Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2008).** pp. 30 – 39

Radical functors in the category of modules over different rings

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Communicated by M. Ya. Komarnytskyj

Abstract. The category $\mathcal{G}$ of all left modules over all rings is studied. Necessary and sufficient conditions for a preradical functor on $\mathcal{G}$ to be radical are given. Radical functors on essential subcategories of $\mathcal{G}$ are investigated.

All categories in our paper are concrete. Recall that a category is called concrete if all objects are (structured) sets, morphisms from A to B are (structure preserving) mappings from A to B, composition of morphisms is the composition of mappings, and the identities are the identity mappings [1].

Let $\mathcal{C}$ be an arbitrary concrete category. (Though all these things we can do in an arbitrary category.)

Definition. A preradical functor (or simply a preradical) on $\mathcal{C}$ is a subfunctor of the identity functor on $\mathcal{C}$. In other words, a preradical functor T assigns to each object A a subobject $T(A)$ in such a way that the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle i_1}\big\uparrow & & {\scriptstyle i_2}\big\uparrow \\ T(A) & \longrightarrow & T(B) \end{array}$$

is commutative.

Definition. A preradical functor T is called idempotent if
$T(T(A)) = T(A)$ for every $A \in Ob(\mathcal{C})$.

Remark 1. We will consider only idempotent preradical functors.

2000 Mathematics Subject Classification: *16S90, 16D90.*

Key words and phrases: *category, functor, radical functor, category of modules, essential subcategory.*

Definition. Let T_1 and T_2 be functors from the category $\mathcal{A}$ to the category $\mathcal{B}$. The functor T_1 is called a subfunctor of the functor T_2 (denote $T_1 \leqslant T_2$) if $T_1(A)$ is a subobject of $T_2(A)$ (denote $T_1(A) \subseteq T_2(A)$) for every $A \in Ob(\mathcal{A})$ and the following diagram

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(\varphi)} & T_1(A_2) \\ {\scriptstyle i_1}\big\downarrow & & {\scriptstyle i_2}\big\downarrow \\ T_2(A_1) & \xrightarrow{T_2(\varphi)} & T_2(A_2) \end{array}$$

is commutative for every morphism $\varphi \colon A_1 \to A_2$, $A_1, A_2 \in Ob(\mathcal{A})$.

Definition. The functor T_1 is called a normal subfunctor of the functor T_2 if $T_1(A)$ is a normal subobject of $T_2(A)$ for every $A \in Ob(A)$.

Recall that A' is called a normal subobject of A (or an ideal) if $A' \to A$ is a kernel of some morphism [2, 3].

As a rule we will consider the cases, when the categories $\mathcal{A}$ and $\mathcal{B}$ coincide.

Definition. Let $\mathcal{A}$ be a category, T_1 and T_2 be functors on $\mathcal{A}$, such that T_1 is a normal subfunctor of T_2. A factor-functor T_2/T_1 is a functor such that $(T_2/T_1)(A) = T_2(A)/T_1(A)$ $\forall A \in Ob(\mathcal{A})$ and the next diagram is commutative

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(\varphi)} & T_1(A_2) \\ {\scriptstyle i_1}\big\downarrow & & {\scriptstyle i_2}\big\downarrow \\ T_2(A_1) & \xrightarrow{T_2(\varphi)} & T_2(A_2) \\ {\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ T_2(A_1)/T_1(A_1) & \longrightarrow & T_2(A_2)/T_1(A_2), \end{array}$$

where i_1, i_2 are normal monomorphisms, π_1, π_2 are canonical epimorphisms.

Definition. A preradical functor T on the category $\mathcal{A}$ is called a radical functor if $T(I/T) = 0$, where I is an identity functor.

Consider a category $\mathcal{G}$, such that its objects are $R-$modules and its morphisms are some semilinear transformations.

Throughout the whole text, all rings are considered to be associative with unit $1 \neq 0$ and all modules are left unitary [5, 6]. Let R be a ring. The category of left R–modules will be denoted by $R-$Mod, radical functor in the category $R-$Mod will be denoted by r_R.

All necessary definitions and theorems of Torsion theory and Category theory can be found in [2, 4, 7, 8].

A pair of mappings $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$, where $\varphi\colon R_1 \to R_2$ is either zero or a surjective ring homomorphism, and $\psi\colon M_1 \to M_2$ is a homomorphism of abelian groups, is called a *semilinear transformation* if $\forall r \in R_1, \forall m \in M_1$

$$\psi(r_1 m_1) = \varphi(r_1)\psi(m_1).$$

Let $\mathcal{G}$ be a category of all left modules over all rings. Or, more precisely, the objects of the category $\mathcal{G}$ are the pairs $(R, M) =_R M$, where R is a ring, M is a left module; the set of morphisms $H(_{R_1}M_1,_{R_2}M_2)$ is defined as a quotient set of a collection of all semilinear transformations $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ by the equivalence relation $\sim$, such that $(\varphi, \psi) \sim (\varphi', \psi')$, if $\psi = \psi'$, and product of morphisms is defined naturally. The class, determined by the semilinear transformation (φ, ψ) will be denoted by $\widetilde{(\varphi, \psi)}$, or, more frequently, (φ, ψ). It is easy to verify that $\mathcal{G}$ is a category. All categories, we consider in the paper, will be subcategories of $\mathcal{G}$. From the definition of equality of morphisms in the category $\mathcal{G}$ it follows

Remark 2. A class $\widetilde{(\varphi, \psi)}$ is a monomorphism (resp., an epimorphism) in the category $\mathcal{G}$ if ψ is a monomorphism (resp., an epimorphism) in the category of abelian groups.

Lemma 1. *If $(0, \psi)$ is a semilinear transformation, then $\psi = 0$.*

Proof. By the definition of a semilinear transformation,

$$\psi(m) = \psi(1m) = \varphi(1)\psi(m) = 0 \quad \forall m \in M.$$

□

The objects $(R, 0)$ and the morphisms $\widetilde{(0, 0)}$ are zero objects and zero morphisms in the category $\mathcal{G}$, respectively.

State some properties of the category $\mathcal{G}$.

Proposition 1. *For arbitrary many of objects (R_i, M_i) of the category $\mathcal{G}$, where $i \in I$, there exists the direct product belonging to G.*

Proof. In fact consider a pair (R, M), where $R = \prod_{i\in I} R_i$ is a direct product of rings R_i and $M = \prod_{i\in I} M_i$ is a direct product of abelian groups M_i. Every abelian group M can be turned into a left R–module putting $rm = (r_1, r_2, \ldots, r_i, \ldots)(m_1, m_2, \ldots, m_i, \ldots) = (r_1m_1, r_2m_2, \ldots, r_im_i, \ldots)$, where $r_i \in R_i$ and $m_i \in M_i$.

Consider the following morphims:

$$(s_i,\pi_i)\colon (\prod_{i\in I} R_i, \prod_{i\in I} M_i) \to (R_i, M_i),$$

where s_i is a projection of $\prod\limits_{i\in I} R_i$ onto R_i and π_i is a projection of $\prod\limits_{i\in I} M_i$ onto M_i. It is easy to see that pairs of homomorphisms (s_i,π_i) belong to the category $\mathcal{G}$. Since R is a direct product of rings R_i and M is a direct product of abelian groups we can verify that the object (R,M) and the morphisms (s_i,π_i) define a direct product of the objects (R_i,M_i) in the category $\mathcal{G}$. □

Proposition 2. *Every morphism of $\mathcal{G}$ has the kernel.*

Proof. In fact, let $(\varphi,\psi)\in H({}_{R_1}M_1, {}_{R_2}M_2)$ be a morphism of the category $\mathcal{G}$. Consider the pair $(R_1, Ker\psi)$, where $Ker\psi$ is the kernel of a homomorphism ψ in the category of abelian groups. Since M_1 is an R_1-module, $Ker\psi$ is an R_1-submodule. Prove that the object $(R_1, Ker\psi)$ with a monomorphism $(1_{R_1}, i)\colon (R_1, Ker\psi)\to(R_1,M_1)$, where i is a canonical injection, is the kernel of the morphism (φ,ψ). As a matter of fact $(\varphi,\psi)(1_{R_1},i) = (\varphi,0)\sim(0,0)$. Now let a morphism $(\varphi',\psi')\colon(R_3,M_3)\to(R_1,M_1)$ be such that $(\varphi,\psi)(\varphi',\psi') = (\varphi\varphi',0)\sim$ $\sim(0,0)$. Since $\psi\psi'=0$ it follows that there exists a homomorphism of abelian groups $\psi_3\colon M_3\to Ker\psi$ satisfying the condition $\psi' = i\psi_3$. Thus, there exists a pair of homomorphisms $(\varphi',\psi_3)\colon(R_3,M_3)\to(R_1,Ker\psi)$ satisfying the condition $(\varphi',\psi') = (1_{R_1},i)(\varphi',\psi_3)$. Verify that (φ',ψ_3) is a semilinear transformation. Let $r_3\in R_3$ and $m_3\in M_3$. Since (φ',ψ') is a semilinear transformations, $\psi'(r_3m_3) = \varphi'(r_3)\psi'(m_3)$. Hence $\psi'(r_3m_3) =$ $= i\psi_3(r_3m_3) = \varphi'(r_3)\psi'(m_3) = \varphi'(r_3)i\psi_3(m_3) = i\varphi'(r_3)\psi_3(m_3)$, i. e. $i\psi_3(r_3m_3) = i(\varphi'(r_3)\psi_3(m_3))$. Since i is a monomorphism in the category of abelian groups it follows that $\psi_3(r_3m_3) = \varphi'(r_3)\psi_3(m_3)$. By the construction of kernel, we see that the ideals of the object (R,M) are of the form (R,N), where N is a submodule of the module M. □

Proposition 3. *Every morphism of $\mathcal{G}$ has the cokernel.*

Proof. Let $(\varphi,\psi)\colon(R_1,M_1)\to(R_2,M_2)$ be a morphism in $\mathcal{G}$. Since φ is either zero or a surjective homomorphism it follows by lemma 1 that the group $\psi(M_1)$ is a submodule of an R_2–module M_2. Using the scheme dual to the scheme of proving proposition 2 it is easy to see that a quotient object $(R_2, M_2/\psi(M_1))$ of the object (R_2,M_2) with an epimorphism $(1_{R_2},\pi)\colon(R_2,M_2)\to(R_2,M_2/\psi(M_1))$, where π is a canonical epimorphism of R_2–modules, is a cokernel of the morphism (φ,ψ) in the category $\mathcal{G}$. □

The construction of the kernel and the cokernel in $\mathcal{G}$ implies

Remark 3. If a subcategory of $\mathcal{G}$ contains each object (R, M) together with the category R–Mod, then it also has properties as in proposition 2 and proposition 3.

Proposition 4. *Every morphism of G has the normal image.*

Proof. In fact, let $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ be a semilinear transformation. In the proof of proposition 3 we recalled that $\psi(M_1)$ is an R_2–module. This R_2–module can be turned into R_1–module.

Consider morphisms $(1_{R_1}, \psi')\colon (R_1, M_1) \to (R_1, \psi(M_1))$ and $(\varphi, i)\colon (R_1, \psi(M_1)) \to (R_2, M_2)$, where i is a canonical injection of abelian groups, and $\psi'(m_1) = \psi(m_1)$ for all $m_1 \in M_1$. It is easy to verify that these transformations are semilinear. By remark 2, morphisms $(1_{R_1}, \psi')$ and (φ, i) are epimorphism and monomorphism in the category $\mathcal{G}$, respectively.

Since $(\varphi, \psi) = (\varphi, i)(1_{R_1}, \psi')$ it remains to show that $(1_{R_1}, \psi')$ is a normal epimorphism in the category $\mathcal{G}$. By the construction of kernel we see that the semilinear transformation $(1_{R_1}, \psi')$ is the cokernel of the semilinear transformation $(1_{R_1}, j)\colon (R_1, Ker\psi) \to (R_1, M_1)$, where j is a canonical injection from $Ker\psi$ to M_1.

Since every cokernel is a normal epimorphism [2] proposition 4 is proved. □

By the construction of a normal image and by the fact that a normal image is determined up to equivalence implies

Remark 4. Every normal epimorphism up to equivalence has the form $(1_R, \psi)$, where ψ is any epimorphism of abelian groups.

Let T be an idempotent preradical functor on the category $\mathcal{G}$. Consider the class

$$\mathcal{T}(T) = \{(R, M) \mid T(R, M) = (R, M)\},\ where (R, M) \in Ob(\mathcal{G}).$$

Proposition 5. *The class $\mathcal{T}$ is closed under epimorphic images.*

Remark 5. Epimorphisms in the category $\mathcal{G}$ are morphisms $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$, such that $\varphi\colon R_1 \to R_2$ is a surjective ring homomorphism, and $\psi\colon M_1 \to M_2$ is an epimorphism of modules (i. e. a surjective homomorphism).

Proof of the proposition 5. Let $(R_1, M_1) \in \mathcal{T}(T)$, $(\varphi,\psi)\colon (R_1, M_1) \to (R_2, M_2)$ be an epimorphism. By the definition of the preradical functor the diagram

$$\begin{array}{ccc} (R_1, M_1) & \xrightarrow{(\varphi,\psi)} & (R_2, M_2) \\ {\scriptstyle i_1}\big\uparrow & & {\scriptstyle i_2}\big\uparrow \\ (R_1, M_1) & \longrightarrow & T(R_2, M_2), \end{array}$$

where i_1, i_2 are monomorphisms, is commutative. Since (φ,ψ) is an epimorphism of the category $\mathcal{G}$ we obtain $T(R_2, M_2) = (R_2, M_2)$. So $(R_2, M_2) \in \mathcal{T}(T)$. □

Proposition 6. *The class $\mathcal{T}$ possesses the following property:*
if $(R, M_1) \in \mathcal{T}(T)$ and $(R, M_2) \in \mathcal{T}(T)$ then $(R, M_1 \oplus M_2) \in \mathcal{T}(T)$.

Proof. Verify that the pair $(R, M_1 \oplus M_2)$ is a direct sum of (R, M_1) and (R, M_2). If we fix the ring then we obtain a subcategory of the category $\mathcal{G}$, which coincides with the category of modules. But in the category of modules class $\mathcal{T}$ is closed under direct sums [4]. □

Remark 6. In the category $\mathcal{G}$ there exist two objects, for which the direct sum does not exist, because the direct sum $(R_1 \oplus R_2, M_1 \oplus M_2)$ must be the greatest object, which contains (R_1, M_1) and (R_2, M_2) as subobjects. But if $R_1 \neq R_2$, then such object does not exist, because a morphism $R_i \to R_j$ must be a surjective ring homomorphism or a zero homomorphism.

Proposition 7. *Let $\mathcal{S}$ be a class of objects of the category $\mathcal{G}$, which is closed under epimorphic images and under direct sums (if they exist). Put*

$$T(R, M) = \sum \{(R, M_i) | (R, M_i) \subseteq (R, M), (R, M_i) \in Ob(\mathcal{S})\}.$$

Then T is an idempotent preradical.

Proof. Let T be a radical functor on $\mathcal{G}$. The restriction of the functor T on the category $R-$Mod is denoted by T_R. So we can write $T(R, M)$ is equal to $(R, T_R(M))$ $\forall (R, M) \in Ob(\mathcal{G})$ or simply to $T_R(M)$. In every category $R-$Mod $T(R, M) = (R, T_R(M))$ is an idempotent preradical functor. So it remains to show that for every $(\varphi,\psi)\colon (R_1,\, M_1) \to (R_2, M_2)$ the next diagram is commutative

$$\begin{array}{ccc} (R_1, M_1) & \xrightarrow{(\varphi,\psi)} & (R_2, M_2) \\ {\scriptstyle i_1}\big\uparrow & & {\scriptstyle i_2}\big\uparrow \\ T(R_1, M_1) & \xrightarrow{T(\varphi,\psi)} & T(R_2, M_2), \end{array}$$

where i_1, i_2 are monomorphisms.

$T(R,M) \in Ob(\mathcal{S})$, so $(R_2, \psi(T_{R_1}(M_1))) \in Ob(\mathcal{S})$ and $\psi(T_{R_1}(M_1)) \subseteq$ $\subseteq T_{R_2}(M_2)$. Hence our diagram is commutative. □

Theorem 1. *A preradical functor on the category $\mathcal{G}$ is a radical functor if and only if its restriction on every category $R-Mod$ is a radical.*

Proof. $(\Rightarrow)$ It is evidently.

$(\Leftarrow)$Let T be an idempotent preradical functor on $\mathcal{G}$, and its restriction T_R on every category $R-$Mod be a radical, i. e. $T_R(M/T_R(M)) = 0$ $\forall M \in R-$Mod. We must prove that $T(I/T) = 0$, where I is an identity functor. For this $T(R,M)$ must be a normal subobject of (R,M), that is $T(R,M) = (R, T_R(M))$. But on the category $R-$Mod T_R is a radical. □

Definition. The surjective ring homomorphism $\varphi\colon R_1 \to R_2$ is called essential in subcategory $\mathcal{K}$ of the category $\mathcal{G}$ if every morphism (φ, ψ) belongs to $\mathcal{K}$.

Definition.A subcategory $\mathcal{K}$ of the category G is called essential if it has such properties:

1) if (R,M) is an object of $\mathcal{K}$, then $R-Mod \subseteq \mathcal{K}$;

2) if $\widetilde{(\varphi_0, \psi_0)}$ is a morphism of $\mathcal{K}$, then $\widetilde{(\varphi_0, \psi_0)} = \widetilde{(\varphi_1, \psi_1)}$, where φ_1 is a surjective homomorphism essential in the category $\mathcal{K}$;

3) if objects $(R_1, M_1), (R_2, M_2) \in Ob(K)$, then zero morphism $\widetilde{(0,0)}\colon (R_1, M_1) \to (R_2, M_2)$ belongs to the category $\mathcal{K}$.

Theorem 2. *Let $\mathcal{K}$ be an essential subcategory of $\mathcal{G}$, r_R be radicals on the categories $R-Mod \subseteq \mathcal{K}$. Radicals r_R generate a radical functor on $\mathcal{K}$ if and only if for every morphism $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ of the category $\mathcal{K}$ $\psi(r_{R_1}(M_1)) \subseteq r_{R_2}(M_2)$.*

Proof. $(\Rightarrow)$ Let radicals r_R generate a radical functor T. So for every $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ the next diagram is commutative

$$\begin{array}{ccc} (R_1, M_1) & \xrightarrow{(\varphi,\psi)} & (R_2, M_2) \\ {\scriptstyle i_1}\uparrow & & {\scriptstyle i_2}\uparrow \\ T(R_1, M_1) & \xrightarrow{T(\varphi,\psi)} & T(R_2, M_2), \end{array}$$

where i_1, i_2 are monomorphisms. But $T(R_1, M_1) = (R_1, r_{R_1}(M_1))$, $T(R_2, M_2) = (R_2, r_{R_2}(M_2))$, so $\psi(r_{R_1}(M_1)) \subseteq r_{R_2}(M_2)$

($\Leftarrow$) 1. We want to show that for every $(\varphi,\psi)\colon (R_1,M_1)\to(R_2,M_2)$ the next diagram is commutative

$$\begin{array}{ccc} (R_1,M_1) & \xrightarrow{(\varphi,\psi)} & (R_2,M_2) \\ {\scriptstyle i_1}\uparrow & & {\scriptstyle i_2}\uparrow \\ T(R_1,M_1) & \xrightarrow{T(\varphi,\psi)} & T(R_2,M_2), \end{array}$$

where i_1, i_2 are monomorphisms. Since $T(R_1,M_1)=(R_1,r_{R_1}(M_1))$, $T(R_2,M_2)=(R_2,r_{R_2}(M_2))$ and $\psi(r_{R_1}(M_1))\subseteq r_{R_2}(M_2)$ it follows the commutativity of the diagram. So T is a preradical functor.

2. It is easy to see that T is an idempotent, because every r_R is an idempotent.

3. $T(I/T)=0$ by the theorem 1. □

Let I be an arbitrary left ideal of the ring R. Define a class $\mathcal{R}_I$ of left $R-$modules in such a way that: $N\in Ob(\mathcal{R}_I)\Leftrightarrow IN=N$, where IN consists of all sums of the form $\sum_{j=1}^{k} i_j n_j$, where $i_j\in I, n_j\in N$ and $k\in\mathbb{N}$. Show that $\mathcal{R}_I$ is a radical class [4, 7].

It is necessary to show that $\mathcal{R}_I$ is closed under 1) epimorphic images, 2) direct sums and 3) extensions.

1). Let $f\colon N\to M$ be an epimorphism of $R-$modules and $N\in$ $\in Ob(\mathcal{R}_I)$ and m be any element of M. There exists $n\in N$ such that $m=f(n)$. Since $N\in Ob(\mathcal{R}_I)$, $n=\sum_{j=1}^{k} i_j n_j$, where $i_j\in I, n_j\in N$ and $k\in\mathbb{N}$.

Therefore $m=f(n)=f(\sum_{j=1}^{k} i_j n_j)=\sum_{j=1}^{k} i_j f(n_j)$. Hence $m\in$ $\in IM$, i. e. $M=IM$.

2). It is clear.

3). We have short exact sequence

$$0\longrightarrow N\xrightarrow{\varphi_1} M\xrightarrow{\varphi_2} M/N\longrightarrow 0$$

and $IN=N$, $I(M/N)=M/N$. We shall show that $IM=M$. Let $m\in M$, so $\varphi_2(m)=m_1=\sum_{j=1}^{n} a_j k_j$, $n\in\mathbb{N}$, $m_1,k_j\in M/N$, $\varphi_2(m_j)=$ $=k_j$. Consider such expression: $m-\sum_{j=1}^{n} a_j m_j$, $m_j\in M$. Then $\varphi_2(m-$ $-\sum_{j=1}^{n} a_j m_j)=\varphi_2(m)-\sum_{j=1}^{n} a_j\varphi_2(m_j)=0$. So $(m-\sum_{j=1}^{n} a_j m_j)\in$ $\in Ker\varphi_2$ implies $(m-\sum_{j=1}^{n} a_j m_j)\in N$, it follows $m-\sum_{j=1}^{n} a_j m_j=$ $=\sum b_j n_j$. So $m\in IM$, i. e. $M=IM$.

A radical functor, defined by the radical class $\mathcal{R}_I$ is called an $I-$radical functor (or simply an $I-$radical).

Let $\mathcal{C}$ be an arbitrary essential subcategory of the category $\mathcal{G}$, such that $R-$Mod $\subseteq\mathcal{C}$ and $I(R)$ be a left ideal of the ring R. Then in every category $R-$Mod $\subseteq\mathcal{C}$ we can define $I(R)-$radical r_R.

Theorem 3. *If $\varphi(I(R_1)) \subseteq I(R_2)$ for every surjective ring homomorphism $\varphi\colon R_1 \to R_2$, which is essential in essential subcategory $\mathcal{C}$, then $I(R)-$radicals generate a radical functor T on the category $\mathcal{C}$.*

Proof. Define a functor T in such a way: $T(R, M) = (R, r_R(M))$ and $T(\varphi, \psi) = (\varphi, \psi_{r_R(M)})$ for every (R, M) and (φ, ψ) belonging to the category $\mathcal{C}$, where $\psi_{r_R(M)}$ is a restriction of the homomorphism ψ on the module $r_R(M)$. It remains to show that inclusions $\psi(r_{R_1}(M_1)) \subseteq r_{R_2}(M_2)$ hold true for every morphism $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ of the category $\mathcal{C}$. The surjective homomorphism φ can be considered as essential in $\mathcal{C}$, because the category $\mathcal{C}$ is essential. Since $\varphi(I(R_1)) \subseteq I(R_2)$, it follows by the definition of an $I(R_2)-$radical in R_2-Mod, $\psi(r_{R_1}(M_1)) =$ $= \psi(I(R_1) r_{R_1}(M_1)) = \varphi(I(R_1))\psi(r_{R_1}(M_1))$. □

Let $I(R)$ be a left ideal of the ring R, $r_{I(R)}$ is an $I(R)-$radical in $R-$Mod.

Definition. A left ideal $J(R)$ is called a maximal left ideal for the $I(R)-$radical $r_{I(R)}$ if $r_{I(R)} = r_{J(R)}$ implies $I(R) \subseteq J(R)$.

Proposition 8. *If $I(R)$ is a maximal left ideal for the radical $r_{I(R)}$, then $\psi(r_{I(R_1)}(M_1)) \subseteq r_{I(R_2)}(M_2)$ for every morphism $(\varphi, \psi)\colon$ $(R_1, M_1) \to (R_2, M_2)$ of the category $\mathcal{G}$ if and only if $\varphi(I(R_1)) \subseteq I(R_2)$.*

Proof. $(\Rightarrow)\psi(r_{R_1}(M_1)) = \varphi(I(R_1))\psi(r_{R_1}(M_1))$ (see the proof of the theorem 3). $r_{I(R_2)}(M_2) = I(R_2) r_{I(R_2)}(M_2)$ implies $\psi(r_{R_1}(M_1)) = I(R_2)\times$ $\times\psi(r_{R_1}(M_1))$. Since φ is a surjective ring homomorphism, R_2- Mod $\subseteq$ $\subseteq R_1-$ Mod and since $I(R_2)$ is a maximal, it follows $\varphi(I(R_1)) \subseteq I(R_2)$

$(\Leftarrow)$ See the proof of the theorem 3.

□

Now let $\mathcal{L}$ be a subcategory of $\mathcal{G}$, where R is a noetherian ring.

For a noetherian ring we can chose a maximal ideal for an $I-$radical functor, so we have

Theorem 4. *Let R be a noetherian ring and $I(R)$ be a left ideal of R, which is maximal for the radical $r_{I(R)}$. Radicals $r_{I(R)}$ in $R-$Mod generate $I(R)-$radical functor on the category $\mathcal{L}$ if and only if $\varphi(I(R_1)) \subseteq I(R_2)$ for every morphism $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ of the category $\mathcal{L}$.*

Proof. $(\Rightarrow)$ Apply Theorem 2 and Proposition 8.

$(\Leftarrow)$ See Theorem 3. □

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Received by the editors: 17.07.2008
and in final form 14.10.2008.