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On some locally finite groups which are sharply triply transitive

О некоторых локально конечных группах, являющихся сильно трижды транзитивными

We classify the groups Γ satisfying the following conditions:
i) Γ is locally finite;
ii) Γ is a sharply triply transitive permutation group;
iii) all elements of Γ have fixpoints.

Мы классифицируем группы Г, удовлетворяющие следующим условиям:
i) Г локально конечна;
ii) Г строго трижды транзитивная группа подстановок;
iii) все элементы из Г имеют неподвижные точки.

Ми класифікуємо групи Г, які задовольняють такі умови:
i) Г локально скінченна;
ii) Г строго тричі транзитивна група підстановок;
iii) всі елементи із Г мають нерухомі точки.

Introduction. In this paper we are concerned with (infinite) locally finite permutation groups of low degree of transitivity. By local finiteness we will be able to use counting arguments which allows us to make statements about finite subgroups. It is well known that the groups PGL (2, K), in its natural operation on the subspaces of the vectorspace of dimension 2 over the field K, operate sharply triply transitively on the one-dimensional subspaces. If furthermore all quadratic polynomials over K are reducible, then PGL (2, K) does not possess fixpoint-free permutations. It is the object of this note to show that these groups are the only such groups. To do this we will consider finite subgroups and also sharply doubly transitive groups.

Notation is mostly standard. We will denote group elements (permutations) by small greek letters like α, β and subgroups by capital letters Γ, Λ, for permuted objects we use latin letters a, b, ..., x, y and for corresponding subsets again capital letters A, B, ...

If Γ is a group, the set of all nonidentity elements of Γ is denoted by Γ^+. A group is sharply doubly transitive on the set S, if it contains one and only one element mapping the given ordered pair a, b onto the pair x, y; it is sharply triply transitive if the same is true for any two ordered triplets. A group is regular if it contains one and only one element mapping x onto y, for any x, y. $N_\Gamma(\Omega) = \langle \alpha | \alpha \in \Gamma, g\alpha\ \Omega = \Omega \rangle$ is the normalizer of Ω in Γ. The index Γ is mostly deleted. This paper is based on results by R. Ritthaler [1] and extends them.

1. Sharply doubly transitive groups. We will see that the situation here is similar to the case of finite groups.

Lemma 1. *Assume that* Γ *is a locally finite group and also a sharply doubly transitive permutation group of the infinite set* R. *Choose two permutations* α, β *in* Γ^+ *having different fixpoints a and b. If* Ω *is any finite subgroup of* Γ *which contains* α *and* β, *then the set of fixpoint-free elements of* Ω, *together with the identity, forms a nontrivial normal subgroup* Ξ *of* Ω.

Proof. Denote by $L = \{x \mid \exists \sigma \in \Omega^+ \text{ such that } \sigma(x) = x\}$ the set of fixpoints connected with Ω. Since Γ is sharply doubly transitive on R, no element of Ω^+ has more than one fixpoint, and $|L| \leqslant |\Omega^+| = |\Omega| - 1$. The subgroup Ω is now a permutation group on L; and L will split into the domains of transitivity $T_1, \ldots, T_k$ with respect to Ω. The set of permutations in Ω which fix x will be denoted by Φ_x. The subgroups Φ_x and Φ_y with $x, y \in L$ are conjugate if and only if x and y belong to the same T_j, so the conjugacy classes of the subgroups Φ_x correspond to the domains of transitivity of L. Now $|T_i| = |\Omega : \Phi_x|$ for x in T_i, and $\Phi_x \cap \Phi_y = 1$ if $x \neq y$. We deduce

$$|\Omega^+| \geqq \sum_{x \in L} |\Phi_x^+| = \sum_{x_i \in T_i} |T_i| |\Phi_{x_i}^+|,$$

and since $|T_i| \cdot |\Phi_{x_i}| = |\Omega|$ and $|\Phi_x^+| \geqq \frac{1}{2} |\Phi_x|$, we have $k = 1$.

This means that Ω operates transitively on the fixpoints in L and that all Φ_x are conjugate, with pairwise trivial intersection. By the theorem of Frobenius (see for instance M. Hall [2, Theorem 16.8.8, p. 293]), the fixpoint-free permutations and the identity element form a normal subgroup, and a simple calculation shows $|L| = |\Xi|$.

Lemma 2. *If* Γ *is a sharply doubly transitive locally finite group, then the identity element and the fixpoint-free elements together form a nontrivial normal subgroup, provided that* $|\Gamma| > 2$.

Proof. It suffices to show that the product of any two fixpoint-free permutations is the identity or fixpoint-free. For this, choose Ω as in Lemma 1 but also containing the two fixpoint-free permutations.

Lemma. 3. *In the situation of Lemma* 2, *the normal subgroup containing all fixpoint-free elements is elementary abelian and regular.*

Proof. Choose two different elements x, y of R, we have to show that there is exactly one fixpoint-free permutation which maps x onto y. Choose first any fixpoint-free element σ, it will map x onto z. Since Γ is doubly transitive, there is a permutation ρ mapping x onto x and z onto y. Now $\rho^{-1}\sigma\rho$ is fixpoint-free and maps x onto y. If ω is another fixpoint-free permutation mapping x onto y, then $\omega^{-1}\rho^{-1}\sigma\rho$ has the fixpoint y, which contradicts Lemma 2. So there is exactly one such permutation and the normal subgroup operates regularly on R.

If α is the only fixpoint-free permutation mapping x onto y and β is any permutation, them $\beta^{-1}\alpha\beta$ is the only fixpoint-free permutation mapping $\beta(x)$ onto $\beta(y)$. Since Γ is doubly transitive we obtain that all fixpoint-free permutations on Γ are conjugate and, in particular, of the same order, which is a prime number.

Lemma 3 is clearly true if the fixpoint-free elements have order 2. If not, apply Lemma 1 to $\langle \alpha, \beta, \delta, \rho, \sigma \rangle$ with α, β fixing a and b respectively, δ exchanging a and b, and ρ, σ fixpoint-free.

Now δ operates by conjugation on the normal subgroup of fixpoint-free permutations, with trivial centralizer. This is only possible if this normaly subgroup is commutative, and σ and ρ commute. Lemma 3 is shown.

We can see now that there is a one-to-one correspondence of the sharply doubly transitive groups and the nearfields, also for the locally finite case. In particular we have.

Lemma 4. *If* Γ *is a locally finite sharply doubly transitive group and* Θ *is an abelian subgroup of* Γ, *then either*

i) Θ *consists of fixpoint-free elements only and* Θ *is elementary abelian, or*

ii) Θ^+ *does not possess fixpoint-free elements, and* Θ *is locally cyclic.*

Furthermore, if Θ is a p-group, then Θ is abelian or locally generalized quaternion (and $p = 2$).

P r o o f. By Schur's Lemma (see Robinson [3; Theorem 8.14, p. 211]) abelian groups of fixpoint-free automorphisms of finite abelian groups are cyclic. The generalized quaternion groups are the only noncyclic groups with all abelian subgroups cyclic (see Huppert [4; Satz 8.2, p. 310]).

C o r o l l a r y 1. *If two elements of a locally finite sharply doubly transitive group have the same order, then they also have the same number of fixpoints.*

2. F i n i t e s u b g r o u p s o f t h e t r i p l y t r a n s i t i v e g r o u p s. Here we consider groups satisfying the following hypothesis (*): Γ is locally finite and sharply triply transitive such that every element has at least one fixpoint.

T h e o r e m 1. *Assume that Γ satisfies hypothesis (*) and*

i) *Ω is a finite subgroup of Γ;*

ii) *no object of R is fixed by all alements of Ω.*

Then Ω is a dihedral group or $|\Omega| = (p^f + 1)\, p^f\, (p^f - 1)$, or $|\Omega| = \frac{1}{2}(p^f + 1)\, p^f\, (p^f - 1)$, where p is a prime number.

R e m a r k. The last case $|\Omega| = \frac{1}{2}(p^f + 1)\, p^f\, (p^f - 1)$ is impossible for $p = 2$.

P r o o f. We proceed by considering two different cases.

C a s e 1. Every element in $\Omega+$ has two fixpoints. Choose some object x in L,—the set of objects fixed by at least one element in $\Omega+$. If there is no second element y of L which is also fixed by all elements of Φ_x, we deduce a contradiction from Lemma 1. So $\Phi_x = \Phi_y$ for some $x \neq y$, and $\Phi_a = \Phi_b$ or $\Phi_a \cap \Phi_b = 1$ for a, b in L.

The finite set L splits into domains of transitivity $T_1, \ldots, T_k$, corresponding to conjugacy classes of the subgroups Φ_x. We choose representatives $\Phi_1, \ldots, \Phi_m$ of these conjugacy classes and obtain

(1) $|\Omega^+| = \sum_{i=1}^{m} |\Omega : N(\Phi_i)|\,|\Phi_i^+|$.

An element of the normalizer $N(\Phi_i)$ of Φ_i which is not contained in Φ_i itself will interchange the two fixpoints of Φ_i, so we have $|N(\Phi_i) : \Phi_i| \leqslant 2$.

Since Φ_i is non-trivial, we find furthermore $|\Phi_i^+| \geqq \frac{1}{2}|\Phi_i|$. Using this in equation 1, we obtain

(2) If s is the number of conjugacy classes of selfnormalizing subgroups Φ_i and r is the number of the remaining ones, then $2s + r = 3$. Assume first $s = 1$, and Φ_1 of order n_1 selfnormalizing, Φ_2 of order n_2 not selfnormalizing, and Ω of order n. Then equation (1) yields

$$1 - \frac{1}{2} = 1 - \frac{1}{n_1} + \frac{1}{2}\left(1 - \frac{1}{n_2}\right),$$

$$\frac{1}{n_1} + \frac{1}{2n_2} = \frac{1}{n} + \frac{1}{2}$$

with $n_1, n_2 \geqq 2$.

From $n_2 \geqq 2$ we deduce $n_1 < 4$; and we obtain the two possibilities $n_1 = 2$, $n = 2n_2$ which leads to the dihedral groups of order $4k + 2$, and $n_1 = 3$, $n^2 = 2$, $n = 12 = \frac{1}{2}(3 - 1)\, 3\, (3 + 1)$ which leads to the alternating group A_4.

As a next step, assume $s = 0$ and $r = m = 3$. Proceeding as defore with three representatives of orders n_1, n_2, n_3 with $n_1 \leqslant n_2 \leqslant n_3$ we obtain from equation (1)

$$1 - \frac{1}{n} = \frac{1}{2}\left(3 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right)$$

and

$$\frac{1}{n}+\frac{1}{2}=\frac{1}{2}\left(\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}\right).$$

We deduce $n_1 = 2$ and $n_2 \leqslant 3$.

For $n_1 = n_2 = 2$ we obtain $n=2n_3$, and this leads to the dihedral groups of order $4k$.

For $n_1 = 2$, $n_2 = 3$ we have $\frac{1}{n}+\frac{1}{12}=\frac{1}{2n_3}$ and $12+n=6|\Omega:\Phi_3|$, $12=|\Omega:\Phi_3|(6-\Phi_3)$ with $n_3=|\Phi_3|\geqq 3$.

We are left with the numerical possibilities

a) $n_3 = 3$, $n = 12$,
b) $n_3 = 4$, $n = 24$,
c) $n_3 = 5$, $n = 60$.

Here there is no group for possibility a) since it would have two conjugacy classes of Sylow-3-subgroups. The other two lead to S_4 and A_5. Case 1 is now completed.

Case 2. There is an element in Ω with only one fixpoint. Choose a fixpoint x of such an element. The subgroup Φ_x of all elements fixing x possesses by Lemma 1 and Lemma 2 an elementary abelian normal subgroup Ψ_x consisting of the identity and all elements fixing x only. Elements outside Ψ_x do not commute with any element in Ψ_x^+.

This yields:

(3) $|\Phi_x:\Psi_x|$ divides $|\psi_x|-1$,

(4) Ψ_x is a Sylow-p-subgroup of Φ_x, and furthermore we know

(5) $\Psi_x \cap \Psi_y = 1$ for $x \neq y$, and from (4) we have

(6) the objects x occurring as signle fixpoints form one domain of transitivity in L.

Let $|\Psi_x| = p^f$; by $\Phi_x = N(\Psi_x)$ and (5) we have

(7) $|\Omega:\Phi_x| = 1 + kp^f$.

We take an object y of L which does not belong to the domain of transitivity mentioned in (6). Then there is an object z in L such that every element in Ω fixing y will also fix z. The corresponding subgroups of elements fixing objects like y fall into conjugacy classes of Ω, and we choose representatives $\Xi_1, \ldots, \Xi_m$ of these classes. Certainly $|N(\Xi_i):\Xi_i| \leqslant 2$.

All complements of Ψ_x in Φ_x are conjugate, and we choose a representative Λ. Again $|N(\Lambda):\Lambda| \leqslant 2$. Now we can count the elements of Ω.

$$|\Omega| = (1+kp^f)(p^f-1) + |\Omega:N(\Lambda)|\,|\Lambda^+| + \sum_{i=1}^{m}|\Omega:N(\Xi_i)|\,|\Xi_i^+| + 1 =$$

$$= (\Lambda + kp^f)\, p^f\, |\Lambda|.$$

Case 2.1. $\Lambda = 1$. In this case we have $m = 1$ and

$$kp^f = |\Omega:N(\Xi)|\,|\Xi^+| \geqq \frac{1}{4}|\Omega|,$$

so $p \leqslant 3$.— For $p^f = 2$ we find that $|\Omega:N(\Xi)|$ divides both $|\Omega| = 3(1+3k)$ and $3k$. If $|\Omega:N(\Xi)| = 3$, then $|\Xi| = k+1$ which would be a divisor of $3(1+3k)$ and further $k+1$ divides 6. This leads to $k=1$ and $\Omega \cong A_4$.

For $|\Omega:N(\Xi)| = 1$ we arrive at a contradiction.

For $p^f = 2$ we find dihedral groups.

Case 2.2. $\Lambda = N(\Lambda)$. Here we obtain again $m = 1$ and

$$|\Omega| = (1+kp^f)(p^f-1) + |\Omega| - (1+kp^f)\,p^f + |\Omega:N(\Xi)|\,|\Xi^+| + 1,$$

$$kp^f = |\Omega:N(\Xi)|\,|\Xi^+| > \frac{1}{4}|\Omega|, \qquad \text{so } p^f|\Lambda| < 4, \text{ which is impossible.}$$

Case 2.3: $1 \neq \Lambda \neq N(\Lambda)$. Here we find

$$|\Omega| = (1+kp^f)(p^f-1) - \frac{1}{2}|\Omega| + \frac{1}{2}(1+kp^f)\,p^f - 1 =$$

$$= \sum_{i=1}^{m} |\Omega : N(\Xi_i)|\,|\Xi_i^+|,$$

$$|\Omega| + (2k-1)\,p^f - kp^{2f} = 2\sum_{i=1}^{m} |\Omega : N(\Xi_i)|\,|\Xi_i^+|,$$

and since the left hand side of the equation is smaller than $|\Omega|$ we have again $m = 1$ and $N(\Xi) \neq \Xi$. This yields

$$|\Omega| + (2k-1)\,p^f - kp^{2f} = |\Omega| - |\Omega : \Xi|$$

and $|\Omega : \Xi| = kp^{2f} - (2k-1)\,p^f$.

This must be a divisor of $(1+kp^f)\,p^f\,(p^f-1)$, which is a multiple of $|\Omega|$. So $kp^f - (2k-1)$ is a divisor of $(1+kp^f)(p^f-1)$. But then $kp^f - (2k-1)$ is a divisor of $2k-2$, and we have as the trivial possibility $k=1$. For this case $|\Omega|$ divides $(p^f+1)\,p^f\,(p^f-1)$ and is a multiple of p^f-1, and $|\Omega|$ is as given in the theorem. On the other hand, if $k \neq 1$, we have $p^f \leqq 3$. For $p^f = 3$ we have that $k+1$ divides 4, therefore $k=3$, and $|\Omega| = 60 = \frac{1}{2}6\cdot5\cdot4$.

For $p^f = 2$ we arrive at a contradiction.

The proof of Theorem 1 is complete.

Lemma 5. *Assume that* i) Γ *satisfies hypothesis* (*); ii) Ω *is a finite subgroup of* Γ; iii) *no object of* R *is fixed by all elements of* Ω; iv) Ω *does not admit a domain of transitivity of length* 2 *in* R; v) $|\Omega| > 120$.

If $\Lambda \neq 1$ *is the set of all elements in* Ω *which fix two different objects* a, b *of* R, *then* Λ *is cyclic.*

Proof. According to Theorem 1, Ω belongs to Case 2.3 of the proof and $|\Omega| = (p^f+1)\,p^f\,(p^f-1)$ or $|\Omega| = \frac{1}{2}(p^f+1)\,p^f\,(p^f-1)$ with $p^f \geqq 5$.

In both cases a subgroup Λ as in Case 2.3 satisfies the inequalities $1 \neq \Lambda \neq N(\Lambda)$, and $N(\Lambda)$ is treated in Case 1 of the proof of Theorem 1 (for p odd) or in Case 2.1 (for $p=2$), and $N(\Lambda)$ is a dihedral group. In both cases Λ must be cyclic.

3. The family of triply transitive groups.

Lemma 6. *If satisfies hypothesis* (*) *and* Φ *is the subgroup all of elements fixing a given object of* R, *then* Φ *is countable and isomorphic to the group of mappings* $x \to ax + b$, $a \neq 0$, a, b *in* F, *where* F *is an infinite field of some characteristic* p, *a prime number*.

Proof. By Lemma 2, Φ possesses an elementary abelian normal subgroup containing the identity and all elements with exactly one fixpoint. This is complemented in Φ by a subgroup of all elements fixing two given objects, and by Lemma 5 this is a locally cyclic group. So this complement is countable and R is countable. The group isomorphism follows since the quotient modulo the elementary abelian normal subgroup is abelian and is sharply doubly transitive.

Corollary 2. *If* Γ *satisfies hypothesis* (*), *then* Γ *is a countable group.*

This follows from the countability of R.

Lemma 7. *The subgroup* Ω *mentioned in Lemma* 5 *is isomorphic to* PGL $(2, p^f)$ *or* PSL $(2, p^f)$.

Proof. We recall that in this situation the elements of order p have exactly one fixpoint and that these fixpoints form one domain of transitivity with respect to Ω in R. Since there are p^f+1 Sylow-p-subgroups in Ω and these have pair-wise trivial intersection, Ω operates on the domain just mentioned as a doubly transitive permutation group, furthermore only the identity fixes more than two of these objects. If $p = 2$ we may apply a result by Zassenhaus [5; Satz 18, p. 39], and we have that Ω is isomorphic to PGL $(2,2^f) =$ $=$ PSL $(2,2^f)$. If there is an element of order 2 in Ω which fixes two of the Sylow-p-subgroup, we apply another result by Zassenhaus [5; Satz 19, p. 39] and we have that Ω is isomorphic to PGL $(2, p^f)$ or to PSL $(2, p^f)$ (since the normalizer of the Sylow-p-subgroup is metabelian, the third alternative is impossible). If no element of order 2 is contained in the normalizer of some Sylow-p-subgroup, we choose an element of order 2 in Ω and consider its fixpoints u, v, and we pick an element p of order ρ which has the fixpoint u. The new subgroup $\langle \Omega, \rho \rangle$ of order $\frac{1}{2}(p^m + 1)\, p^m (p^m - 1)$ allows the application of Satz 19 of Zassenhaus [5] so $\langle \Omega, \rho \rangle$ is isomorphic to PSL $(2, p^m)$, and Ω is isomorphic to PSL $(2, p^f)$.

Theorem 2. *If Ω satisfies hypothesis* $(*)$, *then Ω is isomorphic to some group* PGL $(2, F)$ *where F is a field of characteristic a prime number p in which all polynomials of degree 2 are reducible.*

Proof. We choose some method of labelling $x_1, x_2, \ldots, x_i, \ldots$ all elements of R. This is possible since R is countable. We begin with a subgroup Ω as mentioned in Lemma 5. By Lemma 7 Ω is isomorphic to PGL $(2, p^f)$ or PSL $(2, p^f)$, and its commutator subgroup Λ is isomorphic to PSL $(2, p^f)$.

For each object x_i of R we choose an element ρ_i of order p having x_i as fixpoint, and we obtain the sequence

$$\Lambda \subseteq \langle \Lambda, \rho_1 \rangle \subseteq \langle \Lambda, \rho_1, \rho_2 \rangle \subseteq \ldots\ldots \subseteq \langle \Lambda, \rho_1, \ldots, \rho_i \rangle \subseteq \ldots\ldots$$

By obvious induction on i we see that all subgroups of this series are isomorphic to some PSL $(2, p^{t_i})$, where t_{i-1} is a divisor of t_i. We consider the set-theoretical union of the subgroups constructed

$$\Theta = \bigcup_{i=1}^{\infty} \langle \Omega, \rho_1, \rho_2, \ldots, \rho_i \rangle.$$

Obviously, Θ is isomorphic to some PSL $(2, F)$, where F is an infinite field of characteristic a prime number p, and the normalizers of the maximal p-subgroups fix one object in R, so that Ω is doubly transitive on R since we find a member of the series where a given pair u, v is mapped onto a given pair u', v'. Since Ω does not possess elements without fixpoints, the matrix description of SL $(2, F)$ yields that no polynomial of degree 2 in F is irreducible. But it F has this property, every element in F is a square and PSL $(2, F) =$ $=$ PGL $(2, F) \cong \Theta$.

Also, Θ is sharply triply transitive on R. This proves Theorem 2.

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