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RESEARCH ARTICLE

Serial piecewise domains

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Abstract. A ring A is called a piecewise domain with respect to the complete set of idempotents $\{e_1, e_2, \dots, e_m\}$ if every nonzero homomorphism $e_iA \to e_jA$ is a monomorphism. In this paper we study the rings for which conditions of being piecewise domain and being hereditary (or semihereditary) rings are equivalent. We prove that a serial right Noetherian ring is a piecewise domain if and only if it is right hereditary. And we prove that a serial ring with right Noetherian diagonal is a piecewise domain if and only if it is semihereditary.

1. Introduction

All rings considered in this paper are assumed to be associative with identity $1 \neq 0$, and all modules are unitary right modules, unless otherwise specified.

This paper is devoted to considering the structure of some classes of piecewise domains. These rings first were introduced and studied by R.Gordon and L.W.Small [11]. Piecewise domains extend the notion of hereditary and semihereditary rings.

Recall that a ring A is a right hereditary (resp. semihereditary) if any its ideal (resp. finitely generated ideal) is projective. Any principal ideal domain is hereditary. Any Dedekind ring is hereditary and any

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Prüfer ring is semihereditary. Really, the property being hereditary (resp. semihereditary) is invariant with respect to Morita equivalence. In particular, any full matrix ring over a Dedekind domain (resp. Prüfer ring) is hereditary (semihereditary). It is true the following statement.

Theorem 1.1. ([3]). *A ring A is right hereditary (resp. semihereditary) if and only if any (finitely generated) submodule of a right projective A-module is projective.*

Example 1.1. Let

$$A = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Z} \end{pmatrix},$$

where $\mathbf{Z}$ is the ring of integers and $\mathbf{Q}$ is the field of rational numbers. Then A is a two-sided Noetherian piecewise domain. A is a semihereditary ring, but it is not hereditary.

Example 1.2. ([18], p.46, **Small's example**). Let

$$A = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix},$$

where $\mathbf{Z}$ is the ring of integers and $\mathbf{Q}$ is the field of rational numbers. Then A is a semihereditary, and so it is piecewise domain, it is right hereditary but it is not left hereditary.

The following examples show that there are some very wide class of rings which are piecewise domains but not hereditary rings.

Example 1.3. Let

$$T_2(\mathbf{Z}) = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & \mathbf{Z} \end{pmatrix},$$

where $\mathbf{Z}$ is the ring of integers and $\mathbf{Q}$ is the field of rational numbers. Then $T_2(\mathbf{Z})$ is a two-sided Noetherian piecewise domain, but it is not hereditary.

Example 1.4. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ be a partial ordered connected set with partial order $\preceq$, and let $A = M_n(\mathbf{Z})$ be a full ring of all square matrices of order n, where $\mathbf{Z}$ is the ring of integers. Denote by e_{ij} the matrix units of A $(i, j = 1, \ldots, n)$. Let $B_n(\mathbf{Z}, \mathcal{P})$ be a subring of A which is consisted from the matrices of the following form: $\mathbf{B} = (b_{ij}) \in A$, where $b_{ij} \in e_{ii}Ae_{jj}$, and $b_{ij} = 0$ if $p_i \npreceq p_j$.

The ring $B_n(\mathbf{Z}, \mathcal{P})$ is a piecewise domain, but it is not a hereditary ring for $n \geq 2$, since there exists an idempotent $e = e_{ii} + e_{jj}$ (for $i \neq j$) such that the ring $eB_n(\mathbf{Z}, \mathcal{P})e \simeq T_2(\mathbf{Z})$ is not hereditary.

In this paper we study the rings for which conditions of being piecewise domain and being hereditary (or semihereditary) rings are equivalent.

The central structure theorem which was proved by R.Gordon and L.W.Small states that all piecewise domains are triangular in the sense of definition which will be given below.

Definition 1.1. ([9], p.56). *A ring A is called an FDI-ring if there exists a decomposition of the identity of $1 \in A$ into a finite sum*

$$1 = e_1 + e_2 + \ldots + e_n$$

of pairwise orthogonal primitive idempotents e_i.

A right projective A-module P of an FDI-ring A is called principal if $P \simeq e_i A$ for $i = 1, \ldots, n$.

The important examples of FDI-rings are semiperfect, Artinian, Noetherian, Goldie rings. Note that the decomposition of $1 \in A$, giving in the definition of an FDI-ring, may be non-unique.

Definition 1.2. ([1], p.89). *A finite orthogonal set of idempotents e_1, e_2, ..., $e_m \in A$ is called complete if*

$$e_1 + e_2 + \ldots + e_m = 1 \in A$$

Definition 1.3. ([11]). *A ring A is said to have enough idempotents if it has a complete set of orthogonal primitive idempotents.*

Remark 1.1. Note that the concept of a ring to have enough idempotents coincides with concept of an FDI-ring.

2. Piecewise domains

In this section we consider some properties of piecewise domains.

Definition 2.1. *A ring A is called a piecewise domain with respect to a complete set of idempotents $\{e_1, e_2, \ldots, e_m\}$ if every nonzero homomorphism $e_i A \to e_j A$ is a monomorphism.*

Piecewise domain were first introduced and studied by R.Gordon and L.Small in the paper [11]. Since for a piecewise domain it follows that $e_i A e_i$ is a domain for any $i = 1, \ldots, m$, the set of $\{e_1, e_2, \ldots, e_m\}$ is a complete set of primitive orthogonal idempotents, and so any piecewise domain A is an FDI-ring.

The important examples of piecewise domains are hereditary and semihereditary FDI-rings:

Proposition 2.1. ([8], proposition 10.7.9). *Any right semihereditary FDI-ring is a piecewise domain.*

The following structure theorem describes piecewise domains in the general case.

Theorem 2.1. ([11], main theorem). *If A is a piecewise domain, then*

$$A = \begin{pmatrix} B_1 & B_{12} & \dots & \dots & B_{1r} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & B_{r-1r} \\ 0 & \dots & \dots & 0 & B_r \end{pmatrix},$$

where each B_{ij} is a B_i-B_j-bimodule and each B_i is a prime piecewise domain of the form

$$B = \begin{pmatrix} \mathcal{O}_1 & \dots & \dots & \mathcal{O}_{1t} \\ \vdots & \mathcal{O}_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathcal{O}_{t1} & \dots & \dots & \mathcal{O}_t \end{pmatrix},$$

where each $\mathcal{O}_j$ is a domain and each $\mathcal{O}_{jk}$ is isomorphic as a right $\mathcal{O}_k$-module to a nonzero right ideal in $\mathcal{O}_k$ and as a left $\mathcal{O}_j$-module to a nonzero left ideal in $\mathcal{O}_j$. Moreover, the integer r is uniquely determined by A.

Corollary 2.1. *The prime radical $Pr(A) = N$ of a piecewise domain A is of the form*

$$N = Pr(A) = \begin{pmatrix} 0 & B_{12} & B_{13} & \dots & \dots & B_{1r} \\ 0 & 0 & B_{23} & \ddots & & B_{2r} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & B_{r-2r-1} & B_{r-2r} \\ \vdots & & & \ddots & \ddots & B_{r-1r} \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix},$$

and so N is nilpotent.

Definition 2.2. *A decomposition of identity* $1 = f_1 + f_2 + \cdots + f_m$ *of a ring* A*, where* f_i *are idempotents, is called triangular if* $f_i A f_j = 0$ *for all* $i > j$*. Such a decomposition is called prime if* $f_i A f_i$ *is a prime ring for any* $i = 1, \ldots, m$*.*

A ring A *is called triangular if there exists a triangular decomposition of the identity of* A*.*

A ring A *is said to be a primely triangular ring if there exists a triangular prime decomposition of the identity of* A*, i.e., there exists a decomposition of the identity* $1 \in A$ *into a finite sum* $1 = f_1 + f_2 + \cdots + f_m$ *of pairwise orthogonal idempotents such that* $f_i A f_j = 0$ *for all* $i > j$ *and* $f_i A f_i$ *is a prime ring for any* $i = 1, \ldots, m$*.*

Remark 2.1. Note that the termin "a triangular ring" was first introduced by S.U.Chase in 1961 [2] for a semiprimary ring where all $R_i = e_i A e_i$ are simple Artinian rings. This termin was used by L.W.Small for arbitrary Noetherian rings [23]. M.Harada in 1966 introduced the termin "generalized triangular matrix rings" for rings with triangular decomposition of the identity where $R_i = e_i A e_i$ are arbitrary rings [10]. And he studied the properties of such rings when R_i are semiprimary rings. It is obvious that in this case the generalized triangular matrix rings are also semiprimary. Yu.A.Drozd in 1980 used the termin "a triangular ring" for rings with triangular prime decomposition of the identity [6].

Corollary 2.2. *A piecewise domain is a primely triangular ring. In particular, any semihereditary FDI-ring and any hereditary FDI-ring is a primely triangular ring.*

Definition 2.3. ([8], p.291) *Let* $Pr(A)$ *be the prime radical of a ring* A*. The quotient ring* $\overline{A} = A/Pr(A)$ *is called the diagonal of* A*. A ring* A *is called an FDD-ring if* $\overline{A}$ *is an FD-ring.*

From theorem 2.1, corollary 2.1 and corollary 2.2 it immediately follow the following corollaries.

Corollary 2.3. *The diagonal* $\overline{A} = A/Pr(A)$ *of a piecewise domain* A *is an finite direct product of prime piecewise domains* $\overline{A} = B_1 \times B_2 \times \ldots \times B_r$*. Therefore* $\overline{A}$ *is an FD-ring, and* A *is an FDD-ring.*

Corollary 2.4. *A piecewise domain* A *considered as a group is a direct sum of the ring* $A_0 \simeq \overline{A}$ *and* $N = Pr(A)$*:* $A = A_0 \oplus N$*.*

Since, by corollary 2.3, any piecewise domain A is an FDD-ring, we can build the prime quiver $PQ(A)$ of A (see [8], section 11.7). Since the prime radical of a piecewise domain A is nilpotent, and so T-nilpotent, from theorem 11.7.3, [8], we immediately obtain the following statement.

Proposition 2.2. *Let A be a piecewise domain. Then the prime quiver $PQ(A)$ is connected if and only if the ring A is indecomposable.*

Recall that a quiver without multiple arrows and multiple loops is called simply laced.

Proposition 2.3. *Let A be a piecewise domain. Then the prime quiver $PQ(A)$ is an acyclic simply laced quiver.*

Theorem 2.2. *A right perfect piecewise domain is a semiprimary ring.*

Proof. Let A be a right perfect piecewise domain. Since any one-sided perfect ring is semiperfect, A is semiperfect. Therefore the prime radical of A is nilpotent, by ([9], corollary 4.9.3). Since the prime radical of a one-sided perfect ring coincides with Jacobson radical R of A, by ([9], proposition 4.7.5), R is nilpotent. Thus A/R is Artinian and R is nilpotent, i.e., A is semiprimary. □

Since any right hereditary FDI-ring is a piecewise domain, by proposition 2.1, we obtain as an immediately corollary the following result which was proved by M.Teply.

Theorem 2.3. ([24]). *A right perfect right hereditary ring is semiprimary.*

3. Serial right Noetherian piecewise domains

Definition 3.1. ([8], p.300) *A right A-module is called serial if it is decomposed into a direct sum of uniserial modules, that is, modules possessing a linear lattice of submodules. A ring which is right serial module and left serial module over itself is called a serial ring.*

Rings, over which all modules are serial, were first introduced and studied by G.Köthe [17] and T.Nakayama [20], [21]. Serial Artinian rings was studied by L.A.Skorniakov [22], K.R.Fuller [7], D.Eisenbud, P.Griffith [4], [5], G.Ivanov [13], and others. It was proved that a ring A is serial right Artinian ring if and only if A is a direct sum of uniserial right modules.

Serial non-Artinian rings were first studied and described by V.V.Kirichenko [14], [15] and R.B.Warfield [25]. In particular, they described the structure of serial Noetherian rings.

Theorem 3.1. ([14]). *Any serial Noetherian ring can be decomposed into a finite direct product of an Artinian serial ring and a number of semiperfect Noetherian prime hereditary rings. Conversely, all such rings are serial and Noetherian.*

The structure of semiperfect Noetherian prime hereditary rings was studied by G.Michler, who proved the following main theorem.

Theorem 3.2. ([19]). *A Noetherian semiperfect prime reduced hereditary ring A is either a division ring or it is isomorphic to a ring of the form*

$$H_m(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \dots & \mathcal{O} \end{pmatrix},$$

where $\mathcal{O}$ is a discrete valuation ring, and $\mathcal{M}$ is its unique maximal ideal.

The full description of all serial right Noetherian rings are given by the following theorem.

Theorem 3.3. ([14], [15]). *Any serial right Noetherian ring is Morita equivalent to a direct product of a finite number of rings of the following types:*

1) Artinian serial rings;

2) rings isomorphic to rings of the form $H_m(\mathcal{O})$;

3) rings isomorphic to quotient rings of a ring

$$H(\mathcal{O}, m, n) = \begin{pmatrix} H_m(\mathcal{O}) & M_{m,n}(D) \\ 0 & T_n(D) \end{pmatrix},$$

where $\mathcal{O}$ is a discrete valuation ring with skew field of fractions D, $T_n(D)$ is the ring of upper triangular matrices of order n, and $M_{m,n}(D)$ is a set of all rectangular matrices of size $m \times n$ over the division ring. Conversely, all rings of this form are serial and right Noetherian.

The following theorem gives a description of serial semiprime and right Noetherian ring.

Theorem 3.4. ([8], theorem 13.5.3). *A serial semiprime and right Noetherian ring can be decomposed into a direct product of prime rings. A serial prime and right Noetherian ring is also left Noetherian and two-sided hereditary. In the Artinian case such a ring is Morita equivalent to a division ring and in the non-Artinian case it is Morita equivalent to a ring isomorphic to $H_m(\mathcal{O})$, where $\mathcal{O}$ is a discrete valuation ring. Conversely, all such rings are prime two-sided hereditary and Noetherian.*

From this theorem and corollary 2.3 we immediately obtain the following statement.

Proposition 3.1. *If A is a serial right Noetherian piecewise domain, then its diagonal $\overline{A}$ is Morita equivalent to the finite direct product of rings of the form D, where D is a division ring, or $H_n(\mathcal{O})$, where $\mathcal{O}$ is a discrete valuation ring.*

Proposition 3.2. *An Artinian serial piecewise domain is Morita equivalent to a direct product of rings of the form $T_n(D)$, where D is a division ring. And so A is a hereditary ring.*

Proof. We can assume that a ring A is indecomposable. According to [8], theorems 11.1.9 and 12.1.2, one can assume that the quiver of A is a chain or a simple cycle.

Consider the first case when the quiver $Q(A)$ of A is a chain, then A is Morita equivalent to a ring isomorphic to $T_n(D)/I$, where I is a two-sided ideal of $T_n(D)$ (see [8], pp. 306-308). Let $A_A = P_1 \oplus \ldots \oplus P_n$, where $P_i = e_{ii}A$, $i = 1, \ldots, n$. Obviously, any nonzero two-sided ideal $I \subset T_n(D)$ contains $e_{11}T_n(D)e_{nn}$. So, if $A \neq T_n(D)$ then any homomorphism $\varphi : P_n \to P_1$ is zero, i.e., $e_{11}Ae_{nn} = 0$.

The matrix units $e_{12}, e_{23}, \ldots, e_{n-1n}$ belong A because there is an arrow $i \to i+1$ for each $i = 1, \ldots, n-1$ in the quiver $Q(A)$. Therefore the product $e_{12}e_{23}\ldots e_{n-1n} = e_{1n}$ is nonzero, by [8], proposition 10.7.8, and $e_{1n} \in A$. Thus $e_{11}Ae_{nn} \neq 0$. We obtain a contradiction and so $A = T_n(D)$.

Consider the second case when the quiver $Q(A)$ is a cycle:

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n}{\bullet} \longrightarrow \overset{1}{\bullet} \tag{3.1}$$

Since A is right Artinian piecewise domain, by [8], proposition 11.2.3, the prime radical of A coincides with its Jacobson radical. So, in this case the prime quiver $PQ(A)$ is obtained from the quiver $Q(A)$ by changing all arrows going from one vertex to other vertex by one arrow.

In our case $Q(A)$ is a simple cycle (3.1). Thus $Q(A) = PQ(A)$. By proposition 2.3, the prime quiver $PQ(A)$ of a piecewise domain is an acyclic simply laced quiver. So we obtain a contradiction. □

Theorem 3.5. *For a serial right Noetherian ring A the following conditions are equivalent:*

(i) A is right hereditary;

(ii) A is a piecewise domain.

Proof. (i) ⇒ (ii) follows from proposition 2.1.

(ii) ⇒(i). By [8], theorem 13.4.3, any serial right Noetherian ring A is Morita equivalent to a direct sum of a finite number of rings of the following types:

(1) Artinian serial rings;

(2) rings isomorphic to rings of the form $H_n(\mathcal{O})$;

(3) rings isomorphic to quotient rings of $H(\mathcal{O}, m, n)$, where $\mathcal{O}$ is a discrete valuation ring.

If we have the case (1), then A is hereditary, by proposition 3.2.

So consider case (2). An indecomposable reduced serial ring A is isomorphic to $H_m(\mathcal{O})$, which is hereditary. Therefore all piecewise serial ring of the case (2) are hereditary.

Case (3). We can assume that a ring A is reduced. By theorem 1.2.4, proposition 1.2.14 and theorem 11.7.3, [8], the prime quiver of an indecomposable serial piecewise domain A is a chain. So, there exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents $1 = f_1 + \ldots + f_t$ such that $f_iAf_j = 0$ for $i > j$, all rings $A_{ii} = A_i = f_iAf_i$ are prime and $f_iAf_{i+1} \neq 0$ for $i = 1, \ldots, t-1$. Consequently, A_i is either a division ring or a ring $H_m(\mathcal{O})$ and

$$A = \begin{pmatrix} A_1 & A_{12} & & & * \\ & A_2 & A_{23} & & \\ & & \ddots & \ddots & \\ & 0 & & A_{t-1} & A_{t-1t} \\ & & & & A_t \end{pmatrix} \tag{3.2}$$

Since A is an indecomposable serial reduced piecewise domain of the type (3), $A_1 = H_m(\mathcal{O})$, $A_2 = D$, $\ldots$, $A_{t-1t} = D$. Consequently, matrix units $e_{1,m+1}$, $e_{2,m+1}, \ldots, e_{m,m+1}$, $e_{m+1,m+2}$, $\ldots$, $e_{m+t-2,m+t-1}$ lye in A. Therefore, all Peirce components f_iAf_j for $i \leq j$ are nonzero by [8], proposition 10.7.8, and $A = H(\mathcal{O}, m, t-1)$. Thus, all piecewise domains in the case (3) are right hereditary. The theorem is proved. □

Corollary 3.1. *For a Noetherian serial ring A the following conditions are equivalent:*

(1) A is two-sided hereditary;

(2) A is a piecewise domain.

Recall that a module M is called distributive if for all submodules K, L, N

$$K \cap (L + N) = K \cap L + K \cap N.$$

A module is called semidistributive if it is a direct sum of distributive modules. A ring A is called right (resp. left) semidistributive if the right (resp. left) regular module A_A ($_AA$) is semidistributive. A right and left semidistributive ring is called semidistributive (see [8], p.341). We write an SPSD-ring for a semiperfect semidistributive ring.

The next examples show that the conditions of theorem 3.5 are not equivalent in the case if we change the property being serial by the property being an SPSD-ring neither in the case of a two-sided Artinian ring no in the case of a two-sided Noetherian ring.

Example 3.1. Let

$$A = \begin{pmatrix} D & D & D & D \\ 0 & D & 0 & D \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{3.3}$$

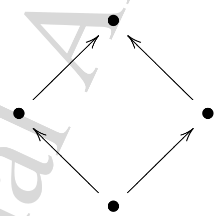
where D is a division ring.

A is obviously an Artinian ring. Since for for any primitive orthogonal idempotents $e, f \in A$ a ring $(e+f)A(e+f)$ is either $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ or $\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$, A is semidistributive, by [8], theorem 14.2.1.

Denote $P_i = e_{ii}A$, $i = 1, \ldots, 4$. Let $\varphi : P_i \to P_j$ be a nonzero homomorphism. Then $\varphi(e_{ii}a) = \varphi(e_{ii})a = e_{jj}a_0e_{ii}a$, where $a_0, a \in A$ and $e_{jj}a_0e_{ii}$ is a nonzero element from $e_{jj}Ae_{ii} = D$. Thus $d_0 = e_{jj}a_0e_{ii}$ defines a monomorphism. Therefore a ring A is a piecewise domain.

But since the right ideal $\mathcal{I} = (0\ D\ D\ D)$ is not projective, A is not right hereditary. Analogously A is not left hereditary.

The quiver $Q(A)$ has the following form:



(3.4)

which is called a diamond.

Example 3.2. Let $\mathcal{O}$ be a discrete valuation ring and $A = T_n(\mathcal{O})$, where

$$T_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O} & \mathcal{O} \\ 0 & \mathcal{O} & \ldots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathcal{O} & \mathcal{O} \\ 0 & 0 & \ldots & 0 & \mathcal{O} \end{pmatrix}$$

is the ring of all upper $n \times n$-matrices with elements from $\mathcal{O}$. Then $T_n(\mathcal{O})$ is a two-sided Noetherian SPSD-ring and a piecewise domain, but $T_n(\mathcal{O})$ is not hereditary for $n \geq 2$.

4. Serial rings with right Noetherian diagonal

If A is a serial ring with right Noetherian diagonal, then by theorem 3.3 this diagonal is two-sided Noetherian ring. Therefore it is possible to say in this case simply about Noetherian diagonal. For this ring we can construct the prime quiver which description gives the following theorem.

Theorem 4.1. ([12], theorem 2.1). *Let A be a serial ring with Noetherian diagonal. Then the prime quiver $PQ(A)$ is a disconnected union of cycles and chains.*

From this theorem and proposition 2.3 we immediately obtain the following statement.

Corollary 4.1. *Let A be a serial piecewise domain with right Noetherian diagonal. Then the prime quiver $PQ(A)$ is a disconnected union of chains.*

The description of right semihereditary serial rings with right Noetherian diagonal was obtained by V.V.Kirichenko, who proved the following theorem.

Theorem 4.2. ([16], theorem 2.8). *A right semihereditary serial indecomposable ring A with right Noetherian diagonal is up to isomorphism Morita equivalent to a ring*

$$H((\Delta_1, n_1), \ldots, (\Delta_k, n_k)) = \left(\begin{array}{c|c|c|c} A_1 & A_{12} & \ldots & A_{1k} \\ \hline O & A_2 & \ldots & A_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline O & O & \ldots & A_k \end{array} \right). \tag{4.1}$$

where $A_{ij} = M_{n_i \times n_j}(D)$, $\Delta_i = D$ or $\Delta_i = \mathcal{O}_i$ a discrete valuation ring with division ring of fractions D, $i, j = 1, 2, \ldots, k$. Moreover $A_i = T_{n_i}(D)$ if $\Delta_i = D$ and $A_i = H_{n_i}(\mathcal{O}_i)$ if $\Delta_i = \mathcal{O}_i$. Conversely, all rings of this form are right semihereditary and serial.

From theorem 3.4 and corollary 2.3 we immediately obtain the statement which is analogous to proposition 3.1.

Proposition 4.1. *If A is a serial piecewise domain with right Noetherian diagonal, then its diagonal $\overline{A}$ is Morita equivalent to the finite direct product of rings of the form D, where D is a division ring, or $H_n(\mathcal{O})$, where $\mathcal{O}$ is a discrete valuation ring.*

Lemma 4.1. *Let A be a serial piecewise domain with Noetherian diagonal which unity is decomposed into two local idempotents. Then A is isomorphic to one of the following rings:*

$$(a)\ T_2(D) = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}; \quad (b)\ H_2(\mathcal{O}) = H((\mathcal{O},1),(D,1)) = \begin{pmatrix} \mathcal{O} & D \\ 0 & D \end{pmatrix};$$

$$(c)\ H((D,1),(\mathcal{O},1),) = \begin{pmatrix} D & D \\ 0 & \mathcal{O} \end{pmatrix};$$

$$(d)\ \ H((\mathcal{O}_1,1),(\mathcal{O}_2,1)) = \begin{pmatrix} \mathcal{O}_1 & D \\ 0 & \mathcal{O}_2 \end{pmatrix},$$

where $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2$ are discrete valuation rings with common division ring of fractions D. All these rings are serial piecewise domains and semihereditary rings.

Proof. We can assume that A is indecomposable and reduced. If A is prime, then A is serial Noetherian prime ring, and, by theorem 3.4, A isomorphic to the ring $H_2(\mathcal{O})$. If A is not prime, then, by theorem 2.1, A is isomorphic to a ring $\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}$, where A_1, A_2 are prime rings. So, by proposition 4.1, A_i is a division ring or a discrete valuation ring $\mathcal{O}_i$ for $i = 1, 2$. If A_1 and A_2 are division rings, then the Jacobson radical of A is equal to the prime radical of A, and so it is nilpotent. In this case A is an Artinian ring. And then, by proposition 3.2, $A \simeq T_2(D)$. Suppose $A_1 = \mathcal{O}$ and $A_2 = D$. By theorem 2.1, A_{12} is an $\mathcal{O}$-D-bimodule, which is not zero, and as a right D module it is isomorphic to D, and as a left $\mathcal{O}$-module is isomorphic to $\mathcal{O}$. Therefore, since A is a serial ring, A_{12} is a uniserial left $\mathcal{O}$-module which is isomorphic to $\mathcal{O}$. Then analogously as in the proof of lemma 2.7 from [16], we can show that in this case $A \simeq H_2(\mathcal{O})$. Analogously we can consider the other cases. □

Theorem 4.3. *For a serial ring A with right Noetherian diagonal the following conditions are equivalent:*

(i) A is semihereditary;

(ii) A is a piecewise domain.

Proof. $(i) \Rightarrow (ii)$ follows from proposition 2.1.

$(ii) \Rightarrow (i)$ We can assume that A is indecomposable reduced ring. Since A is a piecewise domain, by theorem 2.1, there exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents $1 = f_1 + \ldots + f_t$ such that $f_iAf_j = 0$ for $i > j$, all rings $A_{ii} = A_i = f_iAf_i$ are prime and $f_iAf_{i+1} \neq 0$ for $i = 1, \ldots, t-1$. Consequently, by proposition 4.1, A_i is either a division ring or a ring of the form $H_m(\mathcal{O})$. Then, by lemma 4.1, $A_{ij} \subseteq M_{n_i \times n_j}(D)$. And so, by theorem 2.1, we have that $A_{ij} = M_{n_i \times n_j}(D)$. Therefore A has the form (4.1), and thus, by theorem 4.2, A is a semihereditary ring. $\square$

References

[1] F.W.Anderson and K.R.Fuller, Rings and Categories of Modules. Second edition, Graduate Texts in Mathematics, Vol.13, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[2] S.U.Chase, *A generalization of the ring of triangular matrix*, Nagoya Math. J., V.**18**, 1961, pp.13-25.

[3] H.Cartan and S.Eilenberg, Homological algebra. Princeton University Press, Princeton, New York, 1956.

[4] D.Eisenbud, P.Griffith, *The structure of serial rings*, Pacific J. Math., V.**36**, N.**1**, 1971, pp.109-121.

[5] D.Eisenbud, P.Griffith, *Serial rings*, Journal of Algebra, V.**17**, 1971, pp.389-400.

[6] Yu.A.Drozd, *The structure of hereditary rings*, Math. Sbornik, V.**113**(**155**), 1980 (in Russian); English translation: Math. USSR Sbornik, V.**41**, 1982, No.1, pp.139-148.

[7] K.R.Fuller, *On indecomposable injectives over artinian rings*, Pacific J. Math., V.**29**, 1969, pp.115-135.

[8] Michiel Hazewinkel, Nadiya Gubareni, V.V.Kirichenko, *Algebras, Rings and Modules*, vol.1, Kluwer Academic Publishers, 2004.

[9] Michiel Hazewinkel, Nadiya Gubareni, V.V.Kirichenko, *Algebras, Rings and Modules*, vol.2, Springer, 2007.

[10] M.Harada, *Hereditary semi-primary rings and triangular matrix rings*, Nagoya Math.J., V.**27**, 1966, pp.463-484.

[11] R.Gordon, L.W.Small, *Piecewise domains*, Journal of Algebra, V.**23**, 1972, pp.553-564.

[12] N.Gubareni, V.V.Kirichenko, *Serial Rings with T-Nilpotent Prime Radical*, Algebra and Representation Theory, N.**9**, 2006, pp.147-160.

[13] G.Ivanov, *Left Generalized Uniserial Rings*, Jornal of Algebra, V.**31**, 1974, pp.166-181.

[14] V.V.Kirichenko, *Generalized uniserial rings*, Mat. sb. V.**99(141)**, N.**4**, 1976, pp.559-581.

[15] V.V.Kirichenko, *Right Noetherian rings over which all finitely generated modules are semichain modules*, Dokl. Acad. Nauk Ukrain. SSR Ser.A**1**, 1976, pp.9-12.

[16] V.V.Kirichenko, *Hereditary and semihereditary serial rings*, Zapiski Nauchnych Sem. LOMI AN SSSR, Vol.**114**, 1982.

[17] G.Köthe, *Verallgemeinerte Abelsche Gruppen mit Hyperkomplexen Operatorenring*, Math. Z., V.**39**, 1935, pp.31-44.

[18] T.Y.Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, Vol.189, Springer-Verlag, Berlin-Heidelberg-New York, 1999.

[19] G.Michler, *Structure of semiperfect hereditary Noetherian rings*, Journal of Algebra, Vol.**3**, 1969, pp.327-344.

[20] T.Nakayama, *On Frobeniusean algebras* I,II, Ann. of Math., V.**40**, 1939, pp.611-633; V.**42**, 1941, pp.1-21.

[21] T.Nakayama, *Note on uniserial and generalized uniserial rings*, Proc. Imp. Acad. Tokyo, V.**16**, 1940, pp.285-289.

[22] L.A.Skornyakov, *When are all modules semi-chained?*, Mat. Zametki, V.**5**, 1969, pp.173-182.

[23] L.W.Small, *Hereditary rings*, Proc. Nat. Acad. Sci. U.S.A., V.**55**, 1966, pp.25-27.

[24] M.L.Teply, *Right hereditary, right perfects rings are semiprimary*, In: Advances in Ring Theory, 1997, pp.313-316.

[25] R.B.Warfield, *Serial rings and finitely presented modules*, Journal of Algebra, V.**37**, 1975, pp.187-222.

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