^∞ as a subset of continuous functions on the maximal ideal space, denoted by $G \overset{\text{def}}{=} C(M(H^\infty)) = \text{alg}(H^\infty, \overline{H^\infty})$. If $f \in G$, then $f|_{\mathbb{D}}$ has a nontangential limit at almost every point of $\mathbb{T}$. Recall that $M(L^\infty) \subset M(H^\infty) \setminus \mathbb{D}$ is the Shilov boundary of H^∞. The theory of Douglas algebras is associated with the structure of maximal ideal spaces of these algebras [1, Ch. IX].

S. Axler and A. Shields [13] proved that if $f \in C(\mathbb{D})$ is a continuous function on $\mathbb{D} \cup M(L^\infty)$, then f has a nonangential limits at almost every point of $\partial\mathbb{D}$. Our main result in [6] was the following.

Theorem A [6]. *Let f be a continuous function $f : \mathbb{D} \to \overline{\mathbb{C}}$ to the Riemann sphere and let f have nontangential limits at almost every point of $\partial\mathbb{D}$. Then $f|_{\mathbb{D}}$ has a continuous extension $\tilde{f} : \mathbb{D} \cup M(L^\infty) \to \overline{\mathbb{C}}$.*

***Remark* A1.** The condition that the function f is bounded is not necessary. C. Bishop proves this result in case where f is bounded [5].

***Remark* A2.** Note that there exists [17] a function $f_0 \in \mathcal{B}$ ($\mathcal{B}$ is the Bloch class, $\mathcal{B} \overset{\text{def}}{=} \{f : \sup |f'(z)|(1 - |z|^2) < \infty\}$) and $f_0 \in \bigcap_{1 \le p < \infty} H^p$ such that f_0 does not have a continuous extension $\tilde{f}_0 : M(H^\infty) \to \overline{\mathbb{C}}$, where H^p is the Hardy space. Thus, f_0 has nontangential limits a. e. on $\mathbb{T}$ and according to Theorem 1 f has a continuous extension to $M(L^\infty)$. Note that, by the Brown–Gauthier theorem [18], f_0 can be extended to a continuous function (with the values in $\mathbb{C} \cup \{\infty\}$) defined on a union of all nontrivial Gleason parts of $M(H^\infty)$. Therefore, there exists a continuous extension $\tilde{f}_0 : \mathbb{D} \cup M(L^\infty) \cup G \to \overline{\mathbb{C}}$, where G is union of all nontrivial Gleason parts of $M(H^\infty)$. Hence, f_0 does not have a continuous extension to the set $M(H^\infty) \setminus (\mathbb{D} \cup \cup M(L^\infty) \cup G)$.

We consider algebras $\mathcal{H}$ of continuous functions in the open disk which have nontangential limits on $\mathbb{T}$ belonging to the Douglas algebra B. Let $\mathcal{H}_0$ be a closed ideal in $C(\mathbb{D})$ which consists of all continuous functions in the disk having nontangential limits a. e. on $\mathbb{T}$ and these limits are equal to zero. Similarly, let $\mathcal{H}_B$ be a closed subalgebra in $C(\mathbb{D})$ which consists of all continuous functions in the disk which have nontangential limits a. e. on $\mathbb{T}$ belonging to the Douglas algebra B.

Theorem B [11]. *Let $G = \text{alg}(H^\infty, \overline{H^\infty}) = C(M(H^\infty))$. Then $G \subset \mathcal{H}_{L^\infty}$ and $G \neq \mathcal{H}_{L^\infty}$. Moreover, $\mathcal{H}_B \not\subset G$ for all Douglas algebras B.*

Definition. *The uniform algebra B is called a logmodular algebra on Y if the set $\log |B^{-1}| \overset{\text{def}}{=} \{\log |f| : f \in B^{-1}\}$ is everywhere dense in the space $\text{Re}\, C(Y)$.*

Since $\log |e^f| = \text{Re}\, f$, an arbitrary Dirichlet algebra is a logmodular algebra. For example, H^∞ is a logmodular algebra which is not a Dirichlet algebra.

Theorem C [11]. *The algebra $\mathcal{H}_{H^\infty}$ is a logmodular algebra on the maximal ideal space $M(\mathcal{H}_{L^\infty})$ (Re $\mathcal{H}_{L^\infty}$ = close $\log|\mathcal{H}_{H^\infty}^{-1}|$).*

Note that $\mathcal{H}_B/\mathcal{H}_0 = B$, where B is a Douglas algebra. The following Theorem gives a formula for representing the maximal ideal space $M(\mathcal{H}_B)$ of the algebra $\mathcal{H}_B$ as the sum of $M(B)$ and $M(\mathcal{H}_0)$.

Theorem D [11]. *Let B be a Douglas algebra, $H^\infty \subseteq B \subseteq L^\infty$. Then $M(\mathcal{H}_B) = M(B) \cup M(\mathcal{H}_0)$ and $M(B) \cap M(\mathcal{H}_0) = \varnothing$. Moreover, $M(B) \cap M(\mathcal{H}_{L^\infty}) = M(L^\infty)$, where $M(\mathcal{H}_{L^\infty}) = M(\mathcal{H}_0) \cup M(L^\infty)$.*

We now consider $M(\mathcal{H}_{L^\infty}) \setminus (\mathbb{D} \cup M(L^\infty))$. We use the following notation and definition. Let $\beta\mathbb{D}$ be the Stone–Cech compactification of the open disk $\mathbb{D}$. We denote by π the continuous projection from $\beta\mathbb{D}$ onto $M(\mathcal{H}_{L^\infty})$, $\pi(z) = z$ for all $z \in \mathbb{D}$.

Theorem E [12]. *We denote $\Gamma \overset{\text{def}}{=} M(\mathcal{H}_{L^\infty}) \setminus (\mathbb{D} \cup M(L^\infty))$. Then*

$$\Gamma = \beta\mathbb{D} \setminus (\mathbb{D} \cup \pi^{-1} M(L^\infty)),$$

that is any continuous bounded function in the open disk $\mathbb{D}$ has a continuous extension at an arbitrary point $t \in \Gamma$.

Theorem E gives a complete description of the compact set $M(\mathcal{H}_{H^\infty})$ (This compact set is an analog of $M(H^\infty)$). In Theorems A–E, we obtained the description of the maximal ideal space of the algebras $\mathcal{H}_B$. In this paper, we want to study the algebras of bounded functions $\mathcal{H}_B^G \overset{\text{def}}{=} \mathcal{H}_B \cap G$.

In Section 2, we characterize a continuous function in the disk which has nontangential limits belonging to the Douglas algebra on the circle $\mathbb{T} = \{z : |z| = 1\}$, we also describe the maximal ideal spaces of algebras of such functions. Our example, given in Section 3, shows that there exist many algebras $\mathcal{H}$, $H^\infty \subset \mathcal{H} \subset G$, in the disk such that $M(\mathcal{H}) \neq M(H^\infty)$. It is a new fact in case of the disk. This example raises the question about an analog of the Chang–Marchall theorem for the disk. In Section 4, we partially solve this problem.

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2. Subalgebras of Hoffman algebra and the maximal ideal spaces. We consider the maximal ideal space $M(\mathcal{H})$ of uniform Banach algebras $\mathcal{H}$, $H^\infty \subset \mathcal{H} \subset G$. Let $H = \{h_\alpha\}$ denote any collection of complex valued bounded harmonic functions $h_\alpha = \mathrm{Re}\,(h_\alpha) + i\,\mathrm{Im}\,(h_\alpha)$. It is easy to see that every bounded harmonic function on $\mathbb{D}$ can be uniquely extended to a continuous function on $M(H^\infty)$. Thus, we can also regard $A_H \overset{\text{def}}{=} \mathrm{alg}\,(H^\infty, \overline{H})$ as a closed subalgebra of G. For example,

$$\mathrm{alg}(H^\infty, \bar{z}) = \mathrm{alg}(H^\infty, \overline{A}) = H^\infty + C(\mathbb{D}),$$

where A is a disk algebra and $C(\mathbb{D})$ is an algebra of all bounded continuous functions on $\overline{\mathbb{D}}$ [2].

We are interested in the following question. Can we represent any closed algebra between H^∞ and G in the form of A_H? We give the negative answer. The following theorem is a generalization of the result in [2].

Theorem 1. *The maximal ideal of the uniform Banach algebra* A_H *coincides with* $M(H^\infty)$ *for any collection of harmonic functions* H.

Proof. Since for any homomorphism $\varphi \in M(H^\infty)$, $\varphi|_{H^\infty}$ is a homomorphism of the algebra H^∞, there exists a continuous mapping π of the compact set $M(A_H)$ into the compact set $M(H^\infty)$, $\pi(\varphi) = \varphi|_{H^\infty}$. We state that $\pi(M(A_H)) = M(H^\infty)$. We only need to verify that for any $\varphi_0 \in M(H^\infty)$, there exists $\psi_0 \in M(A_H)$ satisfying the following condition: $\pi(\psi_0) = \varphi_0$. Let $\varphi_0 \in M(H^\infty)$. Since $M(H^\infty) = M(G)$, this implies that φ_0 can be uniquely extended to a homomorphism of the algebra G. Denote this homomorphism by $\tilde{\varphi}_0$. Then $\psi_0 = \tilde{\psi}_0|_{A_H}$ is the required homomorphism for which $\pi(\psi_0) = \varphi_0$. We only need to prove that π is a bijection. More precisely, it is necessary to show that each homomorphism on H^∞ has a unique extension to a homomorphism on A_H, and every homomorphism on A_H is obtained is this way. Since linear combinations of the forms $f^n h_\alpha^m$, $f \in H^\infty$, $h_\alpha \in H$ are everywhere dense, we only need to consider $\varphi(f^n h_\alpha^m)$ for all $\varphi \in M(A_H)$. Thus, for all n, m, we have, for any homomorphism φ on A_H,

$$\varphi(f^n h_\alpha^m) = \varphi(f^n)\varphi(h_\alpha^m) = \varphi(f^n)[\varphi(h_\alpha)]^m.$$

To prove this theorem, we need the following lemma which is due to Hoffman.

Lemma 1 [8, p.73]. *Let* u *be a bounded real-valued harmonic function on* $\mathbb{D}$. *Then the function* u *defined on* $M(H^\infty)$ *by*

$$u(\varphi) = \varphi(u) = \log|\varphi(e^{u+i\tilde{u}})| = \log|e^{(u+i\tilde{u})(\varphi)}|,$$

is an extension of u *to a continuous function from* $M(H^\infty)$ *to* $\mathbb{R}$, *where* $\tilde{u}$ *denotes the harmonic conjugate of* u *and* $\tilde{u}(0) = 0$.

Obviously, the function $e^{u+i\tilde{u}} \in H^\infty$. Thus, $\log|e^{(u+i\tilde{u})(\varphi)}|$ is a continuous function on $M(H^\infty)$. Let $h_\alpha = u + iv$. Then

$$\varphi(h_\alpha) = \varphi(u) + i\varphi(v) = \log|\varphi(e^{u+i\tilde{u}})| + i\log|\varphi(e^{v+i\tilde{v}})|.$$

Since $e^{u+i\tilde{u}}, e^{v+i\tilde{v}} \in H^\infty$, we see that $\varphi(h_\alpha)$ is defined on H^∞. Then $\varphi(f^n h_\alpha^m)$ is defined on H^∞, too. Now it follows immediately that $M(A_H) = M(H^\infty)$.

***Corollary* 1.** *Let* $\mathcal{A}, A \subseteq \mathcal{A} \subseteq H^\infty$, *be an analytic subalgebra. Then the maximal ideal space of the algebra* $A_{\overline{\mathcal{A}}} = \mathrm{alg}(H^\infty, \overline{\mathcal{A}})$ *coincides with* $M(H^\infty)$.

We consider the algebras $\mathcal{H}$ of continuous functions in the open disk which have nontangential limits on $\mathbb{T}$ belonging to the Douglas algebra B.

Let $\mathcal{R}_0$ be a closed ideal in $C(\mathbb{D})$, $\mathcal{R}_0 = \{f \in C(\mathbb{D}) : f|_{M(L^\infty)} = 0\}$. For $f \in C(\mathbb{D})$, $f|_{M(L^\infty)} = 0$ denotes that the function f has a continuous extension to $\mathbb{D} \cup M(L^\infty) \subset \subset M(H^\infty)$ equal to zero on $M(L^\infty)$. Similarly, $\mathcal{R}_B = \{f \in C(\mathbb{D}) : f|_{M(L^\infty)} \in B|_{M(L^\infty)}\}$ is a uniform Banach algebra, where B is the Douglas algebra. For $f \in C(\mathbb{D})$, $f|_{M(L^\infty)} \in \in B|_{M(L^\infty)}$ denotes that the function f has a continuous extension to $\mathbb{D} \cup M(L^\infty) \subset$

$\subset M(H^\infty)$ which belongs to the Douglas algebra B.

Let $\mathcal{H}_0$ be a closed ideal in $C(\mathbb{D})$ which consists of all continuous functions in the disk which have nontangential limits a. e. on $\mathbb{T}$ and these limits are equal to zero. Similarly, $\mathcal{H}_B$ is closed subalgebra in $C(\mathbb{D})$ which consists of all continuous functions in the disk having nontangential limits a. e. on $\mathbb{T}$ belonging to the Douglas algebra B. The algebras $\mathcal{H}_B$ were introduced by the author in [9] who proved an analog of the Corona theorem of Carleson for $\mathcal{H}_B$ in [10–12]. In this work, we deal with the special case of subalgebras G, namely $\mathcal{R}_B^G = \mathcal{R}_B \cap G$, $\mathcal{H}_B^G = \mathcal{H}_B \cap G$ and the corresponding ideals $\mathcal{R}_0^G = \mathcal{R}_0 \cap G$, $\mathcal{H}_0^G = \mathcal{H}_0 \cap G$.

Theorem 2. *The equality* $\mathcal{R}_0^G = \mathcal{H}_0^G$ *holds. Moreover,* $\mathcal{R}_B^G = \mathcal{H}_B^G$ *for any Douglas algebra* B.

Proof. Let $f \in \mathcal{R}_0^G$. Then, according to S. Axler and A. Shields [13], $f \in \mathcal{H}_0^G$. Conversely, if $f \in \mathcal{H}_0^G$ and f is not an extension of zero to $M(L^\infty)$, then there exists a point $x \in M(L^\infty)$ for which $\sup\lim_{z\to x} |f(z)| = \alpha > 0$. Consider a set $U_\alpha = \{z : |f(z)| >$ $> \alpha/2\}$. Obviously, x is a limit point for U_α. Therefore, $[U_\alpha]_{M(H^\infty)} \cap M(L^\infty) \neq \varnothing$.

For the proof of the theorem, we need the following brilliant fact that was proved by Gamelin [14, p.23],

Lemma 2. *Let S be any subset of the disk* $\mathbb{D}$. *Then* $[S]_{M(H^\infty)} \cap M(L^\infty) \neq \varnothing \Leftrightarrow$ $\Leftrightarrow m(F(S)) > 0$, *where* m *is the Lebesgue measure and* $F(S)$ *is a set of nontangential cluster points of* S.

If there exists some angle Γ and some sequence $\{z_n\}$ of points of S such that $\{z_n\} \subset \Gamma$ and $\lim_{n\to\infty} |z_n - z_0| = 0$, then the point $z_0 \in \overline{S} \cap \mathbb{T}$ is called a nontangential cluster point of S.

By using Lemma 2, we obtain that the set of nontangential cluster points has a positive measure. According to the definition of the set U_α, this is impossible ($f \in \mathcal{H}_0^G$), and so, $f|_{M(L^\infty)} = 0$. Therefore, $f \in \mathcal{R}_0^G$.

To prove the second statement of Theorem 2, we need the following result.

Lemma 3. *Let B be a Douglas algebra and $\hat{f}$ be the Gelfand transform of $f \in B$. Then for an arbitrary point $m \in M(B)$,*

$$\hat{f}(m) = \lim_{\alpha} \int_{\mathbb{T}} f(\zeta) P_{z_\alpha}(\zeta) dm(\zeta),$$

where z_α is the net of points in D which converge to $m \in M(H^\infty)$, $P_{z_\alpha}(\zeta)$ is the Poisson kernel, and $dm(\zeta)$ is the normalized Lebesgue measure on $\mathbb{T}$.

Proof of Lemma 3. Since the maximal ideal space $M(B)$ of the Douglas algebra B, according to Chang's theorem [1, Th. IX.3.4], is a unique definition of the algebra B, the compact $M(B)$ may be identified with the space of measure in which there exists a weak compactness. By using Corona Theorem of Carleson, we find that for any point $m \in M(B)$, there exists a net $\{z_\alpha\}$ of points in $\mathbb{D}$ such that $z_\alpha \to m$ in $M(H^\infty)$. Therefore, $\mu_{z_\alpha} \to \mu_m$ in the weak topology of the space of measure. The measure μ_{z_α} is induced by the Poisson integral

$$\mu_{z_\alpha}(f) = \int_{M(L^\infty)} f(\zeta) P_{z_\alpha}(\zeta)\, dm(\zeta).$$

where $f \in L^\infty$. If $f \in B$, then

$$\hat{f}(m) = \int_{M(L^\infty)} f\, d\mu_m = \lim_\alpha \int_{M(L^\infty)} f\, d\mu_{z_\alpha} = \lim_\alpha \int_{\mathbb{T}} f(\zeta) P_{z_\alpha}(\zeta)\, dm(\zeta),$$

for any $f \in B$ and any $m \in M(B)$.

We now finish the proof of Theorem 2. If $f \in \mathcal{H}_B^G$, then $f|_{\mathbb{T}} = g \in B$. Thus, taking the harmonic extension $H_g(z)$ of g to the open disk $\mathbb{D}$, we find that $f - H_g \in \mathcal{H}_0^G$. According to the first statement of Theorem 2, $f - H_g \in \mathcal{R}_0^G$, and according to Lemma 3, $H_g = \hat{g}$ on $M(L^\infty)$. Therefore, $f|_{M(L^\infty)} = \hat{g}|_{M(L^\infty)}$ and $f|_{M(L^\infty)} \in B$, $f \in \mathcal{R}_B^G$.

Conversely, let $f \in \mathcal{R}_B^G$. Thus, taking the harmonic extension $H_g(z)$ of g to the open disk $\mathbb{D}$, we obtain $(f - H_g)|_{M(L^\infty)} = 0$. According to the first statement of Theorem 2, $f - H_g \in \mathcal{H}_0^G$. Therefore, $f \in \mathcal{H}_B^G$.

The immediate goal of this section is to describe the maximal ideal space $M(\mathcal{R}_B^G)$ of the algebra $\mathcal{R}_B^G$. According to Theorem 2, we may assume that $\mathcal{H}_B^G$ lies inside $\mathcal{R}_B^G$, and vice versa. Note that $\mathcal{H}_B^G / \mathcal{H}_0^G = B$. Therefore, we claim that $M(\mathcal{H}_B^G) = M(B) \cup M(\mathcal{H}_0^G)$. The main result of this section is the following theorem.

Theorem 3. *Let B be a Douglas algebra, $H^\infty \subseteq B \subseteq L^\infty$. Then $M(\mathcal{H}_B^G) = M(B) \cup M(\mathcal{H}_0^G)$ and $M(B) \cap M(\mathcal{H}_0^G) = \varnothing$; here, $M(B)$ and $M(\mathcal{H}_0^G)$ are subsets of $M(\mathcal{H}_B^G)$. Moreover, $M(\mathcal{H}_0^G) = M(H^\infty) \setminus M(L^\infty)$.*

Proof. Since $\mathcal{H}_B^G / \mathcal{H}_0^G = B$, then, according to [16, Th. 6.2],

$$M(B) = \{m \in M(\mathcal{H}_B^G) : m(\mathcal{H}_0^G) = 0\}.$$

The following result is now necessary.

Lemma 4. If B is Douglas algebra, then

$$M(\mathcal{H}_0^G) = R \overset{\text{def}}{=} \{m \in M(\mathcal{H}_B^G) : m(\mathcal{H}_0^G) \neq 0\}.$$

Proof of Lemma 4. Let us prove that the projection $i : R \to M(\mathcal{H}_0^G)$, $i(m) \overset{\text{def}}{=} m|_{\mathcal{H}_0^G}$ is a homomorphism. Let $m|_{\mathcal{H}_0^G} = \tilde{m}$, and for any $\tilde{m} \in M(\mathcal{H}_0^G)$, we denote

$$m(f) \overset{\text{def}}{=} \tilde{m}(fg)/\tilde{m}(g) \quad (f \in \mathcal{H}_B^G),$$

where $g \in \mathcal{H}_0^G$, $\tilde{m}(g) \neq 0$ and is fixed. Obviously, $m(f)$ is a homomorphism onto $\mathcal{H}_B^G$ and $m(f)$ does not depend on g (otherwise, $\tilde{m}(fg_1)/\tilde{m}(g_1) \neq \tilde{m}(fg_2)/\tilde{m}(g_2)$ implies $\tilde{m}(fg_1g_2)/(g_1g_2))$. Hence, $i^{-1}(\tilde{m}) = m$ and $i^{-1} : M(\mathcal{H}_0^G) \to R$. Let us prove that i is a bijection. We take $m_1, m_2 \in R$ and $m_1 \neq m_2$. Assume that $i(m_1) = i(m_2)$. The equality $\tilde{m}_1(fg)/\tilde{m}_1(g) = \tilde{m}_2(fg)/\tilde{m}_2(g)$ contradicts the assumption that $m_1 \neq m_2$. Therefore, $i(m_1) \neq i(m_2)$. Conversely, suppose that $\tilde{m}_1 \neq \tilde{m}_2$. Then $i^{-1}(\tilde{m}_1) \neq i^{-1}(\tilde{m}_2)$. Hence, i is a bijection of R onto $M(\mathcal{H}_0^G)$. The continuity of i is obvious. We now prove that i^{-1} is a continuous transform. Let $\tilde{m}_\alpha \to \tilde{m}_0$, $\tilde{m}_0 \in M(\mathcal{H}_0^G)$, and let $\{\tilde{m}_\alpha\}$ be a net of points in $M(\mathcal{H}_0^G)$. We take $g \in \mathcal{H}_0^G$ such that $\tilde{m}_0(g) \geq \delta_0 > 0$. Then $\tilde{m}_\alpha(g) \geq \delta/2$. Hence,

$$i^{-1}(\tilde{m}_\alpha)(f) = \tilde{m}_\alpha(fg)/\tilde{m}_\alpha(g) \to \tilde{m}_0(fg)/\tilde{m}_0(g) = i^{-1}(\tilde{m}_0)(f).$$

The last equality implies that i^{-1} is a continuous mapping. Therefore, i is a homomorphism.

We now comlete the proof of Theorem 3. Since $\mathcal{H}_0^G$ is an algebra without $\mathbb{1}$, the maximal ideal space $M(\mathcal{H}_0^G)$ is locally compact [15, p. 236]. Denote $B = L^\infty$. Since $\mathcal{H}_{L^\infty}^G = G$, $M\ (H^\infty) = M(G) = M(L^\infty) \cup M(\mathcal{H}_0^G)$, $M\ (L^\infty) \cap M(\mathcal{H}_0^G) = \varnothing$ (here, $M(L^\infty)$ and $M(\mathcal{H}_0^G)$ are subsets of $M(\mathcal{H}_{L^\infty}^G))$ as in the proof above. Therefore, $M(\mathcal{H}_0^G) = M(H^\infty) / M(L^\infty)$. This completes the proof.

Corollary 2. *The maximal ideal space* $M\ (\mathcal{H}_{H^\infty}^G)$ *is* $M\ (\mathcal{H}_{H^\infty}^G) = L_1 \cup L_2$, *where* $L_1 = L_2 = M(H^\infty)$ *and* $L_1 \cap L_2 = M(L^\infty)$.

3. Pathological algebras on the disk. Recall that $A_{\overline{\mathcal{A}}} = \operatorname{alg}(H^\infty, \overline{\mathcal{A}})$, where $\mathcal{A}$ is the analytic algebra, $A \subseteq \mathcal{A} \subseteq H^\infty$ (A is the disk algebra).

Theorem 4. *For all analytic algebras* $\mathcal{A}$ *and all Douglas algebras* B $(B \neq L^\infty)$ *the maximal ideal spaces* $M\ (A_{\overline{\mathcal{A}}})$ *and* $M\ (\mathcal{H}_B^G)$ *are different. Moreover,* $A_{\overline{\mathcal{A}}} \neq \neq \mathcal{H}_B^G = \mathcal{R}_B^G$.

Proof. The last statement holds since $M(A_{\overline{\mathcal{A}}})$ and $M(\mathcal{H}_B^G)$ are different. According to Corollary 1, $M(A_{\overline{\mathcal{A}}}) = M(H^\infty)$, and according to Theorem 3, $M(\mathcal{H}_B^G) = M(B) \cup (M(H^\infty) \setminus M(L^\infty))$. Hence, if $B \neq L^\infty$, then $M(A_{\overline{\mathcal{A}}}) \neq M(\mathcal{H}_B^G)$.

Remark 1. Obviously, the analog of Theorem 4 holds for the case of an arbitrary collection H of bounded complex harmonic functions.

Example. Let $\mathcal{A}$ be the disk algebra A. According to [2],

$$A_{\overline{\mathcal{A}}} = \operatorname{alg}(H^\infty, \overline{A}) = H^\infty[\bar{z}] = H^\infty + UC(D),$$

where $UC(D) \overset{def}{=} C(\overline{D})|_D$. Clearly, $A_{\overline{\mathcal{A}}}|_{\mathbb{T}} = H^\infty + C$. Therefore, we consider the algebra $\mathcal{H}_{H^\infty + C}^G$. According to Theorem 3, $M(\mathcal{H}_{H^\infty + C}^G) = M(H^\infty + C) \cup (M(H^\infty) \setminus M(L^\infty))$. Then, by using Corollary 1, we get $M(A_{\overline{\mathcal{A}}}) = M(H^\infty)$. Hence, $A_{\overline{\mathcal{A}}} = H^\infty[\bar{z}] = H^\infty + UC(D) \not\subseteq \mathcal{H}_{H^\infty + C}^G$. This example is a good illustration of Theorem 4.

Remark 2. Algebras $A_{\overline{\mathcal{A}}}$ and $\mathcal{H}_{H^\infty + C}^G$ are subalgebras of the Hoffman algebra G and $H^\infty \subset A_{\overline{\mathcal{A}}}$, $H^\infty \subset \mathcal{H}_{H^\infty + C}^G$. Moreover, according to Theorem 2, $A_{\overline{\mathcal{A}}}|_{M(L^\infty)} = \mathcal{H}_{H^\infty + C}^G|_{M(L^\infty)} = H^\infty + C$.

This is an unexpected result. Therefore, a natural analog of the Chang–Marshall theorem in a disk is a difficult problem [2, 3, 5].

4. Description of certain Banach subalgebras of the Hoffman algebra. The most famous theorem on the Douglas algebras is the Chang–Marshall theorem on the description of Banach algebras B such that $H^\infty \subset B \subset L^\infty$ on $\mathbb{T}$. Later, this result was generalized in various ways; one version is presented in this section. We generalize the Chang–Marshall theorem on Banach algebras $\mathcal{H}_B^G$. Note that $A_H \neq \mathcal{H}_B^G$ for all Douglas algebras B $(B \neq L^\infty)$. Therefore, we shall replace Banach algebras H^∞ by the Banach algebra $\mathcal{H}_{H^\infty}^G$. Let $\{u_\alpha\}_B = IB$ be inner functions such that $\bar{u}_\alpha \in B$.

Theorem 5. *For any Douglas algebra* B, $\mathcal{H}_B^G = \operatorname{alg}(\mathcal{H}_{H^\infty}^G, \overline{IB})$.

Proof. Let $f \in \operatorname{alg}(\mathcal{H}^G_{H^\infty}, \overline{IB})$. Thus, for any $\varepsilon > 0$, by taking $f_\varepsilon = \sum_{i=1}^N g_i \overline{u}_i$, $g_i \in \mathcal{H}^G_{H^\infty}$, $u_i \in IB$, we obtain that $\| f - f_\varepsilon \|_{C(\mathbb{D})} < \varepsilon$. Since $g_i, \overline{u}_i \in G$, we have $f_\varepsilon|_{M(L^\infty)} = (\sum_{i=1}^N g_i \overline{u}_i)|_{M(L^\infty)} \in B$. According to Theorem 2, $f_\varepsilon \in \mathcal{H}^G_B$. By using the completeness of $\mathcal{H}^G_B$, we can see that $f \in \mathcal{H}^G_B$. Hence, alg $(\mathcal{H}^G_{H^\infty}, \overline{IB}) \subseteq \mathcal{H}^G_B$.

Conversely, let $f \in \mathcal{H}^G_B$. By using Theorem 2, we see that $f|_{M(L^\infty)} \in B$. Therefore, for any $\varepsilon > 0$, by the Chang-Marshall theorem, we find $\{g_i\}_i^N \in H^\infty$ and $\{u_i\}_i^N \in IB$ such that

$$\| f|_{M(L^\infty)} - \Big(\sum_{i=1}^N g_i \overline{u}_i \Big)|_{M(L^\infty)} \| < \varepsilon. \tag{1}$$

Consider the function $W_\varepsilon \overset{\text{def}}{=} \sum_{i=1}^N g_i \overline{u}_i$ on $\mathbb{D}$. Define the function $\Psi_\varepsilon = f - W_\varepsilon$ on $\mathbb{D}$. Note that $\| \Psi_\varepsilon|_{\mathbb{T}} \| < \varepsilon$ by (1). The harmonic extension of the function $\Psi_\varepsilon|_{\mathbb{T}}$ to the unit disk $\mathbb{D}$ is defined by $P(\Psi_\varepsilon|_{\mathbb{T}})$. According to Maximal Principle for a harmonic function, $\| P(\Psi_\varepsilon|_{\mathbb{T}}) \|_{C(\mathbb{D})} < \varepsilon$. Define the function $R^G_{0,\varepsilon} \overset{\text{def}}{=} \Psi_\varepsilon - P(\Psi_\varepsilon|_{\mathbb{T}}) \in \mathcal{H}^G_0$. Obviously, $W_\varepsilon + R^G_{0,\varepsilon} \in \operatorname{alg}(\mathcal{H}^G_{H^\infty}, \overline{IB})$. We claim that the function $W_\varepsilon + R^G_{0,\varepsilon}$ is the required one. Indeed,

$$\| f - (W_\varepsilon + R^G_{0,\varepsilon}) \|_{C(\mathbb{D})} = \| \Psi_\varepsilon - \Psi_\varepsilon + P(\Psi_\varepsilon|_{\mathbb{T}}) \|_{C(\mathbb{D})} = \| P(\Psi_\varepsilon|_{\mathbb{T}}) \|_{C(\mathbb{D})} < \varepsilon.$$

Since the algebra $\operatorname{alg}(\mathcal{H}^G_{H^\infty}, \overline{IB})$ is closed and $\varepsilon > 0$ is arbitrary, $\mathcal{H}^G_B \subseteq \operatorname{alg}(\mathcal{H}^G_{H^\infty}, \overline{IB})$. This completes the proof.

Remark 3. The theorem which characterizes exactly when a function g on the disk is in $\operatorname{alg}(H^\infty, f)$ and which replaces the Chang-Marshall theorem for these algebras may be found in [5].

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