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Critical phenomena for systems under constraint

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It is well known that the imposition of a constraint can transform the properties of critical systems. Early work on this phemomenon by Essam and Garelick, Fisher, and others, focused on the effects of constraints on the leading critical exponents describing phase transitions. Recent work extended these considerations to critical amplitudes and to exponents governing logarithmic corrections in certain marginal scenarios. Here these old and new results are gathered and summarised. The involutory nature of transformations between the critical parameters describing ideal and constrained systems are also discussed, paying particular attention to matters relating to universality.

Key words: *critical phenomena, Fisher renormalisation, universality*

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1. Introduction

The study of thermodynamic systems subject to constraints has a long history. In 1966, Syozi and Miyazima produced a diluted version of the Ising model and observed that annealed non-magnetic impurities affect the critical behaviour of the model [1]. In particular, the usual infinite critical peak in the specific heat is replaced by a finite cusp. In 1967, Essam and Garelick quantified the nature of this change as [2, 3]

$$\alpha_X = -\frac{\alpha}{1-\alpha}\,. \tag{1.1}$$

Here, α represents the specific heat critical exponent for the ideal (non-diluted) system and α_X is its counterpart for the diluted system. If β and γ similarly represent the magnetisation and susceptibility exponents, Essam and Garelick further showed that these transform to [2, 3]

$$\beta_X = \frac{\beta}{1-\alpha}\,, \qquad \gamma_X = \frac{\gamma}{1-\alpha}\,. \tag{1.2}$$

In 1968, Fisher produced a general theory for critical systems under constraint and the general process linking the ideal critical exponents to those for the constrained system became known as *Fisher renormalisation* [4]. Because of their continued academic importance and relevance to real systems, phase transitions in constrained systems remained a focus of study [5–9]. In recent years the transformation has been extended to deal with other aspects of critical phenomena [10, 11].

Due to their experimental accessibility, amplitude terms are important for the description of critical phenomena. Unsurprisingly, these also change when a constraint is imposed. Perhaps surprisingly, however, the precise nature of this transformation has only recently been studied [11]. Furthermore, in certain marginal circmstances, multiplicative logarithmic corrections also enter the scaling description at continuous phase transitions. Examples include those at the upper critical dimension of spin systems and those at the border to regimes where the transition becomes first-order. The exponents of such logarithmic corrections also transform when the system is subjected to a constraint [11].

To give a compact description of all these various aspects (leading critical exponents, logarithmic corrections and amplitudes), we express the scaling behaviour of an ideal system as follows.

$$C(t,0) = A_\pm |t|^{-\alpha} |\ln|t||^{\hat{\alpha}}, \tag{1.3}$$
$$m(t,0) = B|t|^{\beta} |\ln|t||^{\hat{\beta}} \qquad \text{for } t<0, \tag{1.4}$$
$$\chi(t,0) = \Gamma_\pm |t|^{-\gamma} |\ln|t||^{\hat{\gamma}}, \tag{1.5}$$
$$m(0,h) = D h^{\frac{1}{\delta}} |\ln|t||^{\hat{\delta}}, \tag{1.6}$$
$$\xi(t,0) = N_\pm |t|^{-\nu} |\ln|t||^{\hat{\nu}}. \tag{1.7}$$

Here, t and h refer to the reduced temperature and magnetic field, respectively. The correlation length in the absence an external field is $\xi(t,0)$. The subscripts + and − refer to amplitudes for $t>0$ and $t<0$, respectively. In principle, we could employ subscripts for the critical exponents and their logarithmic counterparts corresponding to those used for the amplitudes, but we suppress these here for simplicity and because the exponents generally coincide on either side of the transition. Note that equation (1.3) for the specific heat corresponds to an internal energy of the leading form

$$e(t,0) = \pm \frac{A_\pm}{1-\alpha} |t|^{1-\alpha} |\ln|t||^{\hat{\alpha}}. \tag{1.8}$$

Finally, and for completeness, we mention that the leading form for the critical correlation function is as follows:

$$G(t=0,h=0;x) = \frac{\Theta}{x^{d-2+\eta}} |\ln x|^{\hat{\eta}}. \tag{1.9}$$

In what follows, we give a comprehensive overview of the effects of the presence of a constraint on the critical exponents (including those of the logarithmic corrections, when present) and the amplitudes. The critical exponents are universal quantities while the amplitudes are not. However, certain combinations of amplitudes are universal. We show that the renormalisation process (Fisher renormalisation) which transforms the universal critical paramenters is involutary in the sense that applying it twice results in the identity transformation. However, quantities which are not universal do not transform as involutions. We also show that the various scaling relations between the critical parameters (exponents and amplitudes) also hold for the transformed quantities.

In the next section, we summarise the scaling relations for the leading exponents, their logarithmic counterparts and the universal amplitude combinations. In section 3 we apply the renormalisation process and study its effects in section 4. We conclude in section 5.

2. Scaling relations and universal amplitude combinations

The four standard scaling relations are (see, e.g., [12] and references therein)

$$\alpha + d\nu = 2, \tag{2.1}$$
$$\alpha + 2\beta + \gamma = 2, \tag{2.2}$$
$$(\delta - 1)\beta = \gamma, \tag{2.3}$$
$$(2-\eta)\nu = \gamma, \tag{2.4}$$

where d represents the dimensionality of the system. The corresponding scaling relations for the logarithmic-correction exponents are as follows:

$$\hat{\alpha} + d\hat{\nu} = d\hat{\text{ϙ}}, \tag{2.5}$$
$$\hat{\alpha} + \hat{\gamma} = 2\hat{\beta}, \tag{2.6}$$
$$(\delta - 1)\hat{\beta} + \hat{\gamma} = \delta\hat{\delta}, \tag{2.7}$$
$$(2-\eta)\hat{\nu} + \hat{\eta} = \hat{\gamma}, \tag{2.8}$$

where $\hat{\alpha}$ is augmented by unity in certain special circumstances described in [13]. The exponent $\hat{\varphi}$ ("koppa-hat") characterises the leading logarithmic correction to the finite-size scaling of the correlation length $\xi_L(0,0) \sim L(\ln L)^{\hat{\varphi}}$, where L is the finite extent of the system [14]. It is the logarithmic counterpart of the exponent φ, recently introduced to characterise the finite-size correlation length above the upper critical dimension: $\xi_L(0,0) \sim L^{\varphi}$ [14]. The relations (2.1)–(2.4) for the leading exponents are derived in the Appendix, where it is also shown that they correspond to the following universal ratios [15]:

$$R_\xi = A_\pm N_\pm^d , \quad (2.9)$$

$$R_c = \frac{A_\pm \Gamma_\pm}{B^2} , \quad (2.10)$$

$$R_\chi = \frac{\Gamma_\pm B^{\delta-1}}{D^\delta} , \quad (2.11)$$

$$Q = \frac{\Theta N_\pm^{2-\eta}}{\Gamma_\pm} . \quad (2.12)$$

For the derivation of the logarithmic scaling relations (2.5)–(2.8), the reader is referred to [13]

In the next section, we examine the effects of constraints on the critical exponents and amplitudes. It will turn out that the renormalised critical exponents obey the same set of scaling relations as their original counterparts and that, when applied to universal quantities, Fisher renormalisation is involutory.

3. Fisher renormalisation

We consider a thermodynamic variable x conjugate to a field u, so that

$$x(t,h,u) = \frac{\partial f_X(t,h,u)}{\partial u} . \quad (3.1)$$

Here, $f_X(t,h,u)$ represents the free energy of the system under constraint and u represents a quantity such as the chemical potential with x representing the density of annealed non-magnetic impurities. The constraint is then expressed in terms of an analytic function as follows:

$$x(t,h,u) = X(t,h,u) . \quad (3.2)$$

One may further assume that the singular part of the free energy of the constrained system is structured analogously to its ideal counterpart f, so that

$$f_X(t,h,u) = f[t^*(t,h,u), h^*(t,h,u)] , \quad (3.3)$$

up to a regular background term and in which t^* and h^* are analytic functions [4]. The ideal free energy $f(t,h)$ is recovered if u is fixed at $u=0$.

We assume that

$$h^*(t,h,u) = h\mathscr{J}(t,h,u) , \quad (3.4)$$

so that $h^* = 0$ when $h = 0$. Then,

$$\frac{\partial h^*(t,0,u)}{\partial t} = 0 , \qquad \frac{\partial h^*(t,0,u)}{\partial u} = 0 , \quad (3.5)$$

and

$$\frac{\partial h^*(t,h,u)}{\partial h} = \mathscr{J}(t,h,u) + h\frac{\partial \mathscr{J}(t,h,u)}{\partial h} , \quad (3.6)$$

so that

$$\frac{\partial h^*(t,0,u)}{\partial h} = \mathscr{J}(t,0,u) . \quad (3.7)$$

For simplicity, we also assume $h \to -h$ symmetry so that t^* is a function of h^2. In that case,

$$\frac{\partial t^*(t,h,u)}{\partial h} \propto h , \quad (3.8)$$

which vanishes at $h = 0$.

3.1. The critical point

To identify the critical point of the constrained system, one first writes the magnetization from equation (3.3) as follows:

$$m_X(t,h,u) = \frac{\partial f_X(t,h,u)}{\partial h} = e(t^*,h^*)\frac{\partial t^*}{\partial h} + m(t^*,h^*)\frac{\partial h^*(t,0,u)}{\partial h}\,. \tag{3.9}$$

From equation (3.8), if the dependency on h is even, the first term on the right hand side of equation (3.9) vanishes at $h=0$. From equation (3.6), then

$$m_X(t,0,u) = m[t^*(t,0,u),0]\mathscr{J}(t,0,u)\,. \tag{3.10}$$

Now, the critical point of the ideal system is given by the vanishing of m. Assuming that $\mathscr{J}(t,0,u)$ is non-vanishing, equation (3.10) gives that $m_X(t,0,u)$ vanishes only when $m[t^*(t,0,u),0]=0$. This means that critical point for the constrained system is given by

$$t^*(t,0,u) = 0\,. \tag{3.11}$$

(The vanishing of $\mathscr{J}(t,0,u)$ would lead to two critical points instead of one for the constrained system.) The Taylor expansion for the function $\mathscr{J}(t,h,u)$ about the critical point is as follows:

$$\mathscr{J}(t,h,u) = J_0 + b_1 t + \dots + c_1 h + \dots + c_1(u-u_{\mathrm{c}}) + \dots\,, \tag{3.12}$$

where u_{c} is the critical value of u for the constrained system. The critical point, therefore, has $\mathscr{J}(0,0,u_{\mathrm{c}}) = J_0$.

3.2. The relation between t^* and t

The constraint (3.2) determines the relation between t^* and t. Equation (3.1) firstly gives

$$x(t,h,u) = \frac{\partial f(t^*,h^*)}{\partial t^*}\frac{\partial t^*}{\partial u} + \frac{\partial f_X(t^*,h^*)}{\partial h^*}\frac{\partial h^*}{\partial u}\,. \tag{3.13}$$

At $h=0$, the second term on the right vanishes after equation (3.5). Therefore,

$$x(t,0,u) = e(t^*,0)\frac{\partial t^*(t,0,u)}{\partial u}\,. \tag{3.14}$$

This will give a non-trivial relationship between t^* and t. Expanding $t^*(t,0,u)$, one has

$$t^*(t,0,u) = a_1(u-u_{\mathrm{c}}) + \dots\,, \tag{3.15}$$

where u_{c} and the coefficients of the expansion are non-universal. Therefore,

$$x(t,0,u) = a_1 e(t^*,0) + \dots\,, \tag{3.16}$$

which, from equation (1.8), is as follows:

$$x(t,0,u) = \pm a_1\frac{A_\pm}{1-\alpha}|t^*|^{1-\alpha}|\ln|t^*||^{\hat{\alpha}} + \dots\,. \tag{3.17}$$

On the other hand, Taylor expansion of the constraining function gives

$$X(t,0,u) = X(0,0,u_{\mathrm{c}}) + d_1(u-u_{\mathrm{c}}) + d_2 t + \dots \tag{3.18}$$

$$= X(0,0,u_{\mathrm{c}}) + \frac{d_1}{a_1}t^* + d_2 t + \dots\,, \tag{3.19}$$

from (3.15). Comparison with equation (3.16) leads to the vanishing of $X(0,0,u_{\mathrm{c}})$ and

$$\pm a_1\frac{A_\pm}{1-\alpha}|t^*|^{1-\alpha}|\ln|t^*||^{\hat{\alpha}} = \frac{d_1}{a_1}t^* + d_2 t + \dots\,. \tag{3.20}$$

If $\alpha < 0$, $t^* \sim t$ and the renormalisation is trivial. In the case where $\alpha > 0$, however, t renormalises to t^* in a non-trivial manner. To describe this, define

$$a = \left[\frac{d_2(1-\alpha)}{a_1}\right]^{\frac{1}{1-\alpha}}. \tag{3.21}$$

Then, the central result is that the constraint renormalises the reduced temperature from t to t^*, whereby

$$|t^*| = a\left(\frac{|t|}{A_\pm}\right)^{\frac{1}{1-\alpha}} |\ln|t||^{-\frac{\hat{\alpha}}{1-\alpha}}. \tag{3.22}$$

3.3. Scaling for the constrained system

Equations (3.3), (3.5) and (3.22) deliver the leading internal energy and specific heat for the constrained system as follows:

$$e_X(t,0,u) = \frac{\partial f_X(t,0,u)}{\partial t} = e(t^*,0)\frac{\partial t^*(t,0,u)}{\partial t} = \pm\frac{a^{2-\alpha}}{(1-\alpha)^2} A_\pm^{\frac{-1}{1-\alpha}} |t|^{\frac{1}{1-\alpha}} |\ln|t||^{-\frac{\hat{\alpha}}{1-\alpha}}, \tag{3.23}$$

and

$$C_X(t,0,u) = \frac{\partial e_X(t,0,u)}{\partial t} = \frac{a^{2-\alpha}}{(1-\alpha)^3} A_\pm^{\frac{-1}{1-\alpha}} |t|^{\frac{\alpha}{1-\alpha}} |\ln|t||^{-\frac{\hat{\alpha}}{1-\alpha}}, \tag{3.24}$$

respectively. We identify the latter as follows:

$$C_X(t,0) = A_{X\pm}|t|^{-\alpha_X}|\ln|t||^{\hat{\alpha}_X}, \tag{3.25}$$

where

$$\alpha_X = -\frac{\alpha}{1-\alpha}, \qquad \hat{\alpha}_X = -\frac{\hat{\alpha}}{1-\alpha}, \qquad A_{X\pm} = a^{1+\frac{1}{1-\alpha_X}}(1-\alpha_X)^3 A_\pm^{\alpha_X - 1}. \tag{3.26}$$

The last relationship is non-universal since, besides $A_\pm$, a is a non-universal constant.

The magnetization for the constrained system is given by equations (1.4), (3.10) and (3.12) as $m_X(t,0,u) = J_0 B|t^*|^\beta |\ln|t^*||^{\hat{\beta}}$ for $t<0$. In terms of t, we write

$$m_X(t,0) = B_X|t|^{\beta_X}|\ln|t||^{\hat{\beta}_X} \quad \text{for } t<0, \tag{3.27}$$

and identify

$$\beta_X = \frac{\beta}{1-\alpha}, \qquad \hat{\beta}_X = \hat{\beta} - \frac{\beta\hat{\alpha}}{1-\alpha}, \qquad B_X = J_0 a^\beta \frac{B}{A_-^{\beta_X}}. \tag{3.28}$$

Differentiating equation (3.9) with respect to h, delivers the susceptibility for the constrained system and, using equation (3.8) at $h=0$, together with equations (3.6) and (3.7), we obtain $\chi_X(t,0,u) = J_0^2\chi(t^*,0) = \Gamma_{X\pm}|t|^{-\gamma_X}|\ln|t^*||^{\hat{\gamma}_X}$, or

$$\chi_X(t,0) = \Gamma_{X\pm}|t|^{-\gamma_X}|\ln|t||^{\hat{\gamma}_X}, \tag{3.29}$$

where

$$\gamma_X = \frac{\gamma}{1-\alpha}, \qquad \hat{\gamma}_X = \hat{\gamma} + \frac{\gamma\hat{\alpha}}{1-\alpha}, \qquad \Gamma_{X\pm} = J_0^2 a^{-\gamma} A_\pm^{\gamma_X}\Gamma_\pm. \tag{3.30}$$

If $\delta > 1$, the critical isotherm $t=0$ has the leading magnetization in the field given by equations (3.6), (3.8) and (3.9) as $m_X(0,h,u) = J_0 D h^{\frac{1}{\delta}}|\ln h|^{\hat{\delta}}$. We identify

$$m_X(0,h) = D_X h^{\delta_X}|\ln h|^{\hat{\delta}_X}, \tag{3.31}$$

with

$$\delta_X = \delta, \qquad \hat{\delta}_X = \hat{\delta}, \qquad D_X = J_0^{1+\frac{1}{\delta}} D. \tag{3.32}$$

The critical exponents are, therefore, unchanged but the amplitude undergoes a transformation.

The correlation length renormalises in a similar way to the susceptibility since

$$\xi_X(t) = \xi(t^*) = N_\pm |t^*|^{-\nu} |\ln|t^*||^{-\hat{\nu}}.$$

We write

$$\xi_X(t,0) = N_{X\pm} |t|^{-\nu_X} |\ln|t||^{-\hat{\nu}_X}, \tag{3.33}$$

where

$$\nu_X = \frac{\nu}{1-\alpha}, \qquad \hat{\nu}_X = \hat{\nu} + \frac{\nu\hat{\alpha}}{1-\alpha}, \qquad N_{X\pm} = a^{-\nu} A_\pm^{\nu_X} N_\pm . \tag{3.34}$$

Finally, the correlation function is obtainable by differentiating the free energy with respect to two local fields $h_1 = h(x_1)$ and $h_2 = h(x_2)$. One obtains

$$G_X(t,h,u;x) = \frac{\partial^2 f_X(t,h,u)}{\partial h_1 \partial h_2} = J_0^2 \frac{\partial^2 f(t^*,h^*)}{\partial h_1^* \partial h_2^*} = J_0^2 G(t*,h^*,x).$$

Setting $t^* = t = h^* = h = 0$, delivers $G_X(0,0,u;x) = J_0^2 G(0,0,x)$. Writing

$$G_X(0,0,x) = \frac{\Theta_X}{x^{d-2+\eta_X}} |\ln x|^{\hat{\eta}_X}, \tag{3.35}$$

we identify

$$\eta_X = \eta, \qquad \hat{\eta}_X = \hat{\eta}, \qquad \Theta_X = J_0^2 \Theta. \tag{3.36}$$

We have observed that neither the in-field magnetisation nor the correlation function exhibit non-trivial renormalisation of the critical exponents. The former is the case by construction and the latter is so because it is defined at the critical point. Likewise, the exponents $ϙ$ and $\hat{ϙ}$ governing finite-size scaling of the correlaton length do not change under Fisher renormalisation, so that $ϙ_X = ϙ$ and $\hat{ϙ}_x = \hat{ϙ}$.

4. Properties of renormalised scaling parameters

It is straightforward to verify that if the critical exponents for the ideal system satisfy the scaling relations (2.1)–(2.4), the renormalised exponents for the constrained system do likewise. (This observation for the Essam-Fisher relation (2.2) was already made in [2].) The same statement applies to the scaling relations for logarithmic corrections (2.5)–(2.8).

Fisher renormalisation applied to the universal critical exponents is involutory. This means that renormalisation of renormalised exponents delivers pure values. For example, $\gamma_{XX} = \gamma_X/(1-\alpha_X) = \gamma$ and $\hat{\gamma}_{XX} = \hat{\gamma}_X + \gamma_X \hat{\alpha}_X/(1-\alpha_X) = \hat{\gamma}$. However, the same starement does not apply to the amplitudes. For example, two successive applications of equation (3.30) give $\Gamma_{XX\pm}$ different from $\Gamma_\pm$.

Of course, the critical exponents, for which the transformation is involutory, are universal, whereas the critical amplitudes are not. This observation prompts one to investigate the nature of the universal combinations (2.9)–(2.12) under Fisher renormalisation. The non-universal terms J_0 and a, which characterise the transformations of the individual amplitude terms, drop out of the transformations of the universal combinations through the scaling relations (2.1)–(2.4). The universal amplitude combinations transform as follows:

$$R_{X\text{c}} = \frac{1}{(1-\alpha)^3} R_\text{c}, \tag{4.1}$$

$$R_{X\chi} = R_\chi, \tag{4.2}$$

$$R_{X\xi} = \frac{1}{(1-\alpha)^3} R_\xi, \tag{4.3}$$

$$Q_X = Q, \tag{4.4}$$

$$Z_X = \frac{Z}{U_0^{\Delta_X}}. \tag{4.5}$$

Two successive applications of these transformations confirm the involutory nature of these universal combinations.

5. Conclusions

Fisher renormalization, which generalises an earlier theory of Essam and Garelick is a staple of the established theory of critical phenomena. The early work by these authors was extended in recent years to encompass critical amplitudes and the exponents which govern logarithmic corrections to scaling, when present. Here, a comprehensive treatment of all of these various elements has been given. We also observe that the involutory nature of the renormalisation process is intrinsically linked to universality.

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A. Appendix: Universal amplitude Combinations

To identify the universal amplitude combinations, we begin with the standard scaling form for the free energy and correlation length [12, 15]

$$f(t,h) = b^{-d} Y(K_t b^{y_t} t, K_h b^{y_h} h), \tag{A.1}$$

$$\xi(t,h) = b X(K_t b^{y_t} t, K_h b^{y_h} h). \tag{A.2}$$

The scaling functions Y and X are universal and all the non-universality is contained in the metric factors K_t and K_h.

Differentiating equation (A.1) with respect to h delivers the scaling form for the magnetization as follows:

$$m(t,h) = b^{-d+y_h} K_h Y^{(h)}(K_t b^{y_t} t, K_h b^{y_h} h), \tag{A.3}$$

where the parenthesized superscript signifies appropriate differentiaton of the scaling function. Setting $h = 0$ and chosing

$$b = K_t^{-\frac{1}{y_t}} |t|^{-\frac{1}{y_t}} \tag{A.4}$$

gives the spontaneous magnetization $m(t,0) = B(-t)^{\beta}$, for $t < 0$, in which

$$\beta = \frac{d - y_h}{y_t} \quad \text{and} \quad B = K_t^{\beta} K_h Y^{(h)}(1,0). \tag{A.5}$$

On the other hand, setting $t = 0$ in equation (A.3) and choosing

$$b = K_h^{-\frac{1}{y_h}} h^{-\frac{1}{y_h}}, \tag{A.6}$$

we obtain $m(0,h) = Dh^{1/\delta}$ in which

$$\frac{1}{\delta} = \frac{d - y_h}{y_h} \quad \text{and} \quad D = K_h^{1+\frac{1}{\delta}} Y^{(h)}(0,1). \tag{A.7}$$

The susceptibility is obtained by differentiating equation (A.3) with respect to h. Again setting $h = 0$ and using equation (A.4), one finds $\chi(t,0) = \Gamma_{\pm}|t|^{-\gamma}$, where

$$\gamma = \frac{2y_h - d}{y_t} \quad \text{and} \quad \Gamma_{\pm} = K_t^{-\gamma} K_h^2 Y^{(hh)}(\pm 1,0). \tag{A.8}$$

For the specific heat, differentiate (A.1) twice with respect to t and again use equation (A.4) to find $C(t,0) = A_{\pm}|t|^{-\alpha}$ with

$$\alpha = 2 - \frac{d}{y_t} \quad \text{and} \quad A_{\pm} = K_t^{2-\alpha} Y^{(tt)}(\pm 1,0). \tag{A.9}$$

From equations (A.5) and (A.7), we can express y_t and y_h in terms of β and δ,

$$y_t = \frac{d}{\beta}\frac{1}{\delta+1} \quad \text{and} \quad y_h = \frac{d\delta}{\delta+1}\,. \tag{A.10}$$

Similarly, using equations (A.5) and (A.7) we can express K_t and K_h in terms of B and D,

$$K_t = \left[\frac{B}{Y^{(h)}(1,0)}\right]^{\frac{1}{\beta}} \left[\frac{D}{Y^{(h)}(0,1)}\right]^{-\frac{1}{\beta}\frac{\delta}{1+\delta}} \quad \text{and} \quad K_h = \left[\frac{D}{Y^{(h)}(0,1)}\right]^{\frac{\delta}{\delta+1}}. \tag{A.11}$$

Here, the $Y^{(h)}$ are universal while the amplitudes B and D are not.

Finally, expressing α and γ in terms of β and δ through equations (A.8) and (A.9), delivers the static scaling relations (2.2) and (2.3). Correspondingly, one can express $A_\pm$ and $\Gamma_\pm$ in terms of B and D,

$$\Gamma_\pm = \frac{Y^{(hh)}(\pm 1,0)}{\left[Y^{(h)}(1,0)\right]^{\frac{1}{\beta}}\left[Y^{(h)}(0,1)\right]^{\delta}} B^{1-\delta} D^{\delta}\,, \tag{A.12}$$

$$A_\pm = \left[Y^{(h)}(1,0)\right]^{-(\delta+1)} \left[Y^{(h)}(0,1)\right]^{\delta} Y^{(tt)}(\pm 1,0)\frac{B^{\delta+1}}{D^{\delta}}\,. \tag{A.13}$$

From the first of these, $\Gamma_\pm B^{\delta-1}/D^{\delta}$ is a universal combination of universal factors. This is R_χ in equation (2.11). From the second, the ratio $A_\pm D^{\delta}/B^{\delta+1}$ is universal. Or, combining with equation (A.7), the quantity $R_{\rm c}$ in equation (2.10) is seen to be universal.

From equations (A.2) and (A.4), the correlation length is $\xi(t,0) = N_\pm |t|^{-\nu}$, where

$$\nu = \frac{1}{y_t} \quad \text{and} \quad N_\pm = K_t^{-\frac{1}{y_t}} X(\pm 1,0). \tag{A.14}$$

From equation (A.9), the first of these is the hyperscaling relation (2.1). To connect $N_\pm$ to the other amplitudes, one can exploit the relatonship between the susceptibility and the correlation function,

$$\chi = \int\limits_0^{\xi} G(x) x^{d-1} \mathrm{d}x = \Theta \xi^{2-\eta}, \tag{A.15}$$

from which Fisher's scaling relation (2.4) follows, along with

$$\Gamma_\pm = \Theta N_\pm^{2-\eta}. \tag{A.16}$$

The combination $Q = \Theta N_\pm^{2-\eta}/\Gamma_\pm$ of equation (2.12) is, therefore, universal. Similarly, the universality of R_ξ in equation (2.9) can be explained through the hyperscaling relation $f(t,0) = A_\pm |t|^{2-\alpha}/(2-\alpha)(1-\alpha) \sim \xi^d(t,0) = (N_\pm |t|^{-\nu})^d$.

References

1. Syozi I., Miyazima S., Prog. Theor. Phys., 1966, **36**, 1083; doi:10.1143/PTP.36.1083.
2. Essam J.W., Garelick H., Proc. Phys. Soc., 1967, **92**, 136; doi:10.1088/0370-1328/92/1/320.
3. Garelick H., Essam J.W., J. Phys. C: Solid State Phys., 1968, **1**, 1588; doi:10.1088/0022-3719/1/6/315.
4. Fisher M.E., Phys. Rev., 1968, **176**, 257; doi:10.1103/PhysRev.176.257.
5. Lushnikov A.A., Phys. Lett. A, 1968, **27**, 158; doi:10.1016/0375-9601(68)91180-8 [Sov. Phys. JETP, 1969, **29**, 120].
6. Aharony A., J. Magn. Magn. Mater., 1978, **7**, 215; doi:10.1016/0304-8853(78)90186-5.
7. Capel H.W., Perk J.H.H., den Ouden L.W.J., Phys. Lett. A, 1978, **66**, 437; doi:10.1016/0375-9601(78)90388-2.
8. Capel H.W., den Ouden L.W.J., Perk J.H.H., Physica A, 1979, **95**, 371; doi:10.1016/0378-4371(79)90024-4.
9. den Ouden L.W.J., Capel H.W., Perk J.H.H., Physica A, 1981, **105**, 53; doi:10.1016/0378-4371(81)90063-7.
10. Kenna R., Hsu H.-P., von Ferber C., J. Stat. Mech., 2008, L10002; doi:10.1088/1742-5468/2008/10/L10002.
11. Izmailian N.Sh., R. Kenna, Preprint arXiv:1402.4673, 2014.
12. Fisher M.E., Rev. Mod. Phys., 1998, 7**0**, 653; doi:10.1103/RevModPhys.70.653.
13. Kenna R., In: Order, Disorder, and Criticality: Advanced Problems of Phase Transition Theory, Yu. Holovatch (Ed.), vol. 3, World Scientific, Singapore, 2012.
14. Kenna R., Berche B., Condens. Matter Phys., 2013, **16**, 23601; doi:10.5488/CMP.16.23601.
15. Privman V., Hohenberg P.C., Aharony A., In: Phase Transitions and Critical Phenomena, vol. **14**, Domb C., Lebowitz J.L. (Eds.), Academic, New York, 1991, pp. 1–134.

Критичні явища для систем з в’язями

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Добре відомо, що накладання в’язей може змінити критичні властивості системи. Ранні роботи Ессіма і Гареліка, Фішера та ін., присвячені цьому явищу, зосереджувалися на впливі в’язей на головні критичні показники, які описують фазові переходи. Недавня робота розширила ці дослідження на випадок критичних амплітуд і показників для логарифмічних поправок для деяких межових сценаріїв. Тут ці старі і нові результати зібрано і підсумовано. Також обговорюється інволютивна природа перетворень між критичними параметрами, які описують ідеальну систему і систему з в’язями, при цьому особлива увага приділяється питанням, пов’язаним з універсальністю.

Ключові слова: *критичні явища, ренормалізація Фішера, універсальність*