Abelian-by-FC-hypercentral groups

Группы, являющиеся расширением абелевых посредством FC-гиперцентрализованных групп

A splitting Theorem for FC-hypercentral group $G$ and $ZG$-module of finite rank is obtained. According to this Theorem, under certain conditions, every extension $E$ of the $ZG$-module $A$ by the group $G$ is split and all the complements to $A$ in $E$ are conjugate in $E$.

Для FC-гиперцентрализованной группы $G$ и $ZG$-модуля конечного ранга получена теорема о расщеплении. Согласно этой теореме при определенных условиях каждое расширение $E$ $ZG$-модуля $A$ посредством группы $G$ расщепляемо и все дополнения к $A$ в $E$ сопряжены в $E$.
For FC-ringenentral'nya gruppa $G$ i ZG-moduli skhshennogo ranga ocherna teorema pro rozspilennya. Zgjado z ideo teoremoi pri pervix umosax kjikne rozroven'nye $E$ ZG-moduli $A$ za dopomoqy gruppy $G$ e rozspilenniya i visi dopolnienia do $A$ v $E$ sprijajeni v $E$.

In recent years a number of results have been obtained which say that, under certain conditions, every extension $E$ of the ZG-module $A$ by the group $G$ split and all the complements to $A$ in $E$ are conjugate in $E$. We say that $E$ splits conjugately over $A$. Most of these results concern a group $G$ which is generalized nilpotent (or supersolvable) and a module $A$ which has no factors which are $G$-trivial (or cyclic as abelian groups). Results of this type may be found in [1-6] and may also be found in [7-9] in a cohomology setting where, in particular, it is shown that $H^2(G, A) = H^1(G, A) = 0$.

In [10, 11] D. I. Zalcik considered these splitting theorems for a hyperfinite group $G$ and a module $A$ which is either artinian or noetherian. In [14] he began to consider modules over FC-hypercentral groups showing that such a module with a finite composition series has a decomposition into a direct sum of a finite submodule and a submodule with no nonzero finite factors. Here we consider a splitting theorem in which $G$ is FC-hypercentral and the ZG-module $A$ has finite rank as an abelian group. (By the rank of the abelian group $A$ we mean Mal'tsev special rank or Prufer rank.) We prove the following.

Theorem. Let $G$ be a locally soluble FC-hypercentral group and let $A$ be a ZG-module which has finite rank as an abelian group.

i). $A$ has no nonzero G-hyperfinite images if and only if $A$ has no nonzero finite G-factors.

ii). If $A$ has no nonzero G-hyperfinite image then every extension $E$ of $A$ by $G$ splits conjugately over $A$.

A G-hyperfinite image of $A$ is a ZG-homomorphic image of $A$ which has an ascending chain of ZG-submodules in which the factors are finite. A G-factor of $A$ is a factor $B/C$ where $B$ and $C$ are ZG-submodules of $A$.

It follows from i) that if $A$ has no nonzero G-hyperfinite images then $A$ must be torsion-free and divisible and so (as an abelian group) $A$ is just the direct sum of finitely many copies of $Q$, the additive group of the rationals. In particular, $G/C_0(A)$ is a $Q$-linear group and so is soluble. The simplest examples in which the hypotheses of part ii) of the theorem occur and in which we have splitting is when $G$ acts faithfully on $A$ so that $A$ is a soluble $Q$-linear group and $A$ has a finite $G$-composition series in which the irreducible factors are all infinite.

However, in the statement of part ii) of the theorem there is no assumption that $G$ acts faithfully on $A$ and the simplest way to construct examples of non-split extensions in which $A$ has finite factors is to include an extension of $A$ by $G_0(A)$ which is non-splitting.

For example, let $M = A \oplus B$ be a sum of two copies of the rationals with an isomorphism $\varphi : A \to B$. Let $x$ be the automorphism of infinite order which fixes each element of $A$ and maps $b \in B$ to $b + \varphi(b)$. Form $E$, the split extension of $M$ by $\langle x \rangle$, and let $G = E/A \cong Q \oplus Z$. Let $C$ be any submodule of $M$ not contained in $A$ so that $C$ contains an element $a + b$ with $a \in A$, $b \in B$ and $b \neq 0$. Then $(a + b) x = a + b + \varphi(b) \in C$ and so $\varphi(b) \in C$. Hence $A \cap C \\ C = 0$ and so $A$ is not a direct summand of $M$. Thus $E$ does not split over $A$.

It should be noted that there is no point in considering extensions by hyperfinite groups. For, if $G$ is hyperfinite, then its irreducible modules are elementary abelian p-groups [1] and so, if $A$ has finite rank, its irreducible G-factors are all finite.

We begin by proving part i) of the theorem.

Let $G$ be a locally soluble FC-hypercentral group and let $A$ be a ZG-module which has finite rank as an abelian group. Then $A$ has a nonzero G-hyperfinite image if and only if it has a nonzero finite G-factor.

Proof. We may assume that $G$ acts faithfully on $A$. It is clear that if $A$ has a nonzero G-hyperfinite image then it has a nonzero finite G-factor. So, conversely, we assume that $A$ has a finite G-factor $U/V$ which may be taken to be irreducible and so is a finite elementary abelian p-group.

Choose a submodule $X$ of $A$ maximal subject to $X \cap U = V$. Replacing $A$ by $A/X$ we may assume that $A$ has a unique minimal submodule $U$, and $U$
is a finite elementary abelian $p$-group. Let $T$ be the torsion part of $A$; then $T$
 is a nonzero $p$-group and, since it has finite rank, $T$ is $G$-hyperfinite. So we
may assume that $A/T$ is nonzero.

We now prove by induction on $r = r(A/T)$, the rank of $A/T$, that $A$ has a
nonzero $G$-image which is a (hyperfinite) $p$-group. If $A/T$ is not rationally
irreducible then there is a submodule $B/T$ of $A/T$ such that $r(A/B) < r$ and
$r(B/T) < r$. By induction, $B$ has a nonzero $G$-hyperfinite $p$-image $B/C$. So the
torsion part of $A/C$ is $B/C$ and, by induction again, $A/C$ has a nonzero $G$-
hyperfinite $p$-image. Therefore we may assume that $A/T$ is rationally irreducible.

We claim that $A/T$ is faithful for $G$. If not, then $C_0(A/T) \neq 1$ and so there
is a nontrivial element $x \in C_0(A/T) \cap \Delta(G)$, where $\Delta(G)$ denotes the
FC-centre of $G$. Let $F = \langle x \rangle$; then $F$ is generated by finitely many conju-
gates of $x$ and, if $K = C_0(F)$, we have $|G/K| < \infty$.

For each $y \in F - 1$, the mapping $\phi_y: a \rightarrow a(y - 1)$ is a $\mathbb{Z}K$- homo-
morphism of $A$ into $T$ and, since $G$ acts faithfully on $A$, $\phi_y$ is a nonzero homo-
morphism. Thus $A/C_A(A) \cong \mathbb{Z}K \otimes A(y - 1)$ is a nonzero $K$-image of $A$
which is a $p$-group. Suppose that $F = \langle x_1, \ldots, x_n \rangle$; then $C_A(F) = \bigcap_{i=1}^n C_A(x_i^n)$
and so $A/C_A(A)$ is a $p$-group. But $C_A(F)$ is a $\mathbb{Z}G$-submodule of $A$ and, since $A$
has finite rank, $A/C_A(F)$ is therefore a nonzero $G$-hyperfinite $p$-image.

Thus we may assume that $A/T$ is faithful for $G$ so that $G$ is an irreducible
$\mathbb{Q}$-linear soluble group and so is abelian-by-finite [13] (Theorem 3.24). Let $H$
be an abelian normal subgroup of finite index in $G$ and let $H_1 = C_H(U)$ so that
$|G/H_1| < \infty$. Now $H_1$ is abelian and $A$ has a nonzero $H_1$-trivial $p$-sub-
module $U$. By Lemma 2.8 of [1], $A$ has a nonzero $H_1$-hypertrivial $p$-image
$A/D$. If $s_1, \ldots, s_m$ is a transversal to $H_1$ in $G$ then $D_0 = \bigcap_{i=1}^m D_{s_i}$ is a $\mathbb{Z}G$
submodule of $A$ and $A/D_0$ is a $p$-group. Since $A$ has finite rank, $A/D_0$ is a
nonzero $G$-hyperfinite $p$-image of $A$.

Lemma 1. Let $G$ be an FC-hypercentral group and let $A$ be a $\mathbb{Z}G$-
module which has finite rank as an abelian group. Let $B$ be a submodule of $A$ such
that $B$ has no nonzero finite $G$-factors and $G$ induces a finite group of automorphisms
on $A/B$. Then there is a unique submodule $C$ of $A$ such that $A = B \oplus C$.

Proof. We may assume that $G$ acts faithfully on $A$ and we proceed by induction
on $r = r(B)$ to show that $B$ has a complement in $A$. If $B$ is not rationally
irreducible then it has a submodule $B_1$ such that $r(B/B_1) < r$ and $r(B_1) < r$.
Then $A/B_1$ contains a submodule $C_1$ such that $A/B_1 = (B/B_1) \oplus (C_1/B_1)$.
Now $C_1/B_1 \cong \mathbb{Z}G/B$ and so $G$ induces a finite group of automorphisms on
$C_1/B_1$. Again by induction $C_1$ has a submodule $C_2$ such that $C_1 = B_1 \oplus C_2$
and hence $A = B \oplus C$. So we may assume that $B$ is rationally irreducible.
This means that every proper $G$-image of $B$ is torsion. But since $A$ has finite rank,
no nonzero torsion factor will have nonzero finite $G$-factors. Thus $B$ has no
proper nonzero $G$-images and so is actually irreducible as a $\mathbb{Z}G$-module.

Since $G$ is FC-hypercentral there is a nontrivial element $x \in C_0(A/B) \cap \Delta(G)$.
Let $F = \langle x \rangle$ and $L = C_0(A/B) \cap C_0(F)$, so that $G/L$ is finite. Then $L$
acts trivially on $A/B$ and $B$ has no finite $L$-factors [14] (Proposition 2).
For each $y \in F$, $A(y - 1)$ is a $\mathbb{Z}L$-submodule of $B$ and so has no finite $L$
factors. Also $A/C_A(A) \cong \mathbb{Z}L \otimes A(y - 1)$ and so $A/C_A(A)$ has no finite $L$-factors.
If $F = \langle x_1, \ldots, x_n \rangle$, then $C_A(F) = \bigcap_{i=1}^n C_A(x_i^n)$ and so $A/C_A(F)$ has no
finite $L$-factors. It follows that $C_A(F) + B = A$.

But $A/C_A(F)$ is a $\mathbb{Z}G$-submodule and so $A(C_F) \cap B$ is equal to either $0$ or $B$.
If $C_A(F) \nrightarrow B$ then, since $C_A(F) + B = A$, we have $C_A(F) + B = A$, contrary to
$G$ acting faithfully on $A$. Therefore $C_A(F) \cap B = 0$ and $A = C_A(F) \oplus B$.

Now suppose that $A = B \oplus C = B \oplus C_0$. Then $G$ induces a finite group
of automorphisms on each of $C$ and $C_0$ and hence also on $C + C_0$. Therefore
every irreducible $G$-factor of $C + C_0$ is finite and so $B \cap (C + C_0) = 0$. It
follows that $C = C + C_0 = C_0$.

Lemma 2. Let $G$ be a locally soluble FC-hypercentral group and let $A$ be
a $\mathbb{Z}G$-module which has finite rank as an abelian group and such that $A$ has no
nonzero finite $G$-factors. Let $E$ be an extension of $A$ by $G$ and let $N = C_E(A)$.
Then there is a normal subgroup $M$ of $E$ such that $N = A \times M$ and $M$ is con-
tained in all supplements to $A$ in $E$. 

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Proof. We show first that there is a normal subgroup $M$ such that $N = A \times M$.

Choose $M$ to be a normal subgroup of $E$ maximal with respect to $M \leq N$ and $M \cap A = 1$. By considering $E/M$, we may assume (*) if $S$ is a nontrivial normal subgroup of $E$ contained in $N$, then $S \cap A \neq 1$. We show that, under this assumption, $N$ must be equal to $A$.

(1) Suppose that $N \neq A$; then, since $E/A$ is FC-hypercentral, there is a nontrivial normal subgroup $L/A$ of $E/A$ with $L \leq N$ and $L/A \leq \Delta (E/A)$. But then $L/A$ is a locally soluble $FC$-group and so contains a nontrivial characteristic abelian subgroup $K/A$. (If $Z(L/A) = 1$ then $L/A$ is periodic and we may take $K/A$ to be the socle of $L/A$.) Let $x \in K - A$ and $F = \langle x^p \rangle$. Then $F/A$ is abelian and $E(C_G(F/A))$ is finite. Let $C = C_G(F/A)$; then $[F, C, F] \leq [A, F] = 1$, since $F \leq \Delta E = C_G(F/A)$, Also $[C, F, F] \leq [A, F] = 1$ and so, by the Three Subgroup Lemma [13] (Lemma 2.3.1), $[F', C] = 1$. Therefore $F'$ is central- ized by $C$ and so $G$ induces a finite group of automorphisms on $F'$. It follows that any irreducible $G$-factors of $F'$ are finite. By the hypothesis on $A$ it follows that $F' = 1$ and so $F$ is abelian. We may therefore consider $F$ as a $ZG$-module. By Lemma 1 there is a normal subgroup $C$ of $E$ such that $F = A \times C$. But this is contrary to (*) and so we have $N = A$.

This completes the proof that $N = A \times M$. Now let $E_1$ be a supplement to $E$ in $N$ so that $E = AB_1$ and $N = N \cap AB_1 = A (N \cap E_1)$. Note that $N \cap E_1 \leq E_1$ and $[N \cap E_1, A] = 1$ so that $N \cap E_1 \leq E_1$.

Now $N/N \cap E_1 \cong \text{ZG}(M/M \cap E_1) \cong \text{ZG}(M/M \cap E_1)$. Therefore $M/N \cap E_1$ is an irreducible $\text{ZG}$-module, and so $E/\text{ZG}(M/M \cap E_1)$ is an irreducible representation of $A$. Hence $A$ is abelian-by-finite [13] (Theorem 3.24). Let $B/N$ be an abelian normal subgroup of finite index in $E/N$. By Proposition 2 of [14], $A$ has no nonzero finite $ZG$-factors and, in particular, $A$ is torsion-free. By induction on the rank of $A$ we may assume that $A$ is rationally irreducible (and, as in Lemma 1, $A$ is irreducible as a $ZG$-module).

Let $N = C_A(A)$; then $E/N$ is an irreducible $ZG$-linear soluble group and so is abelian-by-finite [13] (Theorem 3.24). Let $H/N$ be an abelian normal subgroup of finite index in $E/N$. By Proposition 2 of [14], $A$ has no nonzero finite $ZG$-factors and, in particular, $[A, H] = A$.

By Lemma 2 $N = A \times M$ for some $M \leq E$ and each supplement to $A$ in $E$ contains $M$. Let $\tilde{A} = N/M \cong A$ as a $ZG$-module, where $\tilde{G} = E/N$. If $\tilde{H} = H/N$, then $[\tilde{A}, \tilde{H}] = \tilde{A}$ and so, by Theorem $B$ of [1] $E/M$ splits conjugately over $N/M$.

Let $K/M$ be a complement to $N/M$. Then $KA = E$ and $K \cap A = K \cap M \leq A = M \cap A = 1$ so that $K$ is a complement to $A$ in $E$.

If $K_1$ is any other complement to $A$ then $K_1 \leq M$ and so $K_1/M$ is a complement to $N/M$ and hence is conjugate to $K/M$.


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